**Int. Journal of Math. Analysis, Vol. 1, 2007, no. 23, 1125 - 1130**

# On Bounds of the Fundamental Polynomials Associated with the Hermite-Fejér Interpolation on the Roots of the Jacobi Polynomials

**Abedallah Rababah**

Department of Mathematics and Statistics
Jordan University of Science and Technology
Irbid 22110, Jordan
rababah@just.edu.jo

**Abstract**

In this paper, bounds on the sum of the fundamental polynomials $H_{kn}(\cos\theta)$ associated with the Hermite-Fejér interpolation on the roots $x_{kn} = \cos\theta_{kn}$ of the Jacobi Polynomials $P_n^{(\alpha,\beta)}(\cos\theta), \quad \alpha, \beta > -1$ are presented.

**Mathematics Subject Classification:** 41A05, 41A10

**Keywords:** Orthogonal polynomials, Jacobi polynomials, Hermite-Fejér interpolation

## 1 Introduction

The Jacobi polynomials $P_n^{(\alpha,\beta)}(x), \quad \alpha, \beta > -1$ are orthogonal polynomials on $[-1, 1]$ with respect to the weight function $w(x) = (1-x)^\alpha(1+x)^\beta, \alpha, \beta > -1$. The Jacobi polynomials play very important role in the theory of approximation and interpolation. The roots of the Jacobi polynomials are used to get an optimal placing of the interpolation nodes to minimize the error in the methods of interpolation.

The roots $x_{kn}, \quad k = 1, 2, \ldots, n$ of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x), \quad \alpha, \beta > -1$ are real, simple, and lie in the interior of the interval $[-1, 1]$. The roots $x_{kn} = \cos(\theta_{kn}), \quad k = 1, 2, \ldots, n$ of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x), \quad \alpha, \beta > -1$ are ordered in the following form

$$0 < \theta_{1n} < \theta_{2n} < \cdots < \theta_{nn} < \pi,$$

and

$$1 > x_{1n} > x_{2n} > \cdots > x_{nn} > -1.$$

For more about these topics, see [3, 4, 5, 9, 10].

## 2 Hermite-Fejér Interpolation

Consider the interval $[a, b]$ and the set $S_n = \{x_k\}_{k=1}^n$ of $n$ distinct points from $[a, b]$ that satisfy

$$b \geq x_1 > x_2 > \cdots > x_n \geq a.$$

We also consider the sets $F_n = \{f_k\}_{k=1}^n$ and $F'_n = \{f'_k\}_{k=1}^n$ of arbitrary values. We construct a polynomial $H_{2n-1}(x)$ of lowest degree which satisfies:

$$H_{2n-1}(x_k) = f_k, \quad H'_{2n-1}(x_k) = f'_k, \quad k = 1, 2, \ldots, n.$$

The polynomial of degree $\leq 2n - 1$ corresponding to $S_n$ is given by

$$H_{2n-1}(f, x) = \sum_{k=1}^{n} f_k H_{kn}(x) + \sum_{k=1}^{n} f'_k B_{kn}(x), \tag{1}$$

where

$$H_{kn}(x) = V_k(x) L_k^2(x),$$

$$B_{kn}(x) = (x - x_k)\, L_k^2(x),$$

$$L_k(x) = \frac{W_n(x)}{(x - x_k) W'_n(x_k)},$$

$$V_k(x) = 1 - \frac{W''_n(x_k)}{W'_n(x_k)}(x - x_k),$$

$$W_n(x) = c(x - x_1)(x - x_2) \cdots (x - x_n), \quad c \neq 0.$$

The polynomial $H_{2n-1}(f, x)$ is the well known Hermite interpolating polynomial.

Let $f(x)$ be a real valued function on $[-1, 1]$, and let

$$f_k = f(x_k), \quad f'_k = 0, \quad k = 1, 2, \ldots, n,$$

then the interpolating polynomial becomes

$$H_{2n-1}(f, x) = \sum_{k=1}^{n} f_k H_{kn}(x). \tag{2}$$

This polynomial $H_{2n-1}(f,x)$ is called the Hermite-Fejér interpolating polynomial. If $f(x)$ is continuous and the nodes of interpolation $x_k = x_{kn}, \quad k = 1,2,\ldots,n$ are the roots of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x), \quad \alpha,\beta > -1$ on the interval $[-1,1]$ then $H_{2n-1}(f,x)$ converges uniformly to $f(x)$ over the interval $[-1+\epsilon, 1-\epsilon]$, $\epsilon > 0$. For $\alpha < 0$ it converges uniformly over $[-1+\epsilon, 1]$, $\epsilon > 0$. For $\beta < 0$ it converges uniformly over $[-1, 1-\epsilon]$, $\epsilon > 0$. It is divergent at $x = 1$ if $f(x)$ is merely continuous and $\alpha \geq 0$, and it is divergent at $x = -1$ if $f(x)$ is merely continuous and $\beta \geq 0$.

Since $W_n(x_{kn}) = 0$, we get the simplifications

$$\frac{W_n''(x_{kn})}{W_n'(x_{kn})} = \frac{\alpha-\beta+(\alpha+\beta+2)x_{kn}}{1-x_{kn}^2},$$

and

$$V_k(x) = \frac{1-x(\alpha-\beta+(\alpha+\beta+2)x_{kn})+(\alpha-\beta)x_{kn}+(\alpha+\beta+1)x_{kn}^2}{1-x_{kn}^2}.$$

Thus the polynomial $H_{2n-1}(f,x)$ of degree $\leq 2n-1$ of Hermite-Fejér interpolation based on the zeros $x_{kn}, \quad k = 1,2,\ldots,n$ of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x), \quad \alpha,\beta > -1$ is defined by

$$H_{2n-1}(f,x) = \sum_{k=1}^{n} f(x_{kn})H_{kn}(x), \tag{3}$$

where

$$\begin{aligned} H_{kn}(x) &= \frac{1-x(\alpha-\beta+(\alpha+\beta+2)x_{kn})+(\alpha-\beta)x_{kn}+(\alpha+\beta+1)x_{kn}^2}{1-x_{kn}^2} \times \\ &\times \left(\frac{P_n^{(\alpha,\beta)}(x)}{(x-x_{kn})P_n^{(\alpha,\beta)\prime}(x)}\right)^2. \end{aligned} \tag{4}$$

# 3 Bounds on the Fundamental Polynomials

In this section, we give a bound property of the summation of the fundamental polynomials $H_{kn}(\cos\theta)$ associated with the Hermite-Fejér interpolation on the roots $x_{kn} = \cos\theta_{kn}$ of the Jacobi Polynomials $P_n^{(\alpha,\beta)}(\cos\theta), \quad \alpha,\beta > -1$. Related results are given in [1, 2, 6, 7, 8].

Now we present our main result.

**Theorem:** Let $x \in [-1,1]$ and $x_{kn} = \cos\theta_{kn}$ be the roots of the Jacobi polynomials, $P_n^{\alpha,\beta}(x)$, then for $|\alpha| \leq \frac{1}{2}, |\beta| \leq \frac{1}{2}$, and $0 < \gamma < 1$ we have

$$\lim_{n\to\infty} \sup n^{1-\gamma} \sum_{|\theta-\theta_{kn}|>n^{-\gamma}} H_{kn}(\cos\theta) < \infty.$$

**Proof:** Let $n^{-\gamma} \leq \min\{\frac{\theta}{2}, \frac{\pi-\theta}{2}, (3\pi)^{\frac{-\gamma}{1-\gamma}}\}$ we have

$$\sum_{\theta_{kn}<\theta-n^{-\gamma}} H_{kn}(\cos\theta) = \sum_{\theta_{kn}<\theta-n^{-\gamma}} \left\{\frac{1-\cos\theta(\alpha-\beta+(\alpha+\beta+2)\cos\theta_{kn})}{\sin^2\theta_{kn}}+\right.$$

$$\left.\frac{(\alpha-\beta)\cos\theta_{kn}+(\alpha+\beta+1)\cos^2\theta_{kn}}{\sin^2\theta_{kn}}\right\}\left(\frac{P_n^{\alpha,\beta}(\cos\theta)}{(\cos\theta-\cos\theta_{kn})P_n^{(\alpha,\beta)'}(\cos\theta_{kn})}\right)^2$$

Since

$$\Big(\frac{P_n^{(\alpha,\beta)}(\cos\theta)}{\sin\theta_{kn}P_n^{(\alpha,\beta)'}(\cos\theta_{kn})}\Big)^2 \leq \frac{c(\theta,\alpha,\beta)}{n^2} \tag{5}$$

where $c(\theta,\alpha,\beta)$ is a positive constant depends on $\theta,\alpha,\beta$, see [7]. Thus we have

$$\sum_{\theta_{kn}<\theta-n^{-\gamma}} H_{kn}(\cos\theta) \leq \frac{c(\theta,\alpha,\beta)}{n^2}\sum_{\theta_{kn}<\theta-n^{-\gamma}}\frac{1}{(\cos\theta-\cos\theta_{kn})^2}$$

$$\leq \frac{c(\theta,\alpha,\beta))}{n^2}\sum_{i=1}^{l-1}\frac{1}{(\cos\theta-\cos\theta_{in})^2}$$

where $l$ is the smallest value of $k$ such that $\theta - n^{-\gamma} < \theta_{kn} < \theta$. Condition $n^{-\gamma} \leq (3\pi)^{\frac{-\gamma}{1-\gamma}}$ guarantees that $l \geq 2$. Since

$$\theta_{l-i,n} < \frac{2(l-i)}{2n+1}\pi = \frac{2l-3}{2n+1}\pi - \frac{2i-3}{2n+1}\pi$$

$$< \theta_{l-1,n} - \frac{2i-3}{2n+1}\pi$$

$$< \theta - n^{-\gamma} - \frac{2i-3}{2n+1}\pi.$$

For $i = 1, 2, \ldots, l-1$, we have

$$|\cos\theta_{l-i,n} - \cos\theta| = \cos\theta_{l-i,n} - \cos\theta$$

$$\geq \cos(\theta - n^{-\gamma} - \frac{2i-3}{2n+1}\pi) - \cos\theta$$

$$= 2\sin(\theta - \frac{1}{2}(n^{-\gamma} - \frac{2i-3}{2n+1}\pi))\sin(\frac{1}{2}(n^{-\gamma} + \frac{2i-3}{2n+1}\pi)).$$

Since $\theta > \theta_{l,n} > \frac{2l-1}{2n+1}\pi$, we have

$$n^{-\gamma} + \frac{2i-3}{2n+1}\pi < \frac{\pi-\theta}{2} + \frac{2l-1}{2n+1}\pi$$

$$\leq \frac{\pi - \theta}{2} + \theta = \frac{\pi + \theta}{2}$$

Since $\theta < \pi$ thus we have

$$n^{-\gamma} + \frac{2i-3}{2n+1}\pi < \pi$$

and

$$\theta - \frac{1}{2}(n^{-\gamma} + \frac{2i-3}{2n+1}\pi) \geq \theta - \frac{1}{2}(n^{-\gamma} + \frac{2l-1}{2n+1}\pi)$$
$$\geq \frac{1}{2}(\theta - n^{-\gamma}) \geq \frac{\theta}{4}$$

Thus

$$|\cos\theta_{l-i,n} - \cos\theta| \geq \frac{2}{\pi}(n^{-\gamma} + \frac{2i-3}{2n+1}\pi)M(\theta),$$

where $M(\theta) = \min\{\sin\theta, \sin\frac{\theta}{4}\}$. So our inequality becomes

$$\sum_{\theta_{kn}<\theta-n^{-\gamma}} H_{kn}(\cos\theta) \leq \frac{c(\theta,\alpha,\beta)}{n^2}\sum_{i=1}^{l-1}\frac{\pi^2}{4M^2(\theta)}\frac{1}{(n^{-\gamma}+\frac{(2i-3)\pi}{2n+1})^2}$$
$$\leq \frac{c(\theta,\alpha,\beta))}{n^2}\sum_{i=1}^{\infty}\frac{1}{(n^{1-\gamma}+(2i-3)\pi)^2}.$$

Finally, a simple calculation shows that

$$\sum_{i=1}^{\infty}\frac{1}{(n^{1-\gamma}+(2i-3)\pi)^2} = O(\frac{1}{n^{1-\gamma}}).$$

Hence

$$\lim_{n\to\infty}\sup n^{1-\gamma}\sum_{\theta_{kn}<\theta-n^{-\gamma}} H_{kn}(\cos\theta) < \infty.$$

Similarly we can show that

$$\lim_{n\to\infty}\sup n^{1-\gamma}\sum_{\theta_{kn}>\theta+n^{-\gamma}} H_{kn}(\cos\theta) < \infty.$$

And thus the theorem follows.

# References

[1] Al-Jarrah, R. and Rababah, A. (1990), On the rate of convergence of Hermite-Fejér interpolation to functions of bounded variation on the zeros of certain Jacobi polynomials, Revista Colombiana de Matemáticas, V. 24, 51-64.

[2] Bojanic, R., and Cheng, F. (1983), Estimate for the rate of approximation of functions of bounded variation by Hermite-Fejér polynomials, Second Edmont Conference on Approximation Theory, CMS Conference Proceedings V. 3, 5-17.

[3] Chihara, T. (1978), An introduction to orthogonal polynomials, Gordon and Breach, Science Publ. Inc.

[4] Freud, G. (1971), Orthogonal polynomials, Oxford. Pergamon.

[5] Natanson, I. (1965), Constructive function theory, V. I-III, Ungar.

[6] Rababah, A. (2004), Jacobi-Bernstein basis transformation, Computational Methods in Applied Mathematics 4(2), 206-214.

[7] Rababah, A. (2007), Inequalities for the Jacobi Polynomials and their Derivative, Submitted.

[8] Rababah, A. (2007), Asymptotic properties of the Hermite-Fejér interpolation on the roots of the Legendre polynomials, Submitted.

[9] Rainville, E. (1960), Special functions, Chelsea Publ. Co., Bronx, New York.

[10] Szegö, G. (1959), Orthogonal polynomials, Amer. Math. Soc., Coll. Publ. V. 23, Amer. Math. Soc., New York.

**Received: April 10, 2007**