

Wavelet Bases of the Interval: A New Approach

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Abstract

We introduce a new approach in order to construct a wavelets with arbitrary support on the interval. This result allows us to adapt on the interval some important classes of wavelets (coifflets, symlets, Lemarié's wavelets). In addition, we show how to obtain multiresolution analysis on the interval with better approximation rate in some neighborhood of the interval.

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1 Introduction

Orthonormal wavelet bases have revealed to be powerful tool in applied mathematics and digital signal processing. The possibility of data compression offered by multi-scale decomposition leads to some good results in speech, image coding and fast numerical analysis of operators[5, 2].

Wavelet bases are usually constructed via multiresolution analysis on \mathbb{R} . However, In many applications, we are interested in problems confined on the interval. Examples are numerical analysis with boundary conditions at the edges of the interval[1], image analysis: where the domain of interest is the Cartesian product of two intervals. To fix notations, let us assume that the interval is $[0, 1]$. The extension of the function to the whole real line introduce

some disagreeable effects. (For details see [4]). Several solutions have been proposed in [3, 9, 4]. They all depend on the various ways of adapting the multiresolution analysis to the interval $[0, 1]$. The oldest example of such orthonormal basis of compactly supported wavelets is the well-known Haar basis. A generalization of this basis to a non-orthogonal smooth spline-wavelet bases were constructed on the bounded interval in [3] from spline-multiresolution analysis.

The first smooth orthonormal wavelet bases on an interval was proposed by Meyer [9]. This construction starts from any compactly supported wavelet bases of Daubechies, with N vanishing moments. The result of Meyer's construction is an orthonormal family of wavelets in $[0, 1]$ which they have N vanishing moments and the same regularity as the original ψ . Together with the scaling functions at the coarsest scale under consideration, they form an orthonormal basis for $L^2([0, 1])$. This construction has some disadvantages: firstly, the inequality between the scaling functions number and the wavelets number at a fixed resolution does not enable us to construct a wavelet packets on the interval. The other objection of Meyer's construction is that the explicit construction of the edge functions involves the diagonalization of a matrix becomes ill conditioned (see [4]).

For these reasons, a modification of this construction was proposed by Cohen-Daubechies-Vial (CDV for short)[4]. The resulting wavelets on the interval are derived from **minimal** compactly supported wavelet for \mathbb{R} , with N vanishing moments. The number of scaling function at resolution j is 2^j . Moreover, as in Meyer's case, all polynomials on $[0, 1]$ can be written as linear combinations of the scaling functions at any fixed scale. It then follows that all the corresponding wavelets have N vanishing moments, and an approximation rate of order N .

In this work we give a general method to construct wavelet bases on an interval with arbitrary support. We start with L vanishing moments wavelets with the support $[0, 2N - 1]$ for $N > L$. Using the CDV principle in this case, the number of scaling functions is less than 2^j (number of wavelets).

The paper is organized as follows: in the second section, we describe a generalization of CDV method in order to construct orthogonal wavelets on an interval with arbitrary support. The section 3 proposes a new approach to construct wavelet bases. In particular, we show that one can construct a multiresolution analyzes with a better approximation order in some neighborhood of the edges of the interval.

2 Adapted CDV Method

In this section, we study the construction of arbitrary compactly supported wavelets with L vanishing moments on a real interval.

Our starting point is the L vanishing moment family of [7], [6] or [8]. We consider the scaling function defined by the following two-scale relation:

$$\varphi(x) = \sqrt{2} \sum_{k=0}^{2N-1} h_k \varphi(2x - k). \quad (1)$$

The wavelet ψ is defined by the formula :

$$\psi(x) = \sqrt{2} \sum_{k=0}^{2N-1} g_k \varphi(2x - k), \quad (2)$$

and the reconstruction formula is given by:

$$\varphi_{1,2k}(x) = \sum_{\ell=k-N+1}^k h_{k-2\ell} \varphi(x - \ell) + g_{k-2\ell} \psi(x - \ell). \quad (3)$$

It follows from (1) and (2) above that $\text{supp}\varphi = \text{supp}\psi = [0, 2N - 1]$, where $N > L + 1$. We use the principles outlined in [4] to construct a multiresolution analysis for $L^2([0, 1])$. We have $2^j - 2N + 2$ interior functions, and L added functions at each edge, so that all polynomials up to degree $L - 1$ can be generated. We have $2^j - 2N + 2$ interior wavelets. If we add (as proposed in [4]) to this family the functions defined by

$$\tilde{\psi}_{j\ell}^{\text{left}} = P_{W_j^{[0,1]}}(\varphi_{j+1,\ell}^{\text{left}}),$$

and

$$\tilde{\psi}_{j\ell}^{\text{right}} = P_{W_j^{[0,1]}}(\varphi_{j+1,\ell}^{\text{right}}), \quad \ell = 0, \dots, L - 1$$

then these functions are independent and orthogonal onto the interior wavelets. However,

$$\begin{aligned} \#\{\tilde{\psi}_{jk}^{\text{left}}, \psi_{0\ell}, \tilde{\psi}_{jk}^{\text{right}}, \quad k = 0, \dots, L - 1, \ell = 0, \dots, 2^j - 2N + 1\} \\ = 2^j - 2N + 2L + 2. \end{aligned}$$

This observation means that these functions are not enough to generate the subspace $W_j^{[0,1]}$. The following proposition gives an other method to construct a basis of the subspace $W_j^{[0,1]}$.

Proposition 2.1 *The set $\{P_{W_j^{[0,1]}}(\varphi_{j+1,2k}), \psi_{0\ell}, P_{W_j^{[0,1]}}(\varphi_{j+1,2^j-2N+1-2k}), k = 0, \dots, N - 2, \ell = 0, \dots, 2^j - 2N + 1\}$ is a basis of $W_j^{[0,1]}$.*

Let us show this result for the left edge wavelets, this has to be repeated at the right end. Let consider the half line $[0, +\infty[$. We define W_0^I the subspace of $W_0^{[0,+\infty[}$ generated by the interior wavelets (i.e. $\psi_{0k}, k \geq 0$), and let W_0^B the orthogonal complement of W_0^I in $W_0^{[0,+\infty[}$, namely

$$W_0^{[0,+\infty[} = W_0^B \oplus W_0^I.$$

We have the following lemma :

Lemma 2.2 *The functions defined by*

$$\tilde{\psi}_k = P_{W_0^B}(\varphi_{1,2k}), \quad \text{for } k = 0, \dots, N - 2, \tag{4}$$

constitute a basis of W_0^B .

We can compute the functions $\tilde{\psi}_k, k = 0, \dots, N - 2$, since we dispose an orthonormal basis of $V_0^{[0,+\infty[}$. Indeed,

$$\begin{aligned} \tilde{\psi}_k &= \varphi_{1,2k} - P_{V_0^{[0,+\infty[}}(\varphi_{1,2k}) - P_{W_0^I}(\varphi_{1,2k}) \\ &= \varphi_{1,2k} - \sum_{\ell=0}^{L-1} \langle \varphi_{1,2k}, \varphi_{0\ell}^0 \rangle \varphi_{0\ell}^0 - \sum_{\ell=0}^k h_{2k-2\ell} \varphi_{0\ell} - \sum_{\ell=0}^k g_{2k-2\ell} \psi_{0\ell}. \end{aligned}$$

Proof of lemma 2.2.

First, we remark that $\dim W_0^B = N - 1$. Then, we need just to prove that the functions $\tilde{\psi}_k, k = 0, \dots, N - 2$ are linearly independent. Since they are, by definition, orthogonal onto interior wavelets.

The family $\{f_k, k = 0, \dots, N - 2\}$ defined on the interval $[0, +\infty[$ by

$$f_k = \varphi_{1,2k} - \sum_{\ell=0}^k \langle \varphi_{1,2k}, \varphi_{0\ell} \rangle \varphi_{0\ell} - \sum_{\ell=0}^k \langle \varphi_{1,2k}, \psi_{0\ell} \rangle \psi_{0\ell}$$

is free. To prove this observation, we use (3) to obtain

$$f_k = \sum_{\ell=k-N+1}^{-1} h_{2k-2\ell} \varphi_{0\ell} + g_{2k-2\ell} \psi_{0\ell}, \quad k = 0, \dots, N - 2. \tag{5}$$

Let $(\lambda_k)_{k=0, \dots, N-2}$, any sequence of real numbers, such that

$$\sum_{k=0}^{N-2} \lambda_k f_k = 0. \tag{6}$$

We compute successively $\langle \sum_{k=0}^{N-2} \lambda_k f_k, \varphi_{0\ell} \rangle$, for $\ell = -N + 1, -N + 2, \dots, -1$. The first result is

$$0 = \left\langle \sum_{k=0}^{N-2} \lambda_k f_k, \varphi_{0,-N+1} \right\rangle = \lambda_0 h_{2N-2}. \tag{7}$$

Since $h_{2N-2} \neq 0$ then we get $\lambda_0 = 0$. By induction, we deduce immediately that $\lambda_k = 0$, for $k = 0, \dots, N - 2$. Thus, we get the linear independence of the functions f_k , $k = 0, \dots, N - 2$.

The second observation is that the functions f_k , $k = 0, \dots, N - 2$ are orthogonal onto interior wavelets, and we have :

$$\tilde{\psi}_k = f_k - \sum_{\ell=0}^{L-1} \langle f_k, \varphi_{0\ell}^0 \rangle \varphi_{0\ell}^0, \quad k = 0, \dots, N - 2. \tag{8}$$

Since the family $\{\varphi_{0\ell}^0, f_k; \ell = 0, \dots, L-1, k = 0, \dots, N-2\}$ is free, we deduce that the functions $\{\tilde{\psi}_k, k = 0, \dots, N - 2\}$ are linearly independents. Indeed, the functions f_k and $\varphi_{0\ell}^0$ belong to $V_1^{[0,+\infty[} = V_1^I \oplus V_1^B$. However $f_k \in V_1^I$, for $k = 0, \dots, N - 2$ and $P_{V_1^B}(\varphi_{0\ell}^0) \neq 0$. ■

For the convenience of the reader we repeat the relevant material of the construction from [4] without proofs, thus making our exposition self contained.

The supports of the functions $\tilde{\psi}_k$, $k = 0, \dots, N - 2$ are not staggered. Using the following lemma, we obtain, from this family, $N - 1$ functions with staggered supports.

Lemma 2.3 *If f and g are two linearly independent functions in $V_0^{[0,+\infty[}$ with*

$$\text{supp} f = \text{supp} g = [0, M], \quad M \in \mathbb{N}.$$

Then there exist $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$ such that $\text{supp} (\mu f + \lambda g) = [0, M - 1]$.

Finally, using Gram-Schmidt procedure, we obtain orthonormal functions.

The construction at the right edge is adapted as the following :

- We note $\check{f}(x) = f(-x)$, for $x \in \mathbb{R}$ and

$$\check{V}_j = \{\check{f}, f \in V_j\}, \quad j \in \mathbb{Z}.$$

- Applying the above algorithm on $(\check{V}_j)_{j \in \mathbb{Z}}$, we construct a multiresolution analysis on $[0, +\infty[$. We get then an edge scaling function $\check{\varphi}_\ell^0$ and the edge wavelets $\check{\psi}_\ell^0$, $\ell = 0, \dots, L - 1$.

- If we define the functions $\varphi_{j\ell}^{\text{right}}(x) = 2^{j/2}\varphi_{\ell}^0(-2^jx)$ and $\psi_{j\ell}^{\text{right}}(x) = 2^{j/2}\psi_{\ell}^0(-2^jx)$, $\ell = 0, \dots, L-1$, then the set $\{\varphi_{j\ell}^{\text{right}}, \varphi_{jk}, \ell = 0, \dots, L-1, k \leq -2N + 1\}$ is a basis of $V_j^{]-\infty, 0]}$ (the space of the restriction of the V_j 's functions on $] - \infty, 0]$) and the set $\{\psi_{j\ell}^{\text{right}}, \psi_{jk}, \ell = 0, \dots, L-1, k \leq -2N + 1\}$ form a basis of orthogonal complementary $W_j^{]-\infty, 0]}$ of $V_j^{]-\infty, 0]}$ in $V_{j+1}^{]-\infty, 0]}$.

For $j \geq j_0$, (j_0 is a large integer such that $\text{supp}\varphi_{j_0,0} \subset [0, 1]$), we define

$$\varphi_{j\ell}^1(x) = \varphi_{\ell}^{\text{right}}(2^j(x - 1)) \text{ and } \psi_{j\ell}^1(x) = 2^{j/2}\psi_{\ell}^{\text{right}}(2^j(x - 1)), \ell = 0, \dots, L - 1,$$

and the subspace

$$V_j^{[0,1]} = \text{span}\{\varphi_{j\ell}^0, \varphi_{j\ell}^1, \varphi_{jk}, k = 0, \dots, 2^j - 2N + 1, \ell = 0, \dots, L - 1\}. \tag{9}$$

The sequence $(V_j^{[0,1]})_{j \geq j_0}$ is a multiresolution analysis and we have the following decomposition:

$$L^2([0, 1]) = V_{j_0}^{[0,1]} \oplus \left(\bigoplus_{j \geq j_0} W_j^{[0,1]} \right). \tag{10}$$

In other words, the set

$$\begin{aligned} & \{\varphi_{j_0\ell}^0, \varphi_{j_0\ell}^1, \varphi_{j_0k}, k = 0, \dots, 2_{j_0}^j - 2N + 1, \ell = 0, \dots, L - 1\} \\ & \cup \{\psi_{j\ell}^0, \psi_{j\ell}^1, \varphi_{jk}, k = 0, \dots, 2^j - 2N + 1, \ell = 0, \dots, N - 2, j \geq j_0\} \end{aligned}$$

is a basis of $L^2([0, 1])$.

3 A new approach to construct wavelets on the interval

In the last multiresolution analysis, the dimension of the approximation space at the scale 2^j is less than the dimension of wavelet space at the same scale 2^j . However, in some applications (construction of wavelet packets, image processing, etc), we need to have the dimension 2^j (equals to the dimension of the wavelet subspace).

In this section, we propose a general approach to construct multiresolution analysis $(V_j^{[0,1]})_{j \geq j_0}$, in order to obtain another one with $\dim V_j^{[0,1]} = 2^j$ (and any dimension). The method consists to add a functions at the edge without leaving the multiresolution framework. This theorem gives a characterization of the functions to add.

Theorem 3.1 *Given any $(V_j^{[0,1]})_{j \geq j_0}$ multiresolution analysis of order L on the interval $[0, 1]$ and an integer $n > 0$. Let $\phi_0, \phi_1, \dots, \phi_{n-1}$ be a linearly independent functions of $L^2([0, 1])$ and orthogonal onto $V_{j_0}^{[0,1]}$. Let $M \leq 2N - 2$ be a positive integer such that*

$$\text{supp}\phi_k \subset [0, 2^{-j_0} M], \quad k = 0, \dots, n - 1. \tag{11}$$

We define $(\tilde{V}_j^{[0,1]})_{j \geq j_0}$ a closed vector subspaces, by:

$$\tilde{V}_j^{[0,1]} = V_j^{[0,1]} \oplus \text{span}\{\sqrt{2^{j-j_0}}\phi_k(2^{j-j_0}x), \quad k = 1, \dots, n - 1\}, \quad j \geq j_0.$$

Then the following properties are equivalent:

- (P1) *The sequence $(\tilde{V}_j^{[0,1]})_{j \geq j_0}$ is a multiresolution analysis of $L^2([0, 1])$.*
- (P2) *There exist an integer $p = p_M$ and a square matrix A , of order n such that:*

$$\phi(x) = \sum_{j \geq j_0} (A^{j-j_0} D^{[j_0]}) \Psi_j(x) \tag{12}$$

where $\phi = [\phi_0, \phi_1, \dots, \phi_{n-1}]^T$, $\Psi_j = [\psi_{j,0}, \psi_{j,1}, \dots, \psi_{j,p-1}]^T$ and $D^{[j]}$ denotes the $n \times p$ matrix defined by the entries

$$D_{k,i}^{[j]} = d_{j,i}^k := \langle \phi_k, \psi_{ji} \rangle, \quad k = 0, \dots, n - 1, \text{ and } i = 0, \dots, p - 1.$$

The condition (11) holds in order to preserve the localization property. This involves no loss of generality.

Proof.

We will show that $(\tilde{V}_j^{[0,1]})_{j \geq j_0}$ is a growing sequence (i.e. $\tilde{V}_{j_0}^{[0,1]} \subset \tilde{V}_{j_0+1}^{[0,1]}$). The other properties of multiresolution analysis of $L^2([0, 1])$ are verified since $(V_j^{[0,1]})_{j \geq j_0}$ is a multiresolution analysis.

1. We assume that $\tilde{V}_{j_0}^{[0,1]} \subset \tilde{V}_{j_0+1}^{[0,1]}$. There exist a functions $(F_k)_{k=0, \dots, n-1}$ in $V_{j_0+1}^{[0,1]}$ and a matrix $A = (A_{ks})_{k,s=0, \dots, n-1}$ such that:

$$F_k(x) = \phi_k(x) - \sqrt{2} \sum_{s=0}^{n-1} A_{ks} \phi_s(2x). \tag{13}$$

We assume that $[A_{k1}, A_{k2}, \dots, A_{k,n-1}]$ does not vanish (if not $\phi_k \in W_{j_0}^{[0,1]}$). From the wavelet decomposition (10) of ϕ_k , we have

$$\phi_k = P_{V_{j_0}^{[0,1]}}(\phi_k) + \sum_{j \geq j_0} \sum_{i=0}^{2^j-1} d_{ji}^{(k)} \psi_{ji},$$

where $d_{j,i}^{(k)} = \langle \phi_k, \psi_{ji} \rangle$. Using the orthogonality of ϕ_k onto $V_{j_0}^{[0,1]}$, we get

$$\phi_k = \sum_{j \geq j_0} \sum_{i=0}^{2^j-1} d_{ji}^{(k)} \psi_{ji}. \tag{14}$$

and by dilation,

$$\sqrt{2}f_k(2x) = \sum_{j \geq j_0+1} \sum_{i=0}^{2^j-1} d_{j-1,i}^{(k)} \psi_{ji}(x).$$

Thus, $P_{V_{j_0}^{[0,1]}}(\phi_k(2x)) = 0$ and $P_{W_{j_0}^{[0,1]}}(\phi_k(2x)) = 0$.

Combining (13) with (14), we obtain

$$\begin{aligned} F_k &= \sum_{j \geq j_0} \sum_{i=0}^{2^j-1} d_{ji}^{(k)} \psi_{ji} - \sum_{s=0}^{n-1} A_{ks} \left(\sum_{j \geq j_0+1} \sum_{i=0}^{2^j-1} d_{j-1,i}^{(s)} \psi_{ji} \right) \\ &= \sum_{i=0}^{2^{j_0}-1} d_{j_0,i}^{(k)} \psi_{j_0,i} + \sum_{j \geq j_0+1} \sum_{i=0}^{2^j-1} \left(d_{ji}^{(k)} - \sum_{s=0}^{n-1} A_{ks} d_{j-1,i}^{(s)} \right) \psi_{ji}. \end{aligned} \tag{15}$$

We now decompose F_k in the basis of $V_{j_0+1}^{[0,1]}$ constituted by the union of the bases of $V_{j_0}^{[0,1]}$ and $W_{j_0}^{[0,1]}$. It follows that

$$\begin{aligned} F_k &= P_{W_{j_0}^{[0,1]}}(\phi_k) \\ &= \sum_{i=0}^{2^{j_0}-1} d_{j_0,i}^{(k)} \psi_{j_0,i}. \end{aligned} \tag{16}$$

Subtracting (16) from (15) yields

$$d_{ji}^{(k)} - \sum_{s=0}^{n-1} A_{ks} d_{j-1,i}^{(s)} = 0. \tag{17}$$

We conclude from (11) that $\langle \phi_k, \psi_{j_0,i} \rangle = 0$, for all $i \geq p$, for some p depends on M . Hence, according to our notation, we can rewrite (17) as

$$D_{ki}^{[j]} - \sum_{s=0}^{n-1} A_{ks} D_{si}^{[j-1]} = 0, \quad \text{for } i = 0, \dots, p-1, \tag{18}$$

which establishes the formula

$$D^{[j]} = AD^{[j-1]}, \quad \text{where } D_{k,i}^{[j]} = d_{j,i}^{(k)}.$$

This finishes the first step of the proof.

2. We need now just to prove that the functions $\phi_k(x)$ are in $\tilde{V}_{j_0+1}^{[0,1]}$. This is a direct result of the relation (12). Indeed, we have

$$\begin{aligned} \sqrt{2}A\phi(2x) &= \sqrt{2}A \sum_{j \geq j_0} (A^{j-j_0} D^{[j_0]}) \Psi_j(2x) \\ &= \phi(x) - P_{W_{j_0}^{[0,1]}} \phi(x). \end{aligned}$$

Scaling this equation, we get the embedded relation : $\tilde{V}_j^{[0,1]} \subset \tilde{V}_{j+1}^{[0,1]}$. ■

One can ask how to choose the functions $\{\phi_k\}$? Indeed, the last theorem gives a broad choice : we have just to choose two matrices A and $D^{[j_0]}$ and define the functions ϕ_k with (12). We propose hereafter a *natural* choice in the sense that it is an extension of CDV method. This choice leads to a better approximation near the edges of the interval.

Consider the half line $[0, +\infty[$. For an integer j , we note V_j^I the closed subspaces generated by the scaling functions φ_{jk} , $k \geq 0$ and $W_j^{[0,+\infty[}$ the orthogonal complementary of V_j^I in V_{j+1}^I . Using lemma 2.2 and Gram-Schmidt algorithm, we obtain an orthonormal basis of $W_j^{[0,+\infty[}$, namely

$$\{\tilde{\psi}_{j\ell}(x) = 2^{j/2} \tilde{\psi}_\ell(2^j x), \psi_{jk}, \ell = 0, \dots, N - 1, k \geq 0\}.$$

We note here that

$$\int_0^{+\infty} x^k \tilde{\psi}_k(x) \neq 0, \quad k = 0, \dots, L - 1, \ell = 0, \dots, N - 1. \tag{19}$$

Indeed, the polynomials x^k , $k = 0, \dots, L - 1$, can be written

$$x^k = \sum_{k \geq 0} \langle x^k, \varphi_{0k} \rangle \varphi_{0k}(x) + \sum_{j \geq 0} \sum_{i=0}^{N-1} \langle x^k, \tilde{\psi}_{ji} \rangle \tilde{\psi}_{ji}(x), \quad x \geq 0. \tag{20}$$

Let $D_{k,i}^{[j]} = \langle x^k, \tilde{\psi}_{ji} \rangle$. We get

$$\begin{aligned} D_{k,i}^{[j]} &= 2^{j/2} \int x^k \tilde{\psi}_i(2^j x) dx \\ &= 2^{-j/2-kj} \int x^k \tilde{\psi}_i(x) dx \\ &= (A_{kk})^j D_{k,i}^{[0]}, \quad i = 0, \dots, N - 1, \end{aligned} \tag{21}$$

where A is a diagonal matrix with the entries $A_{kk} = 2^{-1/2-k}$ for $k = 0, \dots, L - 1$. The sequence $D_{k,i}^{[j]} = \langle x^k, \tilde{\psi}_{ji} \rangle$ satisfies (18). We remark that since the

wavelets has a compact support and by (21), the identity (20) converges for any fixed $x \geq 0$. The functions

$$\tilde{\varphi}_k = \sum_{j \geq 0} \sum_{i=0}^{N-1} \langle x^k, \tilde{\psi}_{ji} \rangle \tilde{\psi}_{ji}, \quad k = 0, \dots, L - 1 \tag{22}$$

satisfy (12). Applying Theorem 3.1, the subspaces

$$V_j^{[0,+\infty[} = V_j^I \oplus \text{span}\{\tilde{\varphi}_k, k = 0, \dots, L - 1\}$$

constitute a multiresolution analysis. Actually, it is exactly the CDV multiresolution analysis studied in section 2.

Selection of the edge functions A *natural* generalization is to choose the functions

$$\tilde{\varphi}_k^\# = \sum_{j \geq 0} \sum_{i=0}^{N-1} \langle x^k, \tilde{\psi}_{ji} \rangle \tilde{\psi}_{ji}, \quad k = L, \dots, N - 1. \tag{23}$$

These functions are not sufficient to generate the polynomials x^L, \dots, x^{N-1} in $V_j^{[0,+\infty[}$. Indeed, the L^2 error approximation of these polynomials is given by

$$\|x^k - P_{V_j^{[0,+\infty[}}(x^k)\|_{L^2([0,\beta])} = \sum_{j \geq J} \sum_{i=0}^{2^j\beta-2N+1} |\langle x^k, \psi_{ji} \rangle|^2. \tag{24}$$

However, in this case, the approximation error is small in the neighborhood of 0, since the edge wavelets has N vanishing moments.

To illustrate this property, we consider a positive integer r and define the functions

$$\tilde{\varphi}_k = \sum_{j \geq 0} \sum_{i=0}^{N-1} \langle x^k, \tilde{\psi}_{ji} \rangle \tilde{\psi}_{ji} + \sum_{j \geq 0} \sum_{i=0}^{r-1} \langle x^k, \psi_{ji} \rangle \psi_{ji}, \quad k = L, \dots, N - 1. \tag{25}$$

We formulate this claim by the following :

Proposition 3.2 *Define the subspaces*

$$V_j^{[0,+\infty[} = V_j^I \oplus \text{span}\{2^{j/2}\tilde{\varphi}_k(2^j x), k = 0, \dots, N - 1\}, \quad j \in \mathbb{Z}.$$

Then, for any polynomial p_ℓ of degree $\ell \leq N - 1$, we have

$$p_\ell(x) = P_{V_j^{[0,+\infty[}}(p_\ell)(x) \quad x \in [0, 2^{-j}r]. \tag{26}$$

Proof.

Let p_ℓ denote a polynomial of degree $\ell \leq N - 1$. For any positive real number β , the wavelet decomposition of p_ℓ on $[0, \beta]$ gives us

$$\left(p_\ell = \sum_{k \geq 0} \langle p_\ell, \varphi_{0k} \rangle \varphi_{0k} + \sum_{j \geq 0} \sum_{k=0}^{N-1} \langle p_\ell, \tilde{\psi}_{ji} \rangle \tilde{\psi}_{jk} + \sum_{j \geq 0} \sum_{k \geq 0} \langle p_\ell, \psi_{jk} \rangle \psi_{jk} \right) \chi_{[0, \beta]}. \tag{27}$$

Hence

$$P_{V_j^{[0, +\infty[}}(p_\ell) = \sum_{k \geq 0} \langle p_\ell, \varphi_{0k} \rangle \varphi_{0,k} + \sum_{j \geq 0} \sum_{k=0}^{N-1} \langle p_\ell, \tilde{\psi}_{ji} \rangle \tilde{\psi}_{jk} + \sum_{j \geq 0} \sum_{i=0}^{r-1} \langle p_\ell, \psi_{j,i} \rangle \psi_{ji}.$$

Then, we obtain

$$p_\ell = P_{V_j^{[0, +\infty[}}(p_\ell) + \sum_{j \geq 0} \sum_{k \geq r+1} \langle p_\ell, \psi_{jk} \rangle \psi_{jk}.$$

For $x \in [0, 2^{-j}r]$, we have $\psi_{j,k}(x) = 0$ for any $j \geq 0, k \geq r$ because of ψ 's support. Therefore,

$$p_\ell(x) = P_{V_j^{[j, +\infty[}}(p_\ell)(x), \quad x \in [0, 2^{-j}r].$$

■

This result means that we obtain an exact reproduction of polynomials up to degree $N - 1$ near the neighborhood of 0. The consequence of this claim is an approximation rate of order N on the interval $[0, 2^{-j}]$ for any function $f \in C^{N+1}([0, \epsilon])$, with some $\epsilon > 2^{-j}$.

4 Conclusion

In this paper, we have presented a generalization of CDV construction to compactly supported wavelets with an arbitrary support. It is shown that we can add a functions to any multiresolution analysis without leaving this framework. This result allows us to describe a wavelet packet decomposition on the interval. In addition, we showed how to construct multiresolutions on the interval with a nice approximation properties at the edges . These results may concern some interesting classes of wavelets (coiflets, symlets, Lemarié's wavelets).

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