Some Common Fixed Point Theorems for Mappings Satisfying a General Contractive Condition of Integral Type

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Abstract

Banach contraction principle is a fundamental result in fixed point theory and has been applied and extended in many different directions. In 2002, Branciari [3] obtained a fixed point theorem for a single mapping satisfying an analogue of Banach’s contraction principle for an integral type inequality. Aliouche [2] established a common fixed point theorem for weakly compatible mappings in symmetric spaces satisfying a contractive condition of integral type and a property (E.A) introduced by Aamri and El. Moutawakil [1]. In this paper we prove some common fixed point theorems for pointwise R-weakly commuting mappings in symmetric spaces with at least one pair noncompatible satisfying a contractive condition of integral type. We also obtain some results for weakly compatible mappings which extend some results of Vijayaraju et al. [10], Rhoades [8] and Aamri et al. [1].

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1 Introduction

We recall that a symmetric on a set X is a nonnegative real valued function d on $X \times X$ such that

(i) $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$

Let d be a symmetric on a set X and for $r > 0$ and any $x \in X$, let
$B_r(x) = \{ y \in X : d(x, y) < r \}$. A topology $t(d)$ on $X$ is given by $U \in t(d)$ if and only if for each $x \in U$, $B_r(x) \subseteq U$ for some $r > 0$. A symmetric $d$ is a semi-metric if for each $x \in X$ and each $r > 0$, $B_r(x)$ is a neighbourhood of $x$ in the topology $t(d)$. It is easily seen that $\lim_{n \to \infty} d(x_n, x) = 0$ if and only if $x_n \to x$ in the topology $t(d)$.

Wilson [11] presented the following two axioms:

Let $(X, d)$ be a symmetric space.

(W3) Given $\{x_n\}, x$ and $y$ in $X$, $\lim_{n \to \infty} d(x_n, x) = 0$ and $\lim_{n \to \infty} d(x_n, y) = 0$ imply $x = y$.

(W4) Given $\{x_n\}, \{y_n\}$ and $x$ in $X$, $\lim_{n \to \infty} d(x_n, x) = 0$ and $\lim_{n \to \infty} d(x_n, y_n) = 0$ imply that $\lim_{n \to \infty} d(y_n, x) = 0$.

Sessa [9] introduced the notion of weak commutativity which is defined as follows:

Two self maps $A$ and $B$ of a metric space $(X, d)$ are said to be weakly commuting if $d(ABx, BAx) \leq d(Ax, Bx)$ for all $x \in X$.

Jungck [4] extended this concept in the following way:

Two self maps $A$ and $B$ of a metric space $(X, d)$ are said to be compatible if $\lim_{n \to \infty} d(ABx_n, BAx_n) = 0$ whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n \to \infty} d(Ax_n, t) = \lim_{n \to \infty} d(Bx_n, t) = 0$ for some $t \in X$.

Note that two weakly commuting mappings are compatible but the converse is not true as shown in [4]. Jungck and Rhoades [5] introduced the concept of weakly compatible maps as follows:

Two self maps $A$ and $B$ of a metric space $(X, d)$ are said to be weakly compatible if they commute at their coincidence points, that is, if $Au = Bu$ for some $u \in X$, then $A Bu = B Au$. It is obvious that two compatible maps are weakly compatible but the converse is not true.

Pant [6] introduced the notion of R-weakly commuting maps in the following way:

Two self maps $A$ and $B$ of a metric space $(X, d)$ are called R-weakly commuting at a point $x \in X$ if $d(ABx, BAx) \leq Rd(Ax, Bx)$ for some $R > 0$.

The maps $A$ and $B$ of a metric space $(X, d)$ are called pointwise R-weakly commuting on $X$ if there exists $R > 0$ such that $d(ABx, BAx) \leq Rd(Ax, Bx)$.

Note that $A$ and $B$ cannot fail to be pointwise R-weakly commuting only if there is some $x \in X$ such that $Ax = Bx$ but $ABx \neq BAx$, that is, only if they possess a coincidence point at which they do not commute. Aamri and El Moutawakil [1] established some common fixed point theorems under strict contractive conditions on a metric space for mappings satisfying property (E.A) defined as
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follows:

Pant [7] has proved that R-weak commutativity is equivalent to commutativity at coincidence points. In other words, S and T are pointwise R-weakly commuting if and only if they are weakly compatible.

Two self maps A and B of a metric space \((X,d)\) satisfy property (E.A) if there exists a sequence \(\{x_n\}\) such that \(\lim_{n \to \infty} d(Ax_n, t) = \lim_{n \to \infty} d(Bx_n, t) = 0\) for some \(t \in X\).

Property (H.E) is defined as follows: We say that a symmetric space \((X,d)\) satisfies property (H.E) if given \(\{x_n\}, \{y_n\}\) and \(x\) in \(X\), \(\lim_{n \to \infty} d(x_n, x) = 0\) and \(\lim_{n \to \infty} d(y_n, x) = 0\) imply \(\lim_{n \to \infty} d(x_n, y_n) = 0\)

The main objective of this paper is to give some common fixed point theorems for pointwise R-weakly commuting mappings in symmetric spaces with atleast one pair noncompatible satisfying a contractive condition of integral type. We also obtain some results for weakly compatible mappings which extend some results of Vijayaraju et al. [10], Rhoades [8] and Aamri et al. [1].

2 Main Results

We now present our main results.

Throughout this paper \(\psi\) will be a function defined by: \(\psi : \mathbb{R}^+ \to \mathbb{R}^+\) such that \(\psi(t) < t\) for all \(t > 0\).

**Theorem 1** Let \(d\) be a symmetric for \(X\) that satisfies \((W4)\) and property \((H.E)\). Let \(\{A_k\}, k = 1, 2, 3, \ldots, S\) and \(T\) be self mappings of \((X,d)\) such that

\[
\int_0^{d(A_1x,A_ky)} \phi(t) dt \\
\leq \psi\left(\int_0^{\max\{d(Sx,Ty),d(A_1x,Sx),d(A_ky,Ty),\lfloor d(A_1x,Ty)+d(A_ky,Sx)\rfloor/2\}} \phi(t) dt\right)
\]

for all \(x, y \in X\), where \(\phi : \mathbb{R}^+ \to \mathbb{R}^+\) is a Lebesgue-integrable mapping which is summable, nonnegative and such that

\[
\int_0^\varepsilon \phi(t) dt > 0 \quad \text{for each} \quad \varepsilon > 0.
\]

Suppose that \(A_kX \subset SX\) when \(k > 1\) and \(A_1X \subset TX\). Suppose also that the pairs \(\{A_k,T\}\) for \(k > 1\) and \(\{A_1,S\}\) are pointwise R-weakly commuting with atleast one pair noncompatible. If the range of one of the mappings is a
complete subspace of $X$, then all the $A_k$, $S$ and $T$ have a unique common fixed point in $X$.

Proof:

Suppose that $T$ is noncompatible with $A_k$ for some $k > 1$. Then there exists a sequence $\{z_n\}$ in $X$ such that $\lim_{n \to \infty} d(A_kz_n, t) = \lim_{n \to \infty} d(Tz_n, t) = 0$ for some $t$ in $X$ but $\lim_{n \to \infty} d(A_kTz_n, TA_kz_n)$ is either nonzero or does not exist. By (H.E.) we have $\lim_{n \to \infty} d(A_kz_n, Tz_n) = 0$. Since $A_kX \subset SX$, there exists in $X$ a sequence $\{x_n\}$ such that $A_kx_n = Sx_n$. Thus $\lim_{n \to \infty} d(A_kz_n, t) = \lim_{n \to \infty} d(Sx_n, Tz_n) = 0$. Therefore, by (H.E) we have $\lim_{n \to \infty} d(Sx_n, Tz_n) = 0$ and $\lim_{n \to \infty} d(A_kz_n, Sx_n) = 0$. We need to show that $\lim_{n \to \infty} d(A_1x_n, t) = 0$. Suppose that $\lim sup_{n \to \infty} d(A_1x_n, t) > 0$. Then by virtue of (1), we have

$$\lim_{n \to \infty} \sup_{0 \leq t \leq \infty} d(A_1x_n, Sx_n) = \lim_{n \to \infty} \sup_{0 \leq t \leq \infty} d(A_1x_n, A_kz_n)$$

$$\leq \lim_{n \to \infty} \sup_{0 \leq t \leq \infty} \left( \int_0^{\infty} d(A_1x_n, Sx_n) dt \right)$$

$$= \lim_{n \to \infty} \sup_{0 \leq t \leq \infty} \left( \int_0^{\infty} d(A_1x_n, Sx_n) dt \right)$$

$$= \lim_{n \to \infty} \sup_{0 \leq t \leq \infty} \left( \int_0^{\infty} d(A_1x_n, Sx_n) dt \right)$$

$$< \lim_{n \to \infty} \int_0^{\infty} d(A_1x_n, Sx_n) dt$$

which is a contradiction. Hence $\lim_{n \to \infty} d(A_1x_n, Sx_n) = 0$. By (W4) we deduce that $\lim_{n \to \infty} d(A_1x_n, t) = 0$.

Also, since $A_1X \subset TX$, it follows that for each $\{x_n\}$ in $X$ there exists $\{y_n\}$ in $X$ such that $A_1x_n = Ty_n$ and $\lim_{n \to \infty} d(A_1x_n, t) = \lim_{n \to \infty} d(Ty_n, t) = 0$. We show that $\lim_{n \to \infty} d(A_ky_n, t) = 0$ for each $k > 1$. Suppose that $\lim sup_{n \to \infty} d(A_ky_n, t) > 0$. Then using (1), we get

$$\lim_{n \to \infty} \sup_{0 \leq t \leq \infty} d(A_1x_n, A_ky_n)$$

$$\leq \lim_{n \to \infty} \sup_{0 \leq t \leq \infty} \left( \int_0^{\infty} d(A_1x_n, A_ky_n) dt \right)$$

$$= \lim_{n \to \infty} \sup_{0 \leq t \leq \infty} \left( \int_0^{\infty} d(A_1x_n, A_ky_n) dt \right)$$

$$= \lim_{n \to \infty} \sup_{0 \leq t \leq \infty} \left( \int_0^{\infty} d(A_1x_n, A_ky_n) dt \right)$$

$$< \lim_{n \to \infty} \int_0^{\infty} d(A_1x_n, A_ky_n) dt$$
\[
\begin{align*}
\limsup_{n \to \infty} & \psi \left( \int_0^{\max\{0,0,d(A_ky_n,A_1x_n),0+d(A_ky_n,A_1x_n)/2\}} \phi(t) dt \right) \\
= & \limsup_{n \to \infty} \psi \left( \int_0^{\max\{d(A_ky_n,A_1x_n),d(A_ky_n,A_1x_n)/2\}} \phi(t) dt \right) \\
= & \limsup_{n \to \infty} \psi \left( \int_0^{d(A_ky_n,A_1x_n)} \phi(t) dt \right) \\
< & \limsup_{n \to \infty} \int_0^{d(A_ky_n,A_1x_n)} \phi(t) dt
\end{align*}
\]

which is a contradiction. Thus \(\lim_{n \to \infty} d(A_1x_n,A_ky_n) = 0\) for all \(k > 1\). By (W4) we deduce that \(\lim_{n \to \infty} d(A_ky_n,t) = 0\).

Next we suppose that \(S\) is noncompatible with \(A_1\). Then there exists a sequence \(\{x_n\}\) in \(X\) such that \(\lim_{n \to \infty} d(A_1x_n,t) = \lim_{n \to \infty} d(Sx_n,t) = 0\) for some \(t\) in \(X\) but \(\lim_{n \to \infty} d(A_1x_n,Sx_n)\) is either nonzero or does not exist. Since \(A_1X \subset TX\), there exist a sequence \(\{y_n\}\) in \(X\) such that \(A_1x_n = Ty_n\) and \(\lim_{n \to \infty} d(A_1x_n,t) = \lim_{n \to \infty} d(Ty_n,t) = 0\). By using (1) just as in the previous case, for each \(k > 1\) we obtain that \(\lim_{n \to \infty} d(A_ky_n,t) = 0\). Thus in both cases, we obtain sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that for each \(k > 1\),

\[
\lim_{n \to \infty} d(A_1x_n,t) = \lim_{n \to \infty} d(Sx_n,t) = \lim_{n \to \infty} d(Ty_n,t) = \lim_{n \to \infty} d(A_ky_n,t) = 0
\]

where \(Ty_n = A_1x_n\).

Now suppose that \(SX\), the range of \(S\), is a complete subspace of \(X\). Then, since \(\lim_{n \to \infty} d(Sx_n,t) = 0\), there exists a point \(u\) in \(X\) such that \(t = Su\). If \(A_1u \neq Su\), using (1), we get

\[
\int_0^{d(A_1u,Su)} \phi(t) dt = \limsup_{n \to \infty} \int_0^{d(A_1u,A_ky_n)} \phi(t) dt
\]

\[
\leq \limsup_{n \to \infty} \psi \left( \int_0^{\max\{d(Su,Ty_n),d(A_1u,Su),d(A_ky_n,Ty_n),d(A_1u,Ty_n)+d(A_ky_n,Su)/2\}} \phi(t) dt \right)
\]

\[
= \psi \left( \int_0^{\max\{d(Su,Su),d(A_1u,Su),d(Su,Su),d(A_1u,Su)+d(Su,Su)/2\}} \phi(t) dt \right)
\]
\[= \psi \left( \int_0^{\max \{0,d(A_1u,Su)\}} \phi(t) dt \right)\]

\[= \psi \left( \int_0^{d(A_1u,Su)} \phi(t) dt \right)\]

\[< \int_0^{d(A_1u,Su)} \phi(t) dt\]

which is a contradiction. Hence \(A_1u = Su\). Since \(A_1X \subset TX\), there exists \(w \) in \(X\) such that \(A_1u = Tw\). Thus \(A_1u = Su = Tw\). If \(A_1u \neq A_kw\) for any value of \(k > 1\), using (1) we obtain

\[
\int_0^{d(A_1u,A_kw)} \phi(t) dt \leq \psi \left( \int_0^{\max \{0,d(Su,Tw),d(A_1u,Su),d(A_kw,Tw),[d(A_1u,Tw)+d(A_kw,Su)]/2\}} \phi(t) dt \right)\]

\[= \psi \left( \int_0^{d(A_kw,A_1u)} \phi(t) dt \right)\]

\[< \int_0^{d(A_kw,A_1u)} \phi(t) dt\]

which is a contradiction. Hence \(A_1u = A_kw\). Thus \(Tw = Su = A_1u = A_kw\) for every \(k > 1\). Next let us assume that \(TX\) is a complete subspace of \(X\). Then, since \(\lim_{n \to \infty} d(Ty_n,t) = 0\), there exists a point \(w \) in \(X\) such that \(t = Tw\). If \(A_kw \neq Tw\) for all \(k > 1\), then using (1), we get

\[
\int_0^{d(Tw,A_kw)} \phi(t) dt = \lim_{n \to \infty} \sup \int_0^{d(A_1x_n,A_kw)} \phi(t) dt\]

\[\leq \lim_{n \to \infty} \sup \psi \left( \int_0^{\max \{d(Sx_n,Tw),d(A_1x_n,Sx_n),d(A_kw,Tw)\},[d(A_1x_n,Tw)+d(A_kw,Sx_n)]/2\}} \phi(t) dt \right)\]

\[= \psi \left( \int_0^{\max \{d(Tw,Tw),d(Tw,Tw),d(A_kw,Tw),[d(Tw,Tw)+d(A_kw,Tw)]/2\}} \phi(t) dt \right)\]

\[= \psi \left( \int_0^{\max \{0,0,d(A_kw,Tw),[0+d(A_kw,Tw)]/2\}} \phi(t) dt \right)\]

\[= \psi \left( \int_0^{d(A_kw,Tw)} \phi(t) dt \right)\]
such that \(d(A_k w, Tw)\) which is a contradiction. Hence \(A_k w = Tw\). Since \(A_k X \subseteq SX\), there exists \(u\) in \(X\) such that \(A_k w = Su\). Thus \(Tw = A_k w = Su\). If \(A_k w \neq A_1 u\). Then we have from (1) that

\[
\int_0^{d(A_1 u, A_k w)} \phi(t) dt \leq \psi \left( \int_0^{\max\{d(Su, Tw), d(A_1 u, Su), d(A_k w, Tw), [d(A_1 u, Tw) + d(A_k w, Su)]/2\}} \phi(t) dt \right)
\]

\[
= \psi \left( \int_0^{\max\{d(A_1 u, A_k w), d(A_k w, A_1 u), [d(A_1 u, A_k w) + [0]/2\}} \phi(t) dt \right)
\]

\[
= \psi \left( \int_0^{d(A_k w, A_1 u)} \phi(t) dt \right)
\]

\[
< \int_0^{d(A_1 u, A_k w)} \phi(t) dt
\]

which is a contradiction. Hence \(A_k w = A_1 u\). Thus we have \(Su = A_1 u = Tw = A_k w\) for each \(k > 1\). Thus irrespective of whether \(SX\) or \(TX\) is assumed to be complete, we get

\(Su = A_1 u = Tw = A_k w\) for some \(u, w \in X\) and for all \(k > 1\).

Pointwise R-weak commutativity of \(A_1\) and \(S\) implies that there exists \(R_1 > 0\) such that \(d(A_1 Su, SA_1 u) \leq R_1 d(A_1 u, Su) = 0\), that is, \(A_1 Su = SA_1 u\) and \(A_1 A_1 u = A_1 Su = SA_1 u = SSu\). Similarly, for each \(k > 1\) there exists \(R_k > 0\) such that \(d(A_k Tw, TA_k w) \leq R_k d(A_k w, Tw) = 0\), that is, \(A_k Tw = TA_k w\) and \(A_k A_k w = A_k Tw = TA_k w = TT w\). We claim that \(A_1 A_1 u = A_1 u\). Suppose that this claim is not true. Then using (1), we get

\[
\int_0^{d(A_1 u, A_k u)} \phi(t) dt = \int_0^{d(A_1 u, A_k u)} \phi(t) dt
\]

\[
\leq \psi \left( \int_0^{\max\{d(SA_1 u, Tw), d(A_1 A_1 u, SA_1 u), d(A_k w, Tw), [d(A_1 A_1 u, Tw) + d(A_k w, SA_1 u)]/2\}} \phi(t) dt \right)
\]

\[
= \psi \left( \int_0^{\max\{d(A_1 A_1 u, A_1 u), d(A_1 A_1 u, A_1 A_1 u), d(A_1 A_1 u, A_1 A_1 u), [d(A_1 A_1 u, A_1 A_1 u) + d(A_1 u, A_1 A_1 u)]/2\}} \phi(t) dt \right)
\]

\[
= \psi \left( \int_0^{\max\{d(A_1 A_1 u, A_1 u), d(A_1 u, A_1 A_1 u)/2\}} \phi(t) dt \right)
\]

\[
= \psi \left( \int_0^{d(A_1 u, A_1 u)} \phi(t) dt \right)
\]
\[
\int_0^d(t_1, t_2) \phi(t) dt < \int_0^\max\{d(Sx, Ty), d(A_1x, Sx), d(A_ky, Tx), [d(A_1x, Ty) + d(A_ky, Sx)]/2\} \phi(t) dt.
\]

which is a contradiction. Hence \( A_1u = A_1u \). Thus \( A_1u = A_1A_1u = SA_1u \) and \( A_1u \) is a common fixed point of \( A_1 \) and \( S \). Similarly, it can be shown that \( A_kw = A_kA_kw = TA_kw \) for some \( k > 1 \), that is, \( A_kw = A_1u \) is a common fixed point of \( T \) and \( A_k \) for each \( k > 1 \). Hence \( A_1u \) is a common fixed point of \( A_1, S, T \) and \( A_k \) for each \( k > 1 \).

We now show that \( A_1u = t \) is unique. Let \( z \) be another fixed point of \( A_k, S, T \) and \( A_1 \) for all \( k > 1 \) such that \( t \neq z \). Then from (1)
\[
\int_0^d(t, z) \phi(t) dt = \int_0^d(A_1t, A_kz) \phi(t) dt
\]
\[
\leq \psi \left( \int_0^{\max\{d(Sx, Ty), d(A_1x, Sx), d(A_ky, Tx), [d(A_1x, Ty) + d(A_ky, Sx)]/2\}} \phi(t) dt \right)
\]
\[
= \psi \left( \int_0^{\max\{d(t, z), d(t, t), d(z, z), [d(t, z) + d(z, t)]/2\}} \phi(t) dt \right)
\]
\[
= \psi \left( \int_0^{d(t, z)} \phi(t) dt \right)
\]
\[
< \int_0^{d(t, z)} \phi(t) dt
\]
which is a contradiction. Hence \( t = z \). This proves the uniqueness of \( t = A_1u \).

The proof is similar when \( A_kX \) is assumed complete for some \( k \geq 1 \), since \( A_1X \subset TX \) and \( A_kX \subset SX \) for \( k > 1 \). This completes the proof of the theorem.

Since \( \psi(t) < t \) for all \( t > 0 \), it follows from Theorem 1 that
\[
\int_0^{d(A_1x, A_ky)} \phi(t) dt < \int_0^{\max\{d(Sx, Ty), d(A_1x, Sx), d(A_ky, Ty), [d(A_1x, Ty) + d(A_ky, Sx)]/2\}} \phi(t) dt.
\]

Hence by using similar proof as in Theorem 1 we get the following Theorem.

**Theorem 2** Let \( d \) be a symmetric for \( X \) that satisfies (W4) and property (H.E). Let \( \{A_k\}, k = 1, 2, 3, \ldots \), \( S \) and \( T \) be self mappings of \( (X, d) \) such that
\[
\int_0^{d(A_1x, A_ky)} \phi(t) dt < \int_0^{\max\{d(Sx, Ty), d(A_1x, Sx), d(A_ky, Ty), [d(A_1x, Ty) + d(A_ky, Sx)]/2\}} \phi(t) dt.
\]
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for all \( x, y \in X \), where \( \phi : \mathbb{R}^+ \to \mathbb{R}^+ \) is a Lebesgue-integrable mapping which is summable, nonnegative and such that
\[
\int_0^\varepsilon \phi(t) dt > 0 \quad \text{for each} \quad \varepsilon > 0.
\]

Suppose that \( A_k X \subset SX \) when \( k > 1 \) and \( A_1 X \subset TX \). Suppose also that the pairs \( \{A_k, T\} \) for \( k > 1 \) and \( \{A_1, S\} \) are pointwise R-weakly commuting with at least one pair noncompatible. If the range of one of the mappings is a complete subspace of \( X \), then all the \( A_k, S \) and \( T \) have a unique common fixed point in \( X \).

**Theorem 3** Let \( d \) be a symmetric for \( X \) that satisfies (W4) and property (H.E). Let \( \{A_k\}, k = 1, 2, 3, \ldots \), \( S \) and \( T \) be self mappings of \( (X,d) \) such that
\[
\int_0^d(A_1x, A_ky) \phi(t) dt \leq \psi \left( \int_0^{\max\{d(Sx, Ty), d(A_1x, Sx), d(A_ky, Ty), [d(A_1x, Ty) + d(A_ky, Sx)]/2\}} \phi(t) dt \right) \quad (3)
\]
for all \( x, y \in X \), where \( \phi : \mathbb{R}^+ \to \mathbb{R}^+ \) is a Lebesgue-integrable mapping which is summable, nonnegative and such that
\[
\int_0^\varepsilon \phi(t) dt > 0 \quad \text{for each} \quad \varepsilon > 0.
\]

Suppose that \( A_k X \subset SX \) when \( k > 1 \) and \( A_1 X \subset TX \). Suppose also that the pairs \( \{A_k, T\} \) for \( k > 1 \) and \( \{A_1, S\} \) are pointwise R-weakly commuting with at least one pair compatible and one noncompatible. If one of the mappings in the compatible pair is continuous, then all the \( A_k, S \) and \( T \) have a unique common fixed point in \( X \).

**Proof:**

As in Theorem 1, noncompatibility of one of the R-weak commuting pairs implies the existence of sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \) such that
\[
\lim_{n \to \infty} d(A_1x_n, t) = \lim_{n \to \infty} d(Sx_n, t) = \lim_{n \to \infty} d(Ty_n, t) = \lim_{n \to \infty} d(A_ky_n, t) = 0
\]
for each \( k > 1 \), where \( Ty_n = A_1x_n \) and \( t \in X \).

Suppose that \( S \) is continuous and compatible with \( A_1 \). Then \[ \lim_{n \to \infty} d(SSx_{2n}, St) = \lim_{n \to \infty} d(SA_1x_{2n}, St) = 0 \] and the compatibility of \( A_1 \) and \( S \) implies that \[ \lim_{n \to \infty} d(A_1Sx_{2n}, St) = 0 \]. If we write \( Sx_{2n} = y_n \), then since \( A_1 X \subset TX \), it follows that for each \( \{y_n\} \) there exists \( \{z_n\} \) such that \( A_1y_n = Tz_n \) and
\[
\lim_{n \to \infty} d(A_1Sx_{2n}, St) = \lim_{n \to \infty} d(A_1y_n, St) = \lim_{n \to \infty} d(SSx_{2n}, St) = \lim_{n \to \infty} d(Sy_n, St) = \lim_{n \to \infty} d(Tz_n, St) = 0
\]
If $A_kz_n \neq St$, then using (3), we get
\[
\int_0^{d(St,A_kz_n)} \phi(t) dt = \lim_{n \to \infty} \sup \int_0^{d(A_1y_n,A_kz_n)} \phi(t) dt
\]
\[
\leq \lim_{n \to \infty} \sup \psi \left( \int_0^{\max \{d(Sy_n,Tz_n),d(A_1y_n,Sy_n),d(A_kz_n,Tz_n),[d(A_1y_n,Tz_n)+d(A_kz_n,Sy_n)]/2\}} \phi(t) dt \right)
\]
\[
= \lim_{n \to \infty} \sup \psi \left( \int_0^{\max \{d(St,St),d(St,St),d(A_kz_n,St),[d(St,St)+d(A_kz_n,St)]/2\}} \phi(t) dt \right)
\]
\[
= \lim_{n \to \infty} \sup \psi \left( \int_0^{\max \{0,0,d(A_kz_n,St),[0+d(A_kz_n,St)]/2\}} \phi(t) dt \right)
\]
\[
= \lim_{n \to \infty} \psi \left( \int_0^{d(A_kz_n,St)} \phi(t) dt \right)
\]
\[
< \lim_{n \to \infty} \int_0^{d(A_kz_n,St)} \phi(t) dt
\]

which is a contradiction. Thus $\lim_{n \to \infty} d(St,A_kz_n) = 0$ for all $k > 1$. The remaining part of the proof is similar to that in Theorem 1 when $SX$ was assumed complete. Next, let $A_1$ be continuous and compatible with $S$. Then $\lim_{n \to \infty} d(A_1A_1x_{2n},A_1t) = \lim_{n \to \infty} d(A_1Sx_{2n},A_1t) = 0$ and the compatibility of $A_1$ and $S$ implies that $\lim_{n \to \infty} d(SA_1x_{2n},A_1t) = 0$. Since $A_1X \subset TX$, there exists $w$ in $X$ such that $A_1t = Tw$. Thus $\lim_{n \to \infty} d(A_1A_1x_{2n},Tw) = \lim_{n \to \infty} d(SA_1x_{2n},Tw) = 0$. Now the remaining part of the proof in this case is similar to Theorem 1 when $TX$ was assumed complete. Similar arguments apply when $T$ is assumed compatible with $A_k$ for some $k > 1$ and $T$ or $A_k$ is assumed continuous. This proves the theorem.

Since $\psi(t) < t$ for all $t > 0$, it follows from Theorem 3 that
\[
\int_0^{d(A_1x,A_ky)} \phi(t) dt < \int_0^{\max \{d(Sx,Ty),d(A_1x,Sx),d(A_ky,Ty),[d(A_1x,Ty)+d(A_ky,Sx)]/2\}} \phi(t) dt.
\]
Hence by using similar proof as in Theorem 3 we get the following Theorem.

**Theorem 4** Let $d$ be a symmetric for $X$ that satisfies $(W_4)$ and property $(H.E)$. Let $\{A_k\}, k = 1, 2, 3, \ldots$, $S$ and $T$ be self mappings of $(X,d)$ such that
\[
\int_0^{d(A_1x,A_ky)} \phi(t) dt < \int_0^{\max \{d(Sx,Ty),d(A_1x,Sx),d(A_ky,Ty),[d(A_1x,Ty)+d(A_ky,Sx)]/2\}} \phi(t) dt.
\]
for all \( x, y \in X \), where \( \phi : \mathbb{R}^+ \to \mathbb{R}^+ \) is a Lebesgue-integrable mapping which is summable, nonnegative and such that

\[
\int_0^\varepsilon \phi(t)dt > 0 \quad \text{for each} \quad \varepsilon > 0.
\]

Suppose that \( A_kX \subset SX \) when \( k > 1 \) and \( A_1X \subset TX \). Suppose also that the pairs \( \{A_k, T\} \) for \( k > 1 \) and \( \{A_1, S\} \) are pointwise \( R \)-weakly commuting with at least one pair compatible and one noncompatible. If one of the mappings in the compatible pair is continuous, then all the \( A_k, S \) and \( T \) have a unique common fixed point in \( X \).

**Theorem 5** Let \( d \) be a symmetric for \( X \) that satisfies \((W3)\), \((W4)\) and \((H.E)\). Let \( \{A_k\} \), \( k = 1, 2, 3, \ldots \), \( S \) and \( T \) be self mappings of \((X,d)\) such that

\[
\int_0^{d(A_1x,A_1y)} \phi(t)dt \leq \psi \left( \int_0^{\max\{d(Sx,Ty),d(A_1x,Sx),d(A_1y,Ty),\|d(A_1x,Ty)+d(A_1y,Sx)\|/2\}} \phi(t)dt \right)
\]

for all \( x, y \in X \), where \( \phi : \mathbb{R}^+ \to \mathbb{R}^+ \) is a Lebesgue-integrable mapping which is summable, nonnegative and such that

\[
\int_0^\varepsilon \phi(t)dt > 0 \quad \text{for each} \quad \varepsilon > 0.
\]

Suppose that \( A_kX \subset SX \) when \( k > 1 \) and \( A_1X \subset TX \). Suppose also that the pairs \( \{A_k, T\} \) for \( k > 1 \) and \( \{A_1, S\} \) are weakly compatible and either \( \{A_1, S\} \) or \( \{A_k, T\} \) satisfies property \((E.A)\). If the range of one of the mappings is a complete subspace of \( X \), then all the \( A_k, S \) and \( T \) have a unique common fixed point in \( X \).

**Proof:**

Suppose that \( A_k \) for \( k > 1 \) and \( T \) satisfy property \((E.A)\). Then there exists a sequence \( \{z_n\} \) in \( X \) such that \( \lim_{n \to \infty} d(A_kz_n, t) = \lim_{n \to \infty} d(Tz_n, t) = 0 \) for some \( t \) in \( X \). By \((H.E)\) we have \( \lim_{n \to \infty} d(A_kz_n, Tz_n) = 0 \). Since \( A_kX \subset SX \), there exists in \( X \) a sequence \( \{x_n\} \) such that \( A_kz_n = Sx_n \). Thus \( \lim_{n \to \infty} d(Sx_n, t) = \lim_{n \to \infty} d(A_kz_n, t) = \lim_{n \to \infty} d(Tz_n, t) = 0 \). Therefore, by \((H.E)\) we have \( \lim_{n \to \infty} d(Sx_n, Tz_n) = 0 \) and \( \lim_{n \to \infty} d(A_kz_n, Sx_n) = 0 \). We need to prove that \( \lim_{n \to \infty} d(A_1x_n, t) = 0 \). Suppose that \( \limsup_{n \to \infty} d(A_1x_n, t) > 0 \). Then by virtue of \((5)\), we have

\[
\limsup_{n \to \infty} \int_0^{d(A_1x_n,Sx_n)} \phi(t)dt = \limsup_{n \to \infty} \int_0^{d(A_1x_n,A_kz_n)} \phi(t)dt
\]

\[
\leq \limsup_{n \to \infty} \psi \left( \int_0^{\max\{d(Sx_n,Tz_n),d(A_1x_n,Sx_n),d(A_kz_n,Tz_n),\|d(A_1x_n,Tz_n)+d(A_kz_n,Sx_n)\|/2\}} \phi(t)dt \right)
\]
\[
\begin{align*}
= \limsup_{n \to \infty} \psi \left( \int_0^{\max\{d(Sx_n, Sx_n), d(A_1x_n, Sx_n), d(Sx_n, Sx_n), d(A_1x_n, Sx_n) + d(Sx_n, Sx_n)/2\}} \phi(t) dt \right) \\
= \limsup_{n \to \infty} \psi \left( \int_0^{\max\{d(A_1x_n, Sx_n), d(A_1x_n, Sx_n)/2\}} \phi(t) dt \right) \\
= \limsup_{n \to \infty} \psi \left( \int_0^{d(A_1x_n, Sx_n)} \phi(t) dt \right) \\
< \limsup_{n \to \infty} \int_0^{d(A_1x_n, Sx_n)} \phi(t) dt
\end{align*}
\]

which is a contradiction. Hence \( \lim_{n \to \infty} d(A_1x_n, Sx_n) = 0 \). By (W4) we deduce that

\( \lim_{n \to \infty} d(A_1x_n, t) = 0 \). Suppose that \( SX \) is a complete subspace of \( X \). Then \( t = Su \) for some \( u \in X \). Consequently, we have

\( \lim_{n \to \infty} d(A_1x_n, Su) = \lim_{n \to \infty} d(A_kz_n, Su) = \lim_{n \to \infty} d(Sx_n, Su) = \lim_{n \to \infty} d(Tz_n, Su) = 0 \)

We claim that \( A_1u = Su \). If \( A_1u \neq Su \), using (5), we get

\[
\begin{align*}
\limsup_{n \to \infty} \int_0^{d(A_1u, A_kz_n)} \phi(t) dt \\
\leq \limsup_{n \to \infty} \psi \left( \int_0^{\max\{d(Su, Tz_n), d(A_1u, Su), d(A_kz_n, Tz_n), [d(A_1u, Tz_n) + d(A_kz_n, Su)/2]}} \phi(t) dt \right) \\
= \psi \left( \int_0^{\max\{d(Su, Su), d(A_1u, Su), d(Su, Su), [d(A_1u, Su) + d(Su, Su)/2]}} \phi(t) dt \right) \\
= \psi \left( \int_0^{\max\{0, d(A_1u, Su), 0, [d(A_1u, Su) + 0]/2\}} \phi(t) dt \right) \\
= \psi \left( \int_0^{d(A_1u, Su)} \phi(t) dt \right) \\
< \int_0^{d(A_1u, Su)} \phi(t) dt
\end{align*}
\]

which is a contradiction. Hence \( A_1u = Su \). Since \( A_1 \) and \( S \) are weakly compatible, we have \( A_1Su = SA_1u \), that is, \( At = St \). On the other hand, since
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$A_1X \subset TX$, there exists $w$ in $X$ such that $A_1u = Tw$. Thus $Tw = A_1u = Su$. Suppose that $A_kw \neq Tw$. Then condition (5) gives

$$\int_0^d(Tw, A_kw) \phi(t)dt = \int_0^d(A_1u, A_kw) \phi(t)dt$$

$$\leq \int_0^\psi \left( \max \{ d(Su, Tw), d(A_1u, Su), d(A_kw, Tw), d(A_1u, Tw) + d(A_kw, Su) \} / 2 \right) \phi(t)dt$$

$$= \int_0^\psi \left( \max \{ 0, d(A_kw, Tw), [0 + d(A_kw, Tw)] / 2 \} \right) \phi(t)dt$$

$$= \int_0^\psi \left( \max \{ d(A_kw, Tw), d(A_kw, Tw) / 2 \} \right) \phi(t)dt$$

$$= \int_0^\psi \left( \max \{ d(A_kw, Tw), d(A_kw, Tw) \} \right) \phi(t)dt$$

which is a contradiction. Hence $\int_0^d(Tw, A_kw) \phi(t)dt = 0$ which implies that $d(Tw, A_kw) = 0$ for every $k > 1$. Thus $Tw = A_kw$ and hence $t = Su = A_1u = Tw = A_kw$ for every $k > 1$. The weak compatibility of $A_k$ and $T$ implies that $A_kTw = TA_kw$, that is, $A_kt = Tt$. The weak compatibility of $A_1$ and $S$ imply $A_1A_1u = SA_1u$. We first need to prove that $t = A_1u$ is a common fixed point of $A_k$, $S$ and $T$ for all $k \geq 1$. If $A_1u \neq A_1A_1u$, then using (5), we get

$$\int_0^d(A_1A_1u, A_1u) \phi(t)dt = \int_0^d(A_1A_1u, A_kw) \phi(t)dt$$

$$\leq \int_0^\psi \left( \max \{ d(SA_1u, Tw), d(A_1A_1u, SA_1u), d(A_kw, Tw), d(A_1A_1u, Tw) + d(A_kw, SA_1u) \} / 2 \right) \phi(t)dt$$

$$= \int_0^\psi \left( \max \{ d(SA_1u, A_1u), d(A_1A_1u, SA_1u), d(A_1A_1u, A_1u), [d(A_1A_1u, A_1u) + d(A_1A_1u, SA_1u)] / 2 \} \right) \phi(t)dt$$

$$= \int_0^\psi \left( \max \{ d(A_1A_1u, A_1u), d(A_1A_1u, A_1A_1u), d(A_1A_1u, A_1u), [d(A_1A_1u, A_1u) + d(A_1A_1u, A_1A_1u)] / 2 \} \right) \phi(t)dt$$

$$= \int_0^\psi \left( \max \{ d(A_1A_1u, A_1u), d(A_1A_1u, A_1A_1u) \} \right) \phi(t)dt$$

$$= \int_0^\psi \left( \max \{ d(A_1A_1u, A_1u) \} \right) \phi(t)dt$$
which is a contradiction. Hence $A_1u = A_1A_1u$. Thus $A_1u = A_1A_1u = SA_1u$ and this shows that $A_1u$ is a common fixed point of $A_1$ and $S$. We have already shown that $A_kTw = TA_kw$, that is, $A_kA_1u = TA_1u$. We now show that $A_kw = A_kA_kw$. Suppose $A_kw \neq A_kA_kw$, then using (5), we get

$$\int_0^{d(A_kw,A_kA_kw)} \phi(t)dt = \int_0^{d(A_1u,A_kA_kw)} \phi(t)dt$$

$$\leq \psi \left( \int_0^{\max\{d(Su,TA_kw),d(A_1u,Su),d(A_kA_kw,TA_kw),[d(A_1u,TA_kw)+d(A_kA_kw,Su)]/2\}} \phi(t)dt \right)$$

$$= \psi \left( \int_0^{\max\{d(A_kw,A_kTw),d(A_kA_kw,A_kw),d(A_kA_kw,A_kTw),[d(A_kw,A_kTw)+d(A_kA_kw,A_kw)]/2\}} \phi(t)dt \right)$$

$$= \psi \left( \int_0^{\max\{d(A_kw,A_kA_kw),d(A_kA_kw,A_kw),d(A_kA_kw,A_kA_kw),[d(A_kw,A_kA_kw)+d(A_kA_kw,A_kw)]/2\}} \phi(t)dt \right)$$

$$= \psi \left( \int_0^{d(A_kA_kw,A_kA_kw)} \phi(t)dt \right)$$

$$< \int_0^{d(A_kA_kw,A_kA_kw)} \phi(t)dt$$

which is a contradiction. Hence $A_kw = A_kA_kw$ for all $k > 1$. Thus $A_kw = A_1u$ is a common fixed point of $T$ and $A_k$ for each $k > 1$. But we have already shown that $A_kw = A_1u$ is a common fixed point of $A_1$ and $S$. Thus $A_1u$ is a common fixed point of $A_1, A_k, S$ and $T$ for all $k > 1$. The proof is similar when $TX$ is assumed to be a subspace of $X$.

It remains to prove that $A_1u = t$ is unique. Let $z$ be another fixed point of $A_k, S, T$ and $A_1$ for all $k > 1$ such that $t \neq z$. Then from (5), we have

$$\int_0^{d(t,z)} \phi(t)dt = \int_0^{d(A_1t,A_kz)} \phi(t)dt$$

$$\leq \psi \left( \int_0^{\max\{d(St,Tz),d(A_1t,St),d(A_kz,Tz),[d(A_1t,Tz)+d(A_kz,St)]/2\}} \phi(t)dt \right)$$

$$= \psi \left( \int_0^{\max\{d(t,z),d(t,t),d(z,z),[d(t,z)+d(z,t)]/2\}} \phi(t)dt \right)$$
\[
= \psi \left( \int_0^{d(t,z)} \phi(t)dt \right)
\]
\[
< \int_0^{d(t,z)} \phi(t)dt
\]
which is a contradiction. Hence \( t = z \). This proves the uniqueness of \( t = A_1u \).
The proof is similar when \( A_kX \) is assumed complete for some \( k \geq 1 \), since \( A_1X \subset TX \) and \( A_kX \subset SX \) for \( k > 1 \). This completes the proof of the theorem.

If we take \( A_1 = A \) and \( A_k = B \) for all \( k > 1 \), then from theorem 5 we get the following corollary.

**Corollary 1** Let \( d \) be a symmetric for \( X \) that satisfies (W3), (W4) and (H.E). Let \( A, B, S \) and \( T \) be self mappings of \((X,d)\) such that
\[
\int_0^{d(Ax,By)} \phi(t)dt \leq \psi \left( \int_0^{\max\{d(Sx,Ty),d(Ax,Sx),d(By,Ty)\} / 2} \phi(t)dt \right)
\]
for all \( x, y \in X \), where \( \phi : \mathbb{R}^+ \to \mathbb{R}^+ \) is a Lebesgue-integrable mapping which is summable, nonnegative and such that
\[
\int_0^\varepsilon \phi(t)dt > 0 \quad \text{for each} \quad \varepsilon > 0.
\]
Suppose that \( BX \subset SX \) and \( AX \subset TX \). Suppose also that the pairs \( \{B,T\} \) for \( k > 1 \) and \( \{A,S\} \) are weakly compatible and either \( \{A,S\} \) or \( \{B,T\} \) satisfies property (E.A). If the range of one of the mappings is a complete subspace of \( X \), then \( A, B, S \) and \( T \) have a unique common fixed point in \( X \).

Now let us take \( S = T = Id_X \) where \( Id_X \) is the identity mapping in \( X \). Then from corollary 1 we get

**Corollary 2** Let \( d \) be a symmetric for \( X \) that satisfies (W3), (W4) and (H.E). Let \( A \) and \( B \) be self mappings of \((X,d)\) such that
\[
\int_0^{d(Ax,By)} \phi(t)dt \leq \psi \left( \int_0^{\max\{d(x,y),d(Ax,x),d(By,y)\} / 2} \phi(t)dt \right)
\]
for all \( x, y \in X \), where \( \phi : \mathbb{R}^+ \to \mathbb{R}^+ \) is a Lebesgue-integrable mapping which is summable, nonnegative and such that
\[
\int_0^\varepsilon \phi(t)dt > 0 \quad \text{for each} \quad \varepsilon > 0.
\]
If the range of \( A \) or \( B \) is a complete subspace of \( X \), then \( A \) and \( B \) have a unique common fixed point in \( X \).
We note that Corollary 2 is more general than Theorem 1 of [10].
Again taking \( A = B \) and \( \psi(t) = kt \) for \( k \in [0,1) \) in Corollary 2 we get the following Corollary.

**Corollary 3** Let \( d \) be a symmetric for \( X \) that satisfies (W3), (W4) and (H.E). Let \( A \) be self mapping of \((X,d)\) such that
\[
\int_0^{d(Ax,Ay)} \phi(t) dt \leq k \left( \int_0^{\max\{d(x,y),d(Ax,x),d(Ay,y),[d(Ax,y)+d(Ay,x)]/2\}} \phi(t) dt \right)
\]
for all \( x, y \in X \) and \( k \in [0,1) \), where \( \phi : \mathbb{R}^+ \to \mathbb{R}^+ \) is a Lebesgue-integrable mapping which is summable, nonnegative and such that
\[
\int_0^\varepsilon \phi(t) dt > 0 \quad \text{for each} \quad \varepsilon > 0.
\]
If the range of \( A \) is a complete subspace of \( X \), then \( A \) have a unique fixed point in \( X \).

Clearly, Corollary 3 is more general than Theorem 2 of [8].

**Theorem 6** Let \( d \) be a symmetric space for \( X \) that satisfies (W3). Let \( S \) and \( T \) be two weakly compatible self mappings of \( X \) such that for each distinct \( x, y \in X \),
\[
\int_0^{d(Tx,Ty)} \phi(t) dt < \int_0^{\max\{d(Sx,Sy),\frac{1}{2}[d(Tx,Sx)+d(Ty,Sy)],\frac{1}{2}[d(Ty,Sx)+d(Tx,Sy)]\}} \phi(t) dt
\]
for \( 1 \leq k < 2 \), where \( \phi : \mathbb{R}^+ \to \mathbb{R}^+ \) is a Lebesgue-integrable mapping which is summable, nonnegative and such that
\[
\int_0^\varepsilon \phi(t) dt > 0 \quad \text{for each} \quad \varepsilon > 0.
\]
Suppose that \( T \) and \( S \) satisfy the property (E.A), and \( TX \subset SX \). If \( SX \) or \( TX \) is a complete subspace of \( X \), then \( T \) and \( S \) have a unique common fixed point.

**Proof:**
Suppose that \( T \) and \( S \) satisfy the property (E.A). Then there exists a sequence \( \{x_n\} \) in \( X \) such that
\[
\lim_{n \to \infty} d(Tx_n, z) = \lim_{n \to \infty} d(Sx_n, z) = 0 \quad \text{for some} \quad z \in X
\]
Suppose that \( SX \) is complete. Then \( \lim_{n \to \infty} d(Sx_n, Sa) = 0 \) for some \( a \in X \).
Also,
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\[ \lim_{n \to \infty} d(Tx_n, Sa) = 0. \]

We now show that \( Ta = Sa \). Suppose that \( Ta \neq Sa \). Then using (9), we have

\[
\int_0^{d(Tx_n, Ta)} \phi(t) dt < \int_0^{\max \left\{ d(Sx_n, Sa), \frac{k}{2} [d(Tx_n, Sx_n) + d(Ta, Sa)], \frac{1}{2} [d(Ta, Sx_n) + d(Tx_n, Sa)] \right\}} \phi(t) dt
\]

Taking the limit as \( n \to \infty \) yields,

\[
\int_0^{d(Ta, Sa)} \phi(t) dt < \int_0^{\max \left\{ d(Sa, Sa), \frac{k}{2} [d(Ta, Sa) + d(TTa, STa)], \frac{1}{2} [d(TTa, Sa) + d(Ta, STa)] \right\}} \phi(t) dt
\]

which is a contradiction. Hence \( Ta = Sa \). Since \( T \) and \( S \) are weakly compatible, \( STa = TSa \). Therefore, \( TTa = TSa = STa = SSA \).

We now show that \( Ta \) is a common fixed point of \( S \) and \( T \). Suppose that \( Ta \neq TTa \). Then by (9),

\[
\int_0^{d(Ta, TTa)} \phi(t) dt < \int_0^{\max \left\{ d(Sa, STa), \frac{k}{2} [d(Ta, Sa) + d(TTa, STa)], \frac{1}{2} [d(TTa, Sa) + d(Ta, STa)] \right\}} \phi(t) dt
\]

\[
= \int_0^{d(Ta, TTa)} \phi(t) dt
\]

\[
= \int_0^{d(Ta, TTa)} \phi(t) dt
\]

which is a contradiction. Hence \( TTa = Ta \) and \( STa = TTa = Ta \). Thus \( Ta \) is a common fixed point of \( S \) and \( T \). The proof is similar when \( TX \) is assumed to be a complete subspace of \( X \) since \( TX \subset SX \).

To complete the proof we show the uniqueness of \( Ta \). Suppose that \( Tb \) is also a common fixed point of \( S \) and \( T \) such that \( Ta \neq Tb \). Then by (9),

\[
\int_0^{d(Ta, Tb)} \phi(t) dt < \int_0^{\max \left\{ d(Sa, Tb), \frac{k}{2} [d(Ta, Sa) + d(Tb, Tb)], \frac{k}{2} [d(Tb, Sa) + d(TTa, STa)] \right\}} \phi(t) dt
\]

\[
= \int_0^{\max \left\{ d(Ta, Tb), \frac{k}{2} [d(Tb, Ta) + d(Ta, Tb)] \right\}} \phi(t) dt
\]

\[
= \int_0^{d(Ta, Tb)} \phi(t) dt
\]
which is a contradiction. Hence $Ta = Tb$, proving the uniqueness of $Ta$. □

Setting $\phi(t) = 1$ and $k = 1$ in theorem 6 we get

**Corollary 4** Let $d$ be a symmetric space for $X$ that satisfies (W3). Let $S$ and $T$ be two weakly compatible self mappings of $X$ such that for each distinct $x, y \in X$,

$$d(Tx, Ty) < \max\{d(Sx, Sy), [d(Tx, Sx) + d(Ty, Sy)] / 2, [d(Ty, Sx) + d(Tx, Sy)] / 2\} \quad (11)$$

Suppose that $T$ and $S$ satisfy the property (E.A), and $TX \subset SX$. If $SX$ or $TX$ is a complete subspace of $X$, then $T$ and $S$ have a unique common fixed point.

We note that Corollary is more general than Theorem 1 of [1].

**References**


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