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On Fixed Point Theorems in D-Metric Spaces

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Abstract

In this paper we give a generalization of a contraction principle in generalized metric space. Results generalizing fixed point theorems of Rhoades are established.

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1 Introduction

In 1992, Dhage[1] introduced a generalization of metric space which is called generalized metric space or D-metric space, and proved the existence of unique fixed point of a self map satisfying a contractive condition. Rhoades[4] generalized Dhage's contractive condition and obtained some fixed point theorems. Also, Dhage[3] extended Rhoades' contractive condition to two maps in D-metric space. By using the concept of weak compatibility of self maps on a D-metric space, Dhage[2] obtained a unique common fixed point.

The following theorems have been established in [4].

Theorem 1.1. *Let (X, D) be a complete bounded D-metric space and T be a self map of X satisfying the following condition:*

there exists a $k \in [0, 1)$ such that for all $x, y, z \in X$,

$D(Tx, Ty, Tz)$
$\leq k\ max\{D(x, y, z), D(x, Tx, z), D(y, Ty, z), D(x, Ty, z), D(y, Tx, z)\}$.
Then T has a unique fixed point p in X and T is continuous at p.

Theorem 1.2. *Let (X, D) be a compact D-metric space and T be a continous self map of X satisfying for all $x, y, z \in X$ with $D(x, y, z) \neq 0$,*
$D(Tx, Ty, Tz)$
$< max\{D(x, y, z), D(x, Tx, z), D(y, Ty, z), D(x, Ty, z), D(y, Tx, z)\}$.
Then T has a unique fixed point p in X.

The object of this paper is to generalize Theorem 1.1 and 1.2 by using a contraction satisfying an implicit relation.

2 Prelimimaries

Throughout this paper we denote $\mathbb{N}$ by the set of all natural numbers and $\mathbb{R}_+$ the set of all positive real numbers.

A *generalized metric*(or *D-metric*) on a set X is a function $D : X \times X \times X \to \mathbb{R}_+$ such that for all $x, y, z, w \in X$
(i) $D(x, y, z) \geq 0$ and $D(x, y, z) = 0$ if and only if $x = y = z$;
(ii) $D(x, y, z) = D(p(y, x, z))$, where p is a permutation.
(iii) $D(x, y, z) \leq D(x, y, a) + D(x, a, z) + D(a, y, z)$.

The pair (X, D) is called a *generalized metric* (or *D-metric*) *space.* Geometrically, a D-metric $D(x, y, z)$ is the peridiameter of a triangle whose vertices are x, y and z.

Examples[5] of D-metric are
(1) $D(x, y, z) = Max\{d(x, y), d(y, z), d(x, z)\}$,
(2) $D(x, y, z) = d(x, y) + d(y, z) + d(x, z)$, where, d is a metric on X.

A sequence $\{x_n\}$ in a D-metric space (X, D) is said to be *D-converges*[1] to a point $x \in X$(denoted by $lim_{n\to\infty} x_n = x$) if for any $\epsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that $D(x_n, x_m, x) < \epsilon$ for all $n, m > n_0$.

A sequence $\{x_n\}$ in a D-metric space (X, D) is said to be a *D-Cauchy sequence*[1] if for any $\epsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that $D(x_n, x_{n+m}, x_{n+m+l})$ $< \epsilon$ for all $n > n_0$ and all $m, l \in \mathbb{N}$. A D-metric space (X, D) is said to be *complete*[1] if every D-Cauchy sequence in X converges to a point in X.

For D-metric space (X, D) and $\emptyset \neq S \subset X$, the *diameter*[5] of S is defined as $\delta_D(S) = sup\{D(x, y, z) : x, y, z \in S\}$.

For a bounded sequence $\{y_n\}$ in D-metric space (X, D), let

$r_n = \delta_D(\{y_n, y_{n+1}, y_{n+2}, \cdots\})$ for $n \in \mathbb{N}$. Then r_n is finite for all $n \in \mathbb{N}$ and $\{r_n\}$ is nonincreasing and $r_n \geq 0$ for all $n \in \mathbb{N}$, and so there exists an $r \geq 0$ such that $lim_{n\to\infty} r_n = r$.

From now on, let (X, D) be a D-metric space such that D is continuous.

Note that if D is continuous and the limit of a sequence exists, then the limit of the sequence is unique.

Let Φ be the class of all upper semicontinuous function $\varphi : {\mathbb{R}_+}^5 \times \mathbb{R}_+ \to \mathbb{R}_+$ such that φ is nondecreasing on ${\mathbb{R}_+}^5$ satisfying the following property:

$$\varphi((u, u, u, u, u), v) \geq 0 \quad \text{implies} \quad v \leq \phi(u)$$

where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing upper semicontinuous function with $\phi(0) = 0$ and $\phi(t) < t$ for $t > 0$.

Example 2.1. *Let* $\varphi_m : {\mathbb{R}_+}^5 \times \mathbb{R}_+ \to \mathbb{R}_+$ *defined by*
$\varphi_m((t_1, t_2, t_3, t_4, t_5), t_6) = \phi(max\{t_1, t_2, t_3, t_4, t_5\}) - t_6$*, where* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *is a nondecreasing upper semicontinuous function with* $\phi(0) = 0$ *and* $\phi(t) < t$ *for* $t > 0$*. Then* φ_m *is upper semicontinuous on* ${\mathbb{R}_+}^5 \times \mathbb{R}_+$ *and nondecreasing on* ${\mathbb{R}_+}^5$*. Also,* $\varphi_m((u, u, u, u, u), v) \geq 0$ *implies* $v \leq \phi(u)$*. Thus* $\varphi_m \in \Phi$*.*

3 Main Results

Theorem 3.1. *Let* (X, D) *be a complete bounded* D*-metric space and* T *be a self map of* X *satisfying for all* $x, y, z \in X$*,*

$$(\varphi_T) \quad \varphi((D(x, y, z), D(x, Tx, z), D(y, Ty, z), D(x, Ty, z), D(y, Tx, z)), D(Tx, Ty, Tz)) \geq 0.$$

Then T *has a unique fixed point* p *in* X *and* T *is continuous at* p*.*

Proof. Let $x_0 \in X$ and $Tx_n = x_{n+1}$. Then the orbit $\{x_n\}$ is bounded. let $r_n = \delta_D(\{x_n, x_{n+1}, x_{n+2}, \cdots\}), n \in \mathbb{N}$. Then we know $lim_{n\to\infty} r_n = r$ for some $r \geq 0$.

If $x_{n+1} = x_n$ for some $n \in \mathbb{N}$, then T has a fixed point, say $p \in X$.

Assume that $x_{n+1} \neq x_n$ for each $n \in \mathbb{N}$.

Let $k \in \mathbb{N}$ be fixed.

Taking $x = x_{n-1}, y = x_{n+m-1}$ and $z = x_{n+m+l-1}$ in (φ_T) where $n \geq k$ and $m, l \in \mathbb{N}$, we have

$$\begin{aligned}
&\varphi((D(x_{n-1}, x_{n+m-1}, x_{n+m+l-1}), D(x_{n-1}, Tx_{n-1}, x_{n+m+l-1}),\\
&D(x_{n+m-1}, Tx_{n+m-1}, x_{n+m+l-1}), D(x_{n-1}, Tx_{n+m-1}, x_{n+m+l-1}),\\
&D(x_{n+m-1}, Tx_{n-1}, x_{n+m+l-1})), D(Tx_{n-1}, Tx_{n+m-1}, Tx_{n+m+l-1}))\\
&= \varphi((D(x_{n-1}, x_{n+m-1}, x_{n+m+l-1}), D(x_{n-1}, x_n, x_{n+m+l-1}),\\
&D(x_{n+m-1}, x_{n+m}, x_{n+m+l-1}), D(x_{n-1}, x_{n+m}, x_{n+m+l-1}),\\
&D(x_{n+m-1}, x_n, x_{n+m+l-1})), D(x_n, x_{n+m}, x_{n+m+l})) \geq 0.
\end{aligned}$$

Thus we have

$$\varphi((r_{n-1}, r_{n-1}, r_{n+m-1}, r_{n-1}, r_{n+m-1}), D(x_n, x_{n+m}, x_{n+m+l})) \geq 0.$$

Since φ is nondecreasing on ${\mathbb{R}_+}^5$ and $\{r_n\}$ is nonincreasing, we have

$$\varphi((r_{k-1}, r_{k-1}, r_{k-1}, r_{k-1}, r_{k-1}), D(x_n, x_{n+m}, x_{n+m+l})) \geq 0,$$

which implies
$D(x_n, x_{n+m}, x_{n+m+l}) \leq \phi(r_{k-1})$.

Taking limit sup over $n \geq k$, we have $r_k \leq \phi(r_{k-1})$. Letting $k \to \infty$, we get $r \leq \phi(r)$. If $r > 0$, then $r \leq \phi(r) < r$, which is a contradiction. Thus $r = 0$ and hence $lim_{n\to\infty} r_n = 0$. Thus given $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $r_N < \epsilon$. Then we have for $n \geq N$ and $m, l \in \mathbb{N}$, $D(x_n, x_{n+m}, x_{n+m+l}) < \epsilon$.

Therefore, $\{x_n\}$ is a D-Cauchy sequence in X. By the completeness of X, there exists a $p \in X$ such that $lim_{n\to\infty} x_n = p$. Hence $lim_{n\to\infty} Tx_n = p$.

Taking $x = x_{n-1}, y = x_{n+m-1}$ and $z = p$ in (φ_T), we have

$$\begin{aligned}
&\varphi((D(x_{n-1}, x_{n+m-1}, p), D(x_{n-1}, Tx_{n-1}, p), D(x_{n+m-1}, Tx_{n+m-1}, p),\\
&D(x_{n-1}, Tx_{n+m-1}, p), D(x_{n+m-1}, Tx_{n-1}, p)), D(Tx_{n-1}, Tx_{n+m-1}, Tp))\\
&= \varphi(D(x_{n-1}, x_{n+m-1}, p), D(x_{n-1}, x_n, p), D(x_{n+m-1}, x_{n+m}, p),\\
&D(x_{n-1}, x_{n+m}, p), D(x_{n+m-1}, x_n, p)), D(x_n, x_{n+m}, Tp)) \geq 0.
\end{aligned}$$

Taking limit $n \to \infty$, we have

$$\begin{aligned}
&\varphi((D(p,p,p), D(p,p,p), D(p,p,p),\\
&D(p,p,p), D(p,p,p)), D(p,p,Tp)) \geq 0
\end{aligned}$$

which implies $D(p,p,Tp) \leq \phi(D(p,p,p)) = \phi(0) = 0$. Hence $Tp = p$.

For the uniqueness, let p and w be fixed points of T.

Taking $x = p, y = p$ and $z = w$ in (φ_T), we have

$$\begin{aligned}
&\varphi((D(p,p,w), D(p,Tp,w), D(p,Tp,w),\\
&D(p,Tp,w), D(p,Tp,w)), D(Tp,Tp,Tw)))\\
&= \varphi((D(p,p,w), D(p,p,w), D(p,p,w),\\
&D(p,p,w), D(p,p,w)), D(p,p,w))) \geq 0
\end{aligned}$$

which implies $D(p,p,w) \leq \phi(D(p,p,w)) < D(p,p,w)$ which is a contradiction.

Thus we have $p = w$.

Now, we show that T is continuous at p.

Let $\{y_n\}$ be a sequence in X and $lim_{n\to\infty} y_n = p$.

Taking $x = p, y = p$ and $z = y_n$ in (φ_T), we have

$$\begin{aligned}&\varphi((D(p,p,y_n), D(p,Tp,y_n), D(p,Tp,y_n), D(p,Tp,y_n), D(p,Tp,y_n)),\\ &D(Tp,Tp,Ty_n))\\ &= \varphi((D(p,p,y_n), D(p,p,y_n), D(p,p,y_n), D(p,p,y_n), D(p,p,y_n)),\\ &D(p,p,Ty_n)) \geq 0\end{aligned}$$

which implies $D(p,p,Ty_n) \leq \phi(D(p,p,y_n))$.

Taking limit sup, we have $\overline{lim} D(p,p,Ty_n) \leq \overline{lim}\ \phi(D(p,p,y_n)) \leq \phi(0) = 0$. Hence $lim Ty_n = p = Tp$ and hence T is continuous at p. □

Corollary 3.2. *Let (X,D) be a complete bounded D-metric space, $m \in \mathbb{N}$ and T be a self map of X satisfying for all $x,y,z \in X$,*

$\varphi((D(x,y,z), D(x,T^mx,z), D(y,T^my,z),$
$D(x,T^my,z), D(y,T^mx,z)), D(T^mx,T^my,T^mz)) \geq 0.$

Then T has a unique fixed point p in X and T^m is continuous at p.

Proof. From Theorem 3.1, T^m has a unique fixed point p in X and T^m is continuous at p. Since $Tp = TT^mp = T^mTp$, Tp is also a fixed point of T^m, By the uniqueness it follows $Tp = p$. □

In Theorem 3.1(resp., Corollary 3.2), If we take $\varphi = \varphi_m$ then we have the next result.

Corollary 3.3. *Let (X,D) be a complete bounded D-metric space and T be a self map of X satisfying for all $x,y,z \in X$,*

$D(Tx,Ty,Tz)$
$\leq \phi(max\{D(x,y,z), D(x,Tx,z), D(y,Ty,z), D(x,Ty,z), D(y,Tx,z)\}).$

Then T has a unique fixed point p in X and T is continuous at p.

Corollary 3.4. *Let (X,D) be a complete bounded D-metric space, $m \in \mathbb{N}$ and T be a self map of X satisfying for all $x,y,z \in X$,*

$D(T^mx,T^my,T^mz)$
$\leq \phi(max\{D(x,y,z), D(x,T^mx,z), D(y,T^my,z), D(x,T^my,z), D(y,T^mx,z)\}).$

Then T has a unique fixed point p in X and T^m is continuous at p.

Remark 1. In Corollary 3.3(resp., Corollary 3.4), if we have $\phi(t) = kt$ for $k \in [0,1)$, then Corollary 3.3(resp., Corollary 3.4) becomes same with Theorem 1.1(resp., Corollary 1[4]).

Let Ψ be the class of all upper semicontinuous functions $\psi : {\mathbb{R}_+}^5 \times \mathbb{R}_+ \to \mathbb{R}_+$ such that ψ is nondecreasing on ${\mathbb{R}_+}^5$ satisfying

$$\psi((u,u,u,u,u),v) > 0 \quad \text{implies} \quad v < u.$$

Example 3.5. *Let* $\psi_m : {\mathbb{R}_+}^5 \times \mathbb{R}_+ \to \mathbb{R}_+$ *defined by*
$\psi_m((t_1,t_2,t_3,t_4,t_5),t_6) = max\{t_1,t_2,t_3,t_4,t_5\} - t_6$. *Then* ψ_m *is upper semicontinuous on* ${\mathbb{R}_+}^5 \times \mathbb{R}_+$ *and nondecreasing on* ${\mathbb{R}_+}^5$. *Also, we have that* $\psi_m((u,u,u,u,u),v) > 0$ *implies* $v < u$. *Thus* $\psi_m \in \Psi$.

Theorem 3.6. *Let* (X,D) *be a compact* D*-metric space and* T *be a continuous self map of* X *satisfying for all* $x,y,z \in X$ *with* $D(x,y,z) \neq 0$,

$$\psi((D(x,y,z), D(x,Tx,z), D(y,Ty,z), D(x,Ty,z), D(y,Tx,z)), D(Tx,Ty,Tz)) > 0.$$

Then T *has a unique fixed point* p *in* X.

Proof. We know that for $n = 1,2,\cdots$, T^nX is compact and $T^{n+1}X \subset T^nX$. Let $X_0 = \bigcap_{n=1}^{\infty} T^nX$. Then X_0 is a nonempty compact subset of X and $TX_0 = X_0$.

We claim that X_0 is a singleton set.

Suppose X_0 is not singleton. Then we know the function

$D : X \times X \times X | X_0 \times X_0 \times X_0 \to \mathbb{R}_+$ has a maximum value. That is, there exists a $(x_0,y_0,z_0) \in X_0 \times X_0 \times X_0$ such that $D(x_0,y_0,z_0) \geq D(x,y,z)$ for all $x,y,z \in X_0$. Since $TX_0 = X_0$, there exist $x_1,y_1,z_1 \in X_0$ such that $Tx_1 = x_0, Ty_1 = y_0$ and $Tz_1 = z_0$. Thus we have

$$\psi((D(x_1,y_1,z_1), D(x_1,x_0,z_1), D(y_1,y_0,z_1), D(x_1,y_0,z_1), D(y_1,x_0,z_1)), D(x_0,y_0,z_0)) > 0$$

and so

$$\psi((D(x_0,y_0,z_0), D(x_0,y_0,z_0), D(x_0,y_0,z_0), D(x_0,y_0,z_0), D(x_0,y_0,z_0)), D(x_0,y_0,z_0)) > 0$$

which implies

$D(x_0,y_0,z_0) < D(x_0,y_0,z_0)$. This is a contradiction.

Thus X_0 is singleton and hence T has a fixed point in X.

□

Corollary 3.7. *Let* (X,D) *be a compact* D*-metric space,* $m \in \mathbb{N}$ *and* T *be self map of* X *satisfying for all* $x,y,z \in X$ *with* $D(x,y,z) \neq 0$,

$$\psi((D(x,y,z), D(x,T^mx,z), D(y,T^my,z), D(x,T^my,z), D(y,T^mx,z)), D(T^mx,T^my,T^mz)) > 0.$$

Then T *has a unique fixed point* p *in* X.

Proof. From Theorem 3.6, T^m has a unique fixed point p in X. The proof is similar to that of Corollary 3.2. □

Remark 2. In Theorem 3.6(resp., Corollary 3.7), if we have $\psi = \psi_m$, then Theorem 3.6(resp., Corollary 3.7) becomes same with Theorem 1.2(resp., Corollary 2[4]).

References

[1] B. C. Dhage, Generalized metric spaces and mapping with fixed points, *Bull. Cal. Math. Soc.,* **84** (1992), 329-336.

[2] B. C. Dhage, A common fixed point principle in D-metric space , *Bull. Cal. Math. Soc.,* **91** (1999), 475-480.

[3] B. C. Dhage, Some results on common fixed point I, *Indian J. Pure Appl. Math.,* **30** (1999), 827-837.

[4] B. E. Rhoades, A fixed point theorem for generalized metric space , *Internat. J. Math. Math. Sci.,* **19** (1996), 457-460.

[5] B. Singh et al, Semicompatibility and fixed point theorems in an unbounded D-metric space , *Internat. J. Math. Math. Sci.,* **5** (2005), 789-801.

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