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Arens Product and Topological Center of Some Banach Spaces

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Abstract

Let X be a Banach algebra and X^* be a first dual space of X . Let $f.x$ and fox be the first and second Arens products, respectively. In this paper, we show that mapping T_f from X into X^* is weakly compact for all $f \in X^*$ iff X is Arens regular and also If $dimR(T_f) < \infty$ for all $f \in X^*$ then X is Arens regular. Conversely, we show that if X is Arens regular and $R(T_f)$ is weakly closed in X then $dimR(T_f) < \infty$ for all $f \in X^*$. In finally, we obtain some conclusion in the topological centers. Suppose that the Banach algebra X is wsc and wcc and contains a bai then X is reflexive whenever the mapping $x \longrightarrow f.x$ is weakly compact. Now, suppose that Z_1 and Z_2 are topological centers of X^{**} with respect to the first and second Arens products. We also show the relations between the equality $XX^* = X$ and $Z_1 = Z_2$.

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1 Introduction

Suppose that X is a set and that Ω is a family of functions such that each f in Ω maps X into a topological space(Y_f, τ_f). It is always possible to find a topology for X that makes every member of Ω continuous. Then there is a smallest topology for X with respect to which each member of Ω is continuous and we called its weak topology of X and we show by $\sigma(X, \Omega)$. Let X be a normed space. Then the topology for X induced by X^* (first dual of X) is the weak topology of X and we show by $\sigma(X, X^*)$. That is, the weak topology of a normed space is the smallest topology for the space such that every member of the dual space is continuous with respect to that topology. The norm and weak topologies of a normed space are the same if and only if the space is finite dimensional. For proof, see [12, 2.5.14].

We say that a net $(x_\alpha)_{\alpha\in I}$ is weak convergence to an element x, if the net $(x_\alpha)_{\alpha\in I}$ converges to x with respect to weak topology and we write $x_\alpha \xrightarrow{w} x$ or $w - lim x_\alpha = x$. The closure of a set A can be dented by A^{-w}.

Let X be a normed space and Let Q be the natural map from X into X^{**} (second dual of X). Then the topology for X^* induced by family $Q(X)$ is the $weak^*$ topology of X^* and we denoted by$\sigma(X, X^*)$. A topological property that holds with respect to the $weak^*$ topology is said to hold $weakly^*$ or to be a $weak^*$ property.

Whenever w^* is attached to a topological symbol, it indicates that the reference is to the $weak^*$ topology. If X be a normed space, the function $x \longrightarrow Q(x)$ is the natural map which sometimes we show by Q , that is, for each $f \in X^*$ we write Q(x)(f)=f(x).

Let $x, y \in X$ and $f \in X^*$ and $F, G \in X^{**}$ then for first Arens multiplication, we have $f.x(y) = f(x.y)$, $F.f(x) = F(f.x)$ and F.G(f)=F(G.f). It is clear that $F.G \in X^{**}$, $G.f \in X^*$ and $f.x \in X^*$. The mapping $F \to F.G$, for G fixed in X^{**}, is $weak^* - to - weak^*$ continuous, but the mapping $F \to G.F$ for G fixed in X^{**} is in general not $weak^* - to - weak^*$ continuous. Whence, the first topological center of X^{**} with respect to this product is defined as follows

$$Z_1 = \{G \in X^{**} : \textit{ the mapping } F \longrightarrow G.F \textit{ is } weak^* - to - weak^*$$

$$\textit{continuous on } X^{**}\}.$$

The second Arens product is defined as follows Let $x, y \in X$ and $f \in X^*$ and $F, G \in X^{**}$ then for second Arens multiplication, we have $x.f(y) = f(y.x)$, $foF(x) = F(x.f)$ and FoG(f)=F(Gof). It is clear that $FoG \in X^{**}$, $Gof \in X^*$ and $f.x \in X^*$.

The mapping $F \to GoF$, for G fixed in X^{**}, is $weak^* - to - weak^*$ continuous, but the mapping $F \to FoG$ for G fixed in X^{**} is ,in general, not $weak^* - to - weak^*$ continuous. Whence, the second topological center of X^{**} with respect to second Arense product is defined as follows

$$Z_2 = \{G \in X^{**} : \textit{the mapping } F \longrightarrow FoG \textit{ is } weak^* - to - weak^*$$

$$\textit{continuous on } X^{**}\}.$$

If ,for each $F, G \in X^{**}$, the equality $F.G = FoG$ holds ,then the algebra X is said to be Arens regular.

In the following, we show that the mapping $x \longrightarrow xof$ is weakly continuous if and only if the mapping $x \longrightarrow f.x$ is weakly continuous and we establish that the mapping $x \longrightarrow f.x$ is weakly compact for all $f \in X^*$ iff X is Arens regular. Also, let X be wcc and wsc and includes a bai. Then, X is reflexive whenever the mapping $x \longrightarrow xof$ is weakly compact.

2 Arens products and the topological centers of X^{**}

Definition 1. An element E of X^{**} is said to be mixed unit if E is a right unit for the first Arens product and a left unit for the second Arens product. That is, E is a mixed unit if and only if, for each $F \in X^{**}$, $F.E = EoF = F$. We say that X^{**} is unital with respect to first Arens product , if there exists an element $E \in X^{**}$ such that $F.E = E.F = F$ for all F , and is unital with respect to second Arens product , if there exists an element $E \in X^{**}$ such that $FoE = EoF = F$ for all $F \in X^{**}$. We say that the Banach algebra X is unital if there exists an element as $e \in X$ such that $e.x.e = x$ for each $x \in X$.

Theorem 2. An element E of X^{**} is said to be mixed unit if and only if it is a *weak** cluster point of some bai $(e_\alpha)_{\alpha\in I}$ in X.

Proof: see [3, p.146].

Theorem3. Let X be a commutative Banach algebra. Then the mapping $x \longrightarrow f.x$ is weakly continuous iff the mapping $x \longrightarrow xof$ is weakly continuous. In other hand, the mapping $x \longrightarrow f.x$ is weakly compact iff $x_\alpha \longrightarrow F$ implies $f.x_\alpha \longrightarrow foF$ for each $F \in X^{**}$.

Proof. Let the mapping $x \longrightarrow f.x$ be weakly continuous and a net $(x_\alpha)_{\alpha\in I}$ be weakly convergent to x. Then, we have $f.x_\alpha \longrightarrow f.x$. For each $y \in X$, we have the following statements

$$lim_\alpha x_\alpha of = lim_\alpha f(yx_\alpha) = lim_\alpha f(x_\alpha y) = lim_\alpha f.x_\alpha(y)$$

$$= f.x(y) = f(xy) = f(yx) = xof(y).$$

Therefore, we see that $x_\alpha of \longrightarrow xof$ and so the mapping $x \longrightarrow xof$ is weakly continuous. The proof of the converse is the same. Now, since the mapping $x \longrightarrow f.x$ is weakly compact, there exists a net $(x_\alpha)_{\alpha\in I}$ such that the net$(f.x_\alpha)_{\alpha\in I}$ is weakly convergent in X^*. Therefore

$$lim_\alpha G(f.x_\alpha) = lim_\alpha G.f(x_\alpha) = lim_\alpha x_\alpha(lim_\beta y_\beta.f)lim_\alpha lim_\beta x_\alpha y_\beta of(x_\alpha)$$

$$= lim_\alpha(f.x_\alpha)oG = lim_\alpha x_\alpha oG(f) = FoG(f) = G(foF).$$

Hence $f.x_\alpha \longrightarrow foF$ and the proof is complete.

Theorem4. Let X be a Banach algebra. Then, the mapping $x \longrightarrow f.x$ is weakly compact if and only if X is Arens regular.

Proof. Suppose that the mapping $x \longrightarrow f.x$ is weakly compact and F, G be two arbitrary elements in X^{**} , then by the Goldstins, theorem [4, p. 424, Theorem 5] there exist bounded nets $(x_\alpha)_{\alpha\in I}$ and $(y_\beta)_{\beta\in J}$ in X and Y , respectively, such that $x_\alpha \longrightarrow F$ and $y_\beta \longrightarrow G$. Since the mapping $x \longrightarrow f.x$ is weakly compact, by theorem3 we conclude that $f.x_\alpha \longrightarrow foF$. Therefore, we have

$$F.G(f) = F(G.f) = lim_\alpha x_\alpha(G.f) = x_\alpha(G.f)$$

$$lim_\alpha G.f(x_\alpha) = lim_\alpha G(f.x_\alpha) = G(foF) = FoG(f).$$

Consequently, the above relations imply that X is Arens regular.
Conversely, let X be Arens regular then for each $F, G \in X^{**}$, we have $F.G = FoG$. Therefore, for each $f \in X^*$, we have $(foF)(G) = G(foF) = FoG(f) = F.G(f) = F(G.f)$. Consequently, by theorem 2 from [4, p.482] we see that the mapping $x \longrightarrow f.x$ is weakly compact.

Corollary 5. Let X be wsc and wcc and including a bai. Then, the space X is reflexive whenever the mapping $x \longrightarrow f.x$ is weakly compact.

Proof. By theorem 4, we know that X is Arens regular. Since X is wcc and wsc, by theorem (3,4) from [17] X is reflexive.

Definition6. If T maps X into Y , the null space and the range of T will be denoted by $N(T)$ and R(T) , respectively. That is $N(T) = \{x \in X : Tx = 0\}$ and $R(T) = \{y \in Y : Tx = y \ for\ some\ x \in X\}$

Definition 7. The spectrum $\sigma(T)$ of a linear operator T is the set of all scalars λ such that $T - \lambda I$ is not invertible.

Corollary 8. Let $f \in X^*$ be arbitrary and $T_f(x) = f.x$ be a mapping from X into X^* where $x \in X$. Then we have the following statements

i) If $dimR(T_f) < \infty$ for each $f \in X^*$, then X is Arens regular.
ii) If X is Arens regular and $dimR(T_f)$is weakly closed in X for each $f \in X^*$, then $dimR(T_f) < \infty$.
iii) If X is Arens regular and $\lambda \neq 0$, then $N(T_f - \lambda I) < \infty$ for all $f \in X^*$.
iv) If $dimX = \infty$ and X is Arense regular, then $0 \in \sigma(T)$ for all $f \in X^*$.

Proof. By using Theorem 4, mapping $x \longrightarrow f.x$ is weakly compact if and only if X is Arens regular, and by Theorem (4.18) from [14], the desired results are obtained.

Definition 9. We recall that the topological center of X^{**} can defined to be the set of functional $F \in X^{**}$ which satisfy $F.G = FoG$ for all $G \in X^{**}$. In other words, the topological centers of X^{**} with respect to first and second Arense products are defined as follow:

$$Z_1 = \{F \in X^{**} : \ F.G = FoG \quad \forall G \in X^{**}\},$$

$$Z_2 = \{F \in X^{**} : \ G.F = GoF \quad \forall G \in X^{**}\}$$

Theorem 10. Let X be a Banach algebra and unital. Then the second dual of X , X^{**} is mixed unit.

Proof. Let e be an unit element of X. We show that $\hat{e}$ is a unit element of X^{**} with respect to the first Arens product. By using a property of first Arens product we have the following relations

$$f.e(x) = f(e.x) = f(x),$$

$$\hat{e}.f(x) = \hat{e}(f.x) = f.x(e) = f(x),$$

$$F.\hat{e}(f) = F(\hat{e}.f) = F(f.e) = F(f)$$

Consequently, we conclude $F.\hat{e} = F$. By using

$$\hat{f}(\hat{e}.F) = \hat{e}.F(f) = \hat{e}(F.f) = F.f(e) = F(f.e) = F(f) = \hat{f}(F)$$

we see that $\hat{e}.F = F$ and therefore X^{**} is unital with respect to first Arens product. It is similar that $F = \hat{e}oF$. Then we have $\hat{e}.F = \hat{e}oF = F$. Consequently, the element $\hat{e}$ is a mixed unit in X^{**} which implies that X^{**} is mixed unite.

Corollary 11. Let X be Banach algebra and unital. Then we have $X^*X = XX^* = X$.

Proof. By theorem10, we know that $(X^{**}, .)$ and (X^{**}, o) are unital with respect to the first and second Arens products, respectively. Then the Theorem (2.2) from [17] shows that $XX^* = X^*X = X^*$.

Theorem 12. Let X be a Banach algebra with a bai and $XX^* = X^*$. If $Z_1 = Z_2$, then the following statements hold
i)$(X^{**}, o)$is unital,
ii) $X^*X = X^*$.

Proof. i) Since X has a bai then by theorem 2, X^{**} has a mixed units such as E such that $EoF = F.E = F$ for each $F \in X^{**}$. With notice to $XX^* = X^*$, for any $f \in X^*$, there exist $g \in X^*$ and $x \in X$ such that $xg = f$. On the other hand, the equality $Z_1 = Z_2$ implies that $FoE = F.E$. Therefore, we have .

$$FoE(f) = F.E(f) = F.E(x.g) = F(Ex.g) = F(xg) = F(f).$$

Consequently, the above relation shows that (X^{**}, o) is unital.

ii) Let E be a unit element of X^{**} and let $(e_\alpha)_\alpha \in I$ be a bai in X such that converges to E. Then, for any f of X^* and $F \in X^{**}$ we conclude

$$lim_\alpha F(f.e_\alpha) = lim_\alpha e_\alpha oF(f) = EoF(f) = F(f).$$

So the above relation shows that $f.e_\alpha \longrightarrow f$. Since $f.e_\alpha \in X^*X$ and X^*X is closed subspace of X^* (see Theorem [8,32.33]), we see that $f \in X^*X$ which shows that $X^*X = X^*$.

Theorem 13. Let X be a Banach space and including a bai. Then
i)$XZ_1 = Z_1$
ii) If X is Arens regular then $X^*X = X^*$, $XX^* = X^*$.

Proof. i) Let $F \in Z_1$. We show that $xF \in Z_1$ whenever $x \in X$. Let $(G_\alpha)_{\alpha \in I}$ be an arbitrary net of X^{**} such that converges to some point G. Then for each $f \in X^*$ we have $(xF.G_\alpha)(f) = F.G_\alpha)(f.x)$. Since $F \in Z_1$, we conclude

$$(F.G_\alpha)(f.x) \xrightarrow{w} (F.G)(f.x).$$

Consequently $(xF.G_\alpha)(f) \xrightarrow{w} (F.G)(f.x)$ which implies that $x.F \in Z_1$. So we have $XZ_1 \subseteq Z_1$. On the other hand, since X is included a bai , $Z_1 \subseteq XZ_1$. Therefore, we have $XZ_1 = Z_1$.

ii) Since X is included a bai, by theorem (2,3) there exists an unit element e of X^{**} such that $F.e = F$, $eoF = F$. Since X is Arens regular,$e.F = eoF = F$. Now, let $(e_\alpha)_{\alpha \in I}$ be a bai such that weakly convergent to e. So, we have $(f.e_\alpha)(F) = f(e_\alpha.F)$. Since $e_\alpha.F \xrightarrow{w} e.F$, we have $(f.e)(F) = f(e.F) = f(F)$. With notice to least conclusion shows that $f.e = f$ and consequently we have$f.e_\alpha \xrightarrow{w} f$. We Know that $f.e_\alpha \in X^*X$ and by the theorem (32, 22) from [8] the space X^*X is a weakly closed subspace of X^* which implies that

$f \in X^*$. Therefore, $X^* \subseteq X^*X$. Hence we have$X^* = X^*X$ and proof is complete.

Theorem 14. Let X be a Banach space and wcc , wsc and including a bai as $(e_\alpha)_{\alpha \in I}$. If the mappings $x \longrightarrow f.x$ or $x \longrightarrow xof$ are weakly compact, then we have the following conclusions
i) X is Arens regular,
ii) X is reflexive

Proof. Let $F, G \in X^{**}$. Then there exist the nets $(x_\alpha)_{\alpha \in I}$ and $(s_\beta)_{\beta \in I}$ such that $\hat{x}_\alpha \xrightarrow{w} F$ and $\hat{y}_\beta \xrightarrow{w} G$. Let the mapping $x \longrightarrow f.x$ be weakly compact. Consequently, the net $(f.x_\alpha)_{\alpha \in I}$ converges to some point of X^*. In other words, there exists $h \in X^*$ such that $f.x_\alpha \xrightarrow{w} h$ and we have

$$lim_\alpha G(f.x_\alpha) = G(h) = lim_\beta \hat{y}_\beta(h) = lim_\alpha \hat{y_\beta} lim_\beta (f.x_\alpha) = lim_\alpha lim_\beta f(x_\alpha.y_\beta)$$

$$= lim_\alpha lim_\beta (y_\beta o f)(x_\alpha) = lim_\alpha lim_\beta \hat{x_\alpha}(y_\beta o f) = lim_\beta F(y_\beta o f) = lim_\beta (foF)(y_\beta)$$

$$= lim_\beta \hat{y_\beta}(foF) = G(foF).$$

Therefore the net $(f.x_\alpha)_\alpha$ converges to foF Therefore

$$(FoG)(f) = G(foF) = lim_\alpha G(f.x_\alpha) = lim_\alpha (G.f)(x_\alpha) = lim_\alpha \hat{x_\alpha}(G.F)$$

$$= F(G.F) = F(G.f) = F.G(f)$$

Therefore, $FoG = F.G$,and so X is Arens regular.

ii) By conclusion (i), we have that X is Arens regular. Then, by Corollary (2,8) from [17], X is wcc and we conclude that X is reflexive.

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