Int. Journal of Math. Analysis, Vol. 1, 2007, no. 23, 1131 - 1135

Asymptotic Properties of the Hermite-Fejér Interpolation on the Roots of the Legendre Polynomials

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Abstract

In this paper, we give a result concerning an asymptotic property of $H_{kn}(\cos\theta)$ associated with the Hermite-Fejér interpolation on the roots $x_{kn} = \cos\theta_{kn}$ of the Legendre polynomials.

Mathematics Subject Classification: 41A05, 41A10

Keywords: Hermite-Fejér interpolation, orthogonal polynomials, Legendre polynomials

1 Introduction

There are many ways to define the Legendre polynomials $P_n(x)$. The Legendre polynomials are defined and orthogonal on the interval $[-1, 1]$ with respect to the weight function $w(x) = 1$. They are the solution of the following differential equation:

$$\frac{d}{dx}(1-x^2)\frac{dP_n(x)}{dx} + n(n+1)P_n(x) = 0,$$

that satisfy the condition $P_n(1) = 1$. They satisfy the following recurrence relation:

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x).$$

They satisfy the orthogonality conditions:

$$\int_{-1}^{1} P_n(x)P_m(x)dx = \frac{2}{2n+1}\delta_{n,m}. \tag{1}$$

They also satisfy:

$$|P_n(x)| \leq 1, \quad P_n(\pm 1) = (-1)^n.$$

The Legendre polynomials are special case of the Jacobi polynomials, $P_n^{(\alpha,\beta)}(x)$, for $\alpha = \beta = 0$, see [1, 3].

2 Preliminaries

Let f be a real valued function on $[-1, 1]$. The polynomial $H_{2n-1}(f, x)$ of degree $\leq 2n-1$ of Hermite-Fejér interpolation based on the zeros $x_{kn}, \quad k = 1, 2, \ldots, n$ of the Legendre polynomials $P_n(x)$ is defined by

$$H_{2n-1}(f, x) = \sum_{k=1}^{n} f(x_{kn}) H_{kn}(x), \tag{2}$$

where

$$H_{kn}(x) = \frac{1 - 2xx_{kn} + x_{kn}^2}{1 - x_{kn}^2} \left(\frac{P_n(x)}{(x - x_{kn}) P_n'(x_{kn})} \right)^2. \tag{3}$$

The roots (zeros) $x_{kn}, \quad k = 1, 2, \ldots, n$ of the Legendre polynomial $P_n(x)$ are all real, distinct, and belong to $(-1, 1)$.

In discussing the roots (zeros) $x_{kn} = \cos(\theta_{kn}), \quad k = 1, 2, \ldots, n$ of the Legendre polynomial $P_n(x)$ we use the enumeration:

$$0 < \theta_{1n} < \theta_{2n} < \cdots < \theta_{nn} < \pi, \quad 1 > x_{1n} > x_{2n} > \cdots > x_{nn} > -1.$$

For more, see [5].

3 Asymptotic property

In [2] asymptotic properties of the Hermite-Fejér interpolation based on the zeros $x_{kn}, \quad k = 1, 2, \ldots, n$ of the Chebyshev polynomials of the first kind $T_n(x)$ are studied. Analogously, we give in the following theorem asymptotic property of $H_{kn}(x)$ of the Hermite-Fejér interpolation based on the roots $x_{kn}, \quad k = 1, 2, \ldots, n$ of the Legendre polynomials, for more literature see [5].

Theorem: For $\theta \in (0, \pi)$, $\frac{1}{2} < \gamma < 1$, and all sufficiently large n, we have for the Legendre polynomials, $P_n(x)$,

$$\Big| \sum_{\theta - n^{-\gamma} < \theta_{kn} < \theta} H_{kn}(\cos\theta) - \sum_{\theta - n^{-\gamma} < \theta_{kn} < \theta} \left(\frac{P_n(\cos\theta)}{\sin\theta_{kn}(\theta - \theta_{kn}) P_n'(\cos\theta_{kn})} \right)^2 \Big| \leq \frac{c(\theta)}{n^{2\gamma - 1}} \tag{4}$$

and

$$\Big|\sum_{\theta<\theta_{kn}<\theta+n^{-\gamma}} H_{kn}(\cos\theta)-\sum_{\theta<\theta_{kn}<\theta+n^{-\gamma}}\left(\frac{P_n(\cos\theta)}{\sin\theta_{kn}(\theta-\theta_{kn})P_n'(\cos\theta_{kn})}\right)^2\Big|\le\frac{c(\theta)}{n^{2\gamma-1}}\quad(5)$$

where $c(\theta)$ is a positive constant depending on θ.

Proof: Let

$$E_n(\theta)=\{k:\theta-n^{-\gamma}<\theta_{kn}<\theta\}$$

Then

$$\sum_{k\in E_n(\theta)} H_{kn}(\cos\theta)-\sum_{k\in E_n(\theta)}\left(\frac{P_n(\cos\theta)}{\sin\theta_{kn}(\theta-\theta_{kn})P_n'(\cos\theta_{kn})}\right)^2=$$

$$=\sum_{k\in E_n(\theta)}\frac{\sin^2\theta}{\sin^2\theta_{kn}}\left(\frac{P_n(\cos\theta)}{(\cos\theta-\cos\theta_{kn})P_n'(\cos\theta_{kn})}\right)^2+$$

$$+\sum_{k\in E_n(\theta)}\left(\frac{P_n(\cos\theta)}{\sin\theta_{kn}P_n'(\cos\theta_{kn})}\right)^2-\sum_{k\in E_n(\theta)}\left(\frac{P_n(\cos\theta)}{\sin\theta_{kn}(\theta-\theta_{kn})P_n'(\cos\theta_{kn})}\right)^2$$

$$=\sum_{k\in E_n(\theta)}\left(\frac{P_n(\cos\theta)}{\sin\theta_{kn}P_n'(\cos\theta_{kn})}\right)^2\left(\frac{\sin^2\theta}{(\cos\theta-\cos\theta_{kn})^2}-\frac{1}{(\theta-\theta_{kn})^2}\right)$$

$$+\sum_{k\in E_n(\theta)}\left(\frac{P_n(\cos\theta)}{\sin\theta_{kn}P_n'(\cos\theta_{kn})}\right)^2.$$

Hence

$$\Big|\sum_n^{(1)}(\theta)-\sum_{k\in E_n(\theta)}\left(\frac{P_n(\cos\theta)}{\sin\theta_{kn}(\theta-\theta_{kn})P_n'(\cos\theta_{kn})}\right)^2\le$$

$$\le\sum_{k\in E_n(\theta)}\left(\frac{P_n(\cos\theta)}{\sin\theta_{kn}P_n'(\cos\theta_{kn})}\right)^2\Big|\frac{\sin^2\theta}{(\cos\theta-\cos\theta_{kn})^2}-\frac{1}{(\theta-\theta_{kn})^2}\Big|+$$

$$+\sum_{k\in E_n(\theta)}\left(\frac{P_n(\cos\theta)}{\sin\theta_{kn}P_n'(\cos\theta_{kn})}\right)^2.$$

$=: A_n(\theta)+B_n(\theta)$, say.

Combining inequalities 7.3.8 , 8.9.1 , 8.9.7 in [5], we get

$$\left(\frac{P_n(x)}{\sin\theta_{kn}P_n'(x_{kn})}\right)^2\le\frac{c(\theta)}{n^2}$$

Since the number of k's in $E_n(\theta)$ is $\leq \frac{n^{1-\gamma}}{\pi}$, we have

$$B_n(\theta) \leq \frac{c(\theta)}{n^2} \sum_{k \in E_n(\theta)} 1 \leq \frac{c(\theta)}{n^2} . \frac{n^{1-\gamma}}{\pi} = \frac{c(\theta)}{n^{1+\gamma}}$$

Also since

$$(\theta - \theta_{kn})^2 = \frac{(\cos\theta - \cos\theta_{kn})^2}{\sin^2\bar{\theta}} \quad \text{for some } \bar{\theta} \text{ between } \theta \text{ and } \theta_{kn}$$

then

$$A_n(\theta) = \sum_{k \in E_n(\theta)} \frac{1}{\sin^2\theta_{kn}} \left(\frac{P_n(\cos\theta)}{(\cos\theta - \cos\theta_{kn}) P_n'(\cos\theta_{kn})} \right)^2 . \left| \sin^2\theta - \sin^2\bar{\theta} \right|$$

Also since

$$\left(\frac{P_n(\cos\theta)}{(\cos\theta - \cos\theta_{kn}) P_n'(\cos\theta_{kn})} \right)^2$$

is bounded by a constant c, see inequality 10 in [4], $\sin^2\theta_{kn} \geq M_n^2(\theta)$ and

$$\left| \sin^2\theta - \sin^2\bar{\theta} \right| \leq 2 \left| \theta - \bar{\theta} \right| \leq 2 \left| \theta - \theta_{kn} \right|$$

we get

$$A_n(\theta) \leq \frac{c}{M_n^2(\theta)} \sum_{k \in E_n(\theta)} \left| \theta - \theta_{kn} \right| \leq \frac{cn^{-\gamma}}{M_n^2(\theta)} \sum_{k \in E_n(\theta)} 1 \leq c(\theta) n^{1-2\gamma} = \frac{c(\theta)}{n^{2\gamma-1}}.$$

Thus (3) follows, and (4) can be shown similarly, which completes the proof.

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Received: July 1, 2007