On the Radial Solutions of a Degenerate Elliptic Equation with Convection Term

Arij Bouzelmate and Abdelilah Gmira

Département de Mathématiques et Informatique
Faculté des Sciences, B-P-2121 Téouan, Maroc
gmira@fst.ac.ma

Guillermo Reyes

Departamento de Matemáticas.Universidad Carlos III de Madrid, Leganés, Madrid 28911, Spain
greyes@math.uc3m.es

Abstract

In this paper, we study existence and uniqueness of radial solutions for the degenerate elliptic equation

$$\Delta_p U + \alpha x \cdot \nabla U + \beta x \cdot \nabla(|U|^{q-1}U) + U = 0 \quad \text{in } \mathbb{R}^N$$

where $p > q+1 > 2$, $N \geq 1$, $\alpha \in \mathbb{R}$, $\beta \leq 0$. We give also a classification of solutions and the behaviour of those which are positive.

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1 Introduction

In this paper, we are concerned with the following degenerate elliptic equation

$$\Delta_p U + \alpha x \cdot \nabla U + \beta x \cdot \nabla(|U|^{q-1}U) + U = 0 \quad \text{in } \mathbb{R}^N$$

where $p > 2$, $q > 1$, $\alpha \in \mathbb{R}$ and $\beta \leq 0$. More precisely, the main motivation of this work is to continue the study of (1.1); introduced by authors [4] in the study of self-similar solutions of the following parabolic equation

$$u_t = \Delta_p u + x \cdot \nabla(|u|^{q-1}u) \quad \text{in } \mathbb{R}^N \times (0, +\infty)$$
which may be viewed as a non-linear Ornstein-Uhlenbeck equation. The corresponding linear equation appears in [8] and [5]. It is worth mentioning that if the convection term in the parabolic equation (1.2) is replaced by $|u|^{q-1}u$, we get the p-Laplace heat equation with a source term. This equation was studied by [6],[10],[3] and [1]. As, if the convection term is replaced by $|\nabla u|^q$, we obtain the generalised KPZ equation, whose self-similar solutions were studied by [12] for $p = 2$ and by [7] for $p > 2$. Equation (1.2) has a special scaling invariance in the sense that $u$ is a solution if and only if $u_\lambda$ defined by 

$$u_\lambda(x, t) = \lambda^\gamma u(\lambda^\sigma x, \lambda t)$$

is a solution for all $\lambda > 0$. A solution $u$ is said to be self-similar if $u_\lambda = u$, for all $\lambda > 0$.

It can be easily seen that $u$ is a self-similar solution to (1.2) if and only if $u$ has the form 

$$u(x, t) = t^{-\gamma} U(c x t^{-\sigma}), \quad (1.3)$$

defined for $x \in \mathbb{R}^N$ and $t > 0$; with

$$\gamma = \frac{1}{q-1}, \quad \sigma = \frac{q + 1 - p}{p(q-1)}, \quad c = (q-1)^{-1/p} \quad (1.4)$$

and $U$ satisfies equation (1.1) with

$$\alpha = \frac{q + 1 - p}{p}, \quad \beta = (q - 1).$$

Looking for radial solutions $U(x) = u(|x|)$, where $u : \mathbb{R}^+ \rightarrow \mathbb{R}$ it is easy to see that the elliptic equation (1.1) is reduced to the following O.D.E

$$(|u'|^{p-2} u')' + \frac{N-1}{r} |u'|^{p-2} u' + \alpha r u' + \beta r (|u|^{q-1})' + u = 0. \quad (1.5)$$

If $\beta = 0$, equation (1.5) becomes

$$(|u'|^{p-2} u')' + \frac{N-1}{r} |u'|^{p-2} u' + \alpha r u' + u = 0. \quad (1.6)$$

By a simple scaling, this last equation can be written in the following form

$$(|u'|^{p-2} u')' + \frac{N-1}{r} |u'|^{p-2} u' + \epsilon (r u' + \gamma u) = 0. \quad (1.7)$$

with $\epsilon = \pm 1$ and $\gamma \in \mathbb{R}$. Let us mention that equation (1.7) have been considered by [11] for $1 < p < 2$, $\gamma > 0$ and $\epsilon = 1$. Recently a complete study was presented in [2] for $1 < p < 2$, $\epsilon = \pm 1$ and $\gamma \in \mathbb{R}$. Note also that for $p > 2$ a carefully analysis of radial solutions of equation (1.1) was made by the authors in [4] when $\beta > 0$. The main purpose of this paper is to continue the
study of the case $\beta \leq 0$. More precisely we are concerned with the following initial-value problem

$$\begin{cases}
\frac{(|u'|^{p-2}u')'}{r} + \frac{N-1}{r}|u'|^{p-2}u' + \alpha u' + \beta r(|u|^{q-1}u)' + u = 0 \text{ in } [0, +\infty[, \\
u(0) = A, \quad u'(0) = 0,
\end{cases}$$

(E1) (E2)

where $p > q + 1 > 2$, $N \geq 1$, $\alpha \in \mathbb{R}$, $\beta \leq 0$, $A \neq 0$. We will mainly discuss: Existence and uniqueness of solutions of (P) as well as their classification. Also, qualitative behavior of positives solutions are presented. This paper will be divided into three sections. In section 2 we prove existence and uniqueness of solutions of problem (P). First we recall some known result about local existence obtained by the authors [4]; this is derived using the Banach Fixed Point Theorem. Secondly for global existence we write equation (E1) as a first order system in three space dimensions for which we establish a suitable tubular surfaces used as a barrier for the solutions in this space. Section 3 concerns the asymptotic behaviour of positive solutions as $r \to \infty$, where the sign of some explicit functional depending of the solution and its derivative plays an important part. Finally section 4 examines the classification of solutions according to their behaviour at infinity depending on the parameters $\alpha$ and $\beta$.

2 Existence and uniqueness

Unless otherwise specified, we assume throughout that

$$p > q + 1 > 2, \quad N \geq 1, \quad \alpha \in \mathbb{R}, \quad \beta \leq 0.$$ 

In this section, we investigate existence and uniqueness of solutions of problem (P). First of all note that equation (E1) is invariant under the change of unknown $u \to v = -u$, i.e., if $u$ solves (P) with $u(0) = A$, then $v = -u$ solves the same problem with $v(0) = -A$. Then we don’t lose of generality we can restrict ourselves to the case $A > 0$. By a solution of (P) we mean a function $u$ defined in $[0, +\infty]$ such that $|u'|^{p-2}u' \in C^1([0, +\infty])$ and satisfies (E1) and (E2).

**Theorem 2.1** *Problem (P) has a unique solution $u(\cdot, A, \alpha, \beta)$.*

**Proof:** First of all we note that local existence and uniqueness can be established exactly in the same way as given in [4] for $\beta > 0$. In fact we have just to convert the initial value problem (P) to a fixed problem of some operator. To this end we remark that solving (P) is equivalent to find a function $u \in [0, +\infty]$ such that $|u'|^{p-2}u' \in C^1([0, +\infty])$ and satisfies

$$\mathcal{T}[u](r) = A - \int_0^r G(F[u](s))ds$$
where
\[ G(s) = |s|^{(2-p)/(p-1)}s, \quad s \in \mathbb{R}. \]

and
\[
F[u](s) = \alpha su(s) + \beta s |u|^{q-1} u(s) + s^{1-N} \int_0^s \sigma^{N-1} \left[-\beta N |u|^{q-1}(\sigma) + (1-N\alpha)\right] u(\sigma) d\sigma.
\]

Now take \( R > 0, A, M > 0 \) and consider the following complete metric space:
\[ X = \{ \varphi \in C([0, R]) : \| \varphi - A \|_0 \leq M \}. \]

where \( C([0, R]) \) is the Banach space of real continuous functions on \([0, R]\) with the uniform norm, denoted by \( \| \cdot \|_0 \). In order to conclude we prove that \( T \) is a contraction mapping from \( X \) into itself. The Banach Fixed Point Theorem then establishes the existence of a fixed point i.e a solution to (P). Note that it easy to prove that \((|u'|^{p-2}u')'(0) = -A/N. It remains to prove global existence. So we must to extend the local solutions to the whole \( \mathbb{R}_+ \). Recall that global existence is derived in [4] from the energy function for the case \( \beta > 0 \) and \( \alpha \geq 0 \). But this method is not valid here. So, we transform the problem (P) to the following first order autonomous system in the space \((X, Y, r)\)
\[
\begin{cases}
X' = |Y|^{-\frac{p-2}{p-1}} Y, \\
Y' = -\frac{N-1}{r} Y - \alpha r |Y|^{-\frac{p-2}{p-1}} Y - \beta qr |X|^{q-1} |Y|^{-\frac{p-2}{p-1}} Y - X, \\
r' = 1
\end{cases}
\]  

where \( X = u, \ Y = |u'|^{p-2}u', \ \prime := d/dt \) and \((X(0), Y(0), r(0)) = (A, 0, 0)\). The idea of the proof is that of constructing a suitable tubular surface which serves as a barrier for the solutions in this space. Below, \( r_A > 0 \) is such that the local solution exists on \([0, r_A]\). We claim that there exist \( \sigma, \gamma > 0 \) and sufficiently large \( B, C > 0 \) such that
\[
|X(r)| \leq B(1 + r^\sigma), \quad |Y(r)| \leq C(1 + r^\gamma)
\]  

for \( r \geq r_A \). This clearly implies our assertion. Indeed, consider the boundary \( S \) of the region defined by (2.2) and \( r \geq r_A \). Our goal is to prove that the flux vector
\[
F(X, Y, r) := \left( |Y|^{-\frac{p-2}{p-1}} Y, -\frac{N-1}{r} Y - \alpha r |Y|^{-\frac{p-2}{p-1}} Y - \beta qr |X|^{q-1} |Y|^{-\frac{p-2}{p-1}} Y - X, 1 \right)
\]  

(2.3)
in (2.1) points inward this region. Denote
\begin{align*}
S_1 &:= \{(X, Y, r) : X(r) = B(1 + r^{\sigma}), |Y(r)| \leq C(1 + r^\gamma); r \geq r_A\}; \\
S_2 &:= \{(X, Y, r) : X(r) = -B(1 + r^{\sigma}), |Y(r)| \leq C(1 + r^\gamma); r \geq r_A\}; \\
S_3 &:= \{(X, Y, r) : |X(r)| \leq B(1 + r^{\sigma}), Y(r) = C(1 + r^\gamma); r \geq r_A\}; \\
S_4 &:= \{(X, Y, r) : |X(r)| \leq B(1 + r^{\sigma}), Y(r) = -C(1 + r^\gamma); r \geq r_A\}; \\
T &:= \{(X, Y, r) : |X(r)| \leq B(1 + r^{\sigma}), |Y(r)| \leq C(1 + r^\gamma), r = r_A\}.
\end{align*}
Clearly, \( S = (\cup S_i) \cup T \). Let \( N_i \) denote exterior (with respect to the region) normal vectors to \( S_i \), \( i = 1, 2, 3, 4 \), and \( N \) an exterior normal vector to \( T \). An elementary calculation shows that we can take
\begin{align*}
N_1 &= (1, 0, -\sigma Br^{\sigma - 1}), & N_2 &= (-1, 0, -\sigma Br^{\sigma - 1}); \\
N_3 &= (0, 1, -\gamma Cr^{\gamma - 1}), & N_4 &= (0, -1, -\gamma Cr^{\gamma - 1}); \\
N &= (0, 0, -1).
\end{align*}
First, choose \( B, C \) large, such that \( (X(r_A), Y(r_A), r_A) \in T \). Next, we impose
\begin{equation}
N_i \cdot F < 0 \quad \text{on } S_i; \quad N \cdot F < 0 \quad \text{on } T. \tag{2.6}
\end{equation}
The last inequality in (2.6) trivially holds. For \( i = 1, 2 \), (2.6) is implied by
\begin{equation}
C^{1/(p-1)}(1 + r^\gamma)^{1/(p-1)} - \sigma Br^{\sigma - 1} < 0 \quad \text{for } r \geq r_A. \tag{2.7}
\end{equation}
Let
\begin{equation}
G_1(r) = r^{1-\sigma}(1 + r^\gamma)^{1/(p-1)}. \tag{2.8}
\end{equation}
Then, inequality (2.7) is equivalent to
\begin{equation*}
G_1(r) < \frac{\sigma B}{C^{1/(p-1)}} \quad \text{for } r \geq r_A.
\end{equation*}
It’s easy to see that \( G_1 \) is strictly decreasing if
\begin{equation}
\sigma > 1 + \gamma/(p - 1). \tag{2.9}
\end{equation}
Hence, if we choose \( B \) and \( C \) such that
\begin{equation}
G_1(r_A) < \frac{\sigma B}{C^{1/(p-1)}}, \tag{2.10}
\end{equation}
then, condition (2.7) follows easily by (2.9) and (2.10). Concerning \( i = 3, 4, \) (2.6) holds if
\begin{align*}
&-C\frac{N - 1}{r}(1 + r^\gamma) + |\alpha|C^{1/(p-1)} r(1 + r^\gamma)^{1/(p-1)} + \\
&+|\beta|qC^{1/(p-1)} B^{\gamma - 1} r(1 + r^\gamma)^{1/(p-1)}(1 + r^\sigma)^{\sigma - 1} + B(1 + r^\sigma) - \gamma Cr^{\gamma - 1} < 0
\end{align*}
for $r \geq r_A$, which in turn holds if
\begin{align*}
-\gamma - (N-1)r^{-\gamma}(1 + r^{\gamma}) + |\alpha|C^{(2-p)/(p-1)}r^{2-\gamma}(1 + r^{\gamma})^{1/(p-1)} + \\
+|\beta|qC^{(2-p)/(p-1)}B^{q-1}r^{2-\gamma}(1 + r^{\gamma})^{1/(p-1)}(1 + r^{\sigma})^{q-1} + \frac{B}{C}r^{1-\gamma}(1 + r^{\sigma}) < 0
\end{align*}
(2.11)

Let
\begin{align*}
G_2(r) &= r^{2-\gamma}(1 + r^{\gamma})^{1/(p-1)}, \\
G_3(r) &= r^{1-\gamma}(1 + r^{\sigma}),
\end{align*}
(2.12)
(2.13)
and
\begin{align*}
G_4(r) &= r^{2-\gamma}(1 + r^{\gamma})^{1/(p-1)}(1 + r^{\sigma})^{q-1}.
\end{align*}
(2.14)

The idea is to give sufficient conditions to have for $r \geq r_A$
\begin{align*}
|\alpha|C^{(2-p)/(p-1)}G_2(r) < \frac{\gamma}{3},
\end{align*}
\begin{align*}
\frac{B}{C}G_3(r) < \frac{\gamma}{3},
\end{align*}
and
\begin{align*}
|\beta|qC^{(2-p)/(p-1)}B^{q-1}G_4(r) < \frac{\gamma}{3}
\end{align*}

In the same way as the first case concerning $i = 1, 2$, we look for conditions which mean that functions $G_2(r)$, $G_3(r)$ and $G_4(r)$ are strictly decreasing for $r > 0$. By a simple calculation, this holds if
\begin{align*}
\gamma > \max\{\sigma + 1, \frac{2(p-1)}{p-2} + \frac{\sigma(q-1)(p-1)}{p-2}\}.
\end{align*}
(2.15)

Hence, inequality (2.11) is implied by choosing $B$ and $C$ such that
\begin{align*}
C^{(p-2)/(p-1)} > \frac{3|\alpha|}{\gamma}G_2(r_A)
\end{align*}
(2.16)
\begin{align*}
\frac{C}{B} > \frac{3}{\gamma}G_3(r_A)
\end{align*}
(2.17)
and
\begin{align*}
\frac{C^{(p-2)/(p-1)}}{B^{q-1}} > \frac{3q|\beta|}{\gamma}G_4(r_A)
\end{align*}
(2.18)

In order to accomplish with (2.9) and (2.15), observe that the conditions on $\gamma$ and $\sigma$ are compatible, since taking $\gamma > 2(p-1)/(p-2)$, one can choose $\sigma \in (1 + \gamma/(p-1), \gamma - 1)$. Then, the condition (2.15) is achieved since $q < p - 1$ by choosing
\begin{align*}
\gamma > \frac{(p-1)(q+1)}{p-q-1}.
\end{align*}
Concerning the parameters $B, C$, to have conditions (2.10), (2.16), (2.17) and (2.18), we fix the relation

$$C^{1/(p-1)}/B = K = K(r_A)$$

Then the quantities

$$\frac{C}{B} = K^{p-1}B^{p-2} \quad \frac{C(p-2)/(p-1)}{B^{q-1}} = K^{p-2}B^{p-q-1}$$

satisfy (2.17) and (2.18) by taking $B$ (and hence $C$) large enough. Note that here we have made use of the hypothesis $p - q - 1 > 0$ once again.

3 Behaviour at infinity

This section deals with some qualitative properties of solutions of problem (P).

**Theorem 3.1** Assume $\alpha \geq 0$, $\beta = 0$ and $N > 1$. Let $u$ be a solution of (P). Then,

$$\lim_{r \to +\infty} u(r) = \lim_{r \to +\infty} u'(r) = 0.$$ 

*Proof:* We define the energy function

$$E(r) = \frac{p-1}{p} |u'|^p(r) + \frac{1}{2} u^2(r). \quad (3.1)$$

According to equation $(E_1)$, the energy satisfies

$$E'(r) = -r u'^2 \left\{ \alpha + \frac{N-1}{r^2} |u'|^{p-2}(r) \right\}. \quad (3.2)$$

Hence, it is enough to show that $\lim_{r \to +\infty} E(r) = 0$.

Since $E'(r) \leq 0$ and $E(r) \geq 0$ for all $r > 0$, there exists a constant $l \geq 0$ such that $\lim_{r \to +\infty} E(r) = l$. Suppose $l > 0$. Then, there exists $r_1 > 0$ such that

$$E(r) \geq l/2 \quad \text{for} \quad r \geq r_1. \quad (3.3)$$

Now we introduce a function depending on the energy, for which we can control its derivative (the same idea was used by [3] and [9]). More precisely we set

$$D(r) = E(r) + \frac{N-1}{2r} |u'|^{p-2} u'(r) u(r) + \frac{\alpha(N-1)}{4} u^2(r) + \alpha \int_0^r s u'^2(s) ds.$$
Then,
\[ D'(r) = -\frac{(N-1)}{2r} \left[ |u'|^p + \frac{N}{r} |u'|^{p-2} u' u + u^2 \right]. \]

Recalling that \( u \) and \( u' \) are bounded,
\[ \lim_{r \to +\infty} \frac{|u'|^{p-2} u' u(r)}{r} = 0. \]

Moreover, by (3.3) we have
\[ |u'(r)|^p + u^2(r) = \frac{p-1}{p} |u'(r)|^p + \frac{u^2(r)}{2} = E(r) > l/2 \quad \text{for} \quad r \geq r_1. \]

Consequently, there exist two constants \( c > 0 \) and \( r_2 \geq r_1 \) such that
\[ D'(r) \leq -c/r \quad \text{for} \quad r \geq r_2. \]

Integrating this last inequality between \( r_2 \) and \( r \), we get
\[ D(r) \leq D(r_2) - c \ln(r/r_2) \quad \text{for} \quad r \geq r_2. \]

In particular we obtain \( \lim_{r \to +\infty} D(r) = -\infty. \) Since
\[ E(r) + \frac{N-1}{2r} |u'|^{p-2} u' u(r) \leq D(r), \]
we get \( \lim_{r \to +\infty} E(r) = -\infty. \) This is impossible, hence the conclusion.

**Theorem 3.2** Let \( u \) be a solution of (P). If \( L = \lim_{r \to +\infty} u(r) \) exists and is finite, then necessarily \( L = 0. \) Moreover, if \( \alpha \neq 0 \) then \( \lim_{r \to +\infty} u'(r) = 0. \)

**Proof:** The proof will be done in three steps.

**STEP 1.** \( L = 0. \) Assume by contradiction \( L \neq 0. \) Equation \((E_1)\) can be written in the following form:
\[ r^{1-N} \left[ r^{N-1} |u'|^{p-2} u' + \alpha r^N u + \beta r^N |u|^{q-1} u \right]' = (\alpha N - 1) u + \beta N |u|^{q-1} u. \]

Then
\[ \lim_{r \to +\infty} r^{1-N} \left[ r^{N-1} |u'|^{p-2} u' + \alpha r^N u + \beta r^N |u|^{q-1} u \right]' = (\alpha N - 1) L + \beta N |L|^{q-1} L. \]

Let \( K = (\alpha N - 1) L + \beta N |L|^{q-1} L. \) We assert that \( |u'|^{p-2} u' \sim -\frac{L}{N} r \) as \( r \to +\infty. \) In fact, if \( K \neq 0, \) the following holds for large \( r: \)
\[ r^{N-1} |u'|^{p-2} u' + \alpha r^N u + \beta r^N |u|^{q-1} u \sim \frac{K}{N} r^N. \]
Thus, in both cases we have
\[ |u'|^{p-2} u' + \alpha u + \beta |u|^{q-1} u \sim (\alpha - 1/N)L + \beta |L|^{q-1} L, \]
hence
\[ |u'|^{p-2} u' \sim -\frac{L}{N} r \quad \text{as} \quad r \to +\infty. \]  \hspace{1cm} (3.5)

On the other hand, if \( K = 0 \) (which means \( (\alpha N - 1) + \beta N |L|^{q-1} = 0 \)) then for any \( \varepsilon > 0 \), there exists a large \( r_\varepsilon \) such that for any \( r > r_\varepsilon \) the following estimate holds
\[ -\varepsilon \frac{r^N}{N} \leq r^{N-1} |u'|^{p-2} u' + \alpha r^N u + \beta r^N |u|^{q-1} u \leq \varepsilon \frac{r^N}{N}. \]

Then
\[ \frac{|u'|^{p-2} u'}{r} \sim -L (\alpha + \beta |L|^{q-1}) = -\frac{L}{N}. \]

Thus, in both cases we have \( u' \to -\infty \) as \( r \to +\infty \), and this contradicts the convergence of \( u \). Consequently \( L = 0 \).

**Step 2.** \( \lim_{r \to +\infty} |u'|^{p-2} u'(r)/r = 0. \) From Step 1 and equation (3.4) it follows
\[ \lim_{r \to +\infty} r^{1-N} f'(r) = 0, \]  \hspace{1cm} (3.6)

where
\[ f(r) = r^{N-1} |u'|^{p-2} u'(r) + \alpha r^N u + \beta r^N |u|^{q-1} u(r). \]  \hspace{1cm} (3.7)

L'Hopital's rule then implies \( \lim_{r \to +\infty} f(r)/r^N = 0. \) This means
\[ \lim_{r \to +\infty} \frac{|u'|^{p-2} u'(r)}{r} + \alpha \lim_{r \to +\infty} u(r) + \beta \lim_{r \to +\infty} |u|^{q-1} u(r) = 0, \]
from where the assertion readily follows.

**Step 3.** \( \lim_{r \to +\infty} u'(r) = 0. \) First, we assert that \( u' \) is bounded. Indeed, assume this is false. As \( \lim_{r \to +\infty} u(r) = 0 \), then there exists a sequence \( \eta_i \to +\infty \) such that \( \{\eta_i\} \) are local maxima of \( |u'|^{p-2} u' \) and \( \lim_{i \to +\infty} |u'|^{p-2} u'(\eta_i) = \infty \). Hence \( (|u'|^{p-2} u')(\eta_i) = 0 \). Dividing equation \( (E_1) \) by \( ru'(r) \), taking \( r = \eta_i \) and recalling Step 1 and Step 2, we get \( \alpha = 0 \) in the limit \( i \to \infty \), contradicting our hypotheses. Then necessarily \( u' \) is bounded. Next, we claim that \( |u'|^{p-2} u'(r) \to 0 \) as \( r \to \infty \). Note that as \( u(r) \) converges as \( r \to \infty \), it suffices to prove that \( |u'|^{p-2} u'(r) \) converges. We argue by contradiction. Suppose that
\[ -\infty < m = \lim_{r \to +\infty} \inf r |u'|^{p-2} u'(r) < \lim_{r \to +\infty} \sup r |u'|^{p-2} u'(r) = M < \infty. \]  \hspace{1cm} (3.8)
Then there exist two sequences $\eta_i \to +\infty$ and $\zeta_i \to +\infty$ such that $\eta_i$ and $\zeta_i$ are local minima and local maxima of $|u'|^{p-2}u'$, respectively, satisfying $\eta_i < \zeta_i < \eta_{i+1}$, $i = 1, 2, \ldots$ and $\lim_{i \to \infty} |u'|^{p-2}u'(\eta_i) = m$, $\lim_{i \to \infty} |u'|^{p-2}u'(\zeta_i) = M$.

Hence $(|u'|^{p-2}u')'(\eta_i) = (|u'|^{p-2}u')'(\eta_i) = 0$. Dividing equation $(E_1)$ by $r$, taking $r = \eta_i$ and $r = \zeta_i$ and letting $i \to \infty$, we obtain $\alpha \lim_{r \to +\infty} u'(\eta_i) = \alpha \lim_{r \to +\infty} u'(\zeta_i) = 0$. This contradicts (3.8), thus completing the proof.

**Theorem 3.3** The strictly positives solutions are strictly decreasing.

**Proof:** We argue by contradiction. Let $r_0 > 0$ be the first zero of $u'$. Then, it follows from $(E_1)$ that $(|u'|^{p-2}u')'(r_0) = -u(r_0) < 0$. On the other hand, we know that $u' < 0$ for $r \sim 0$. By continuity and the definition of $r_0$, there exists a left neighborhood $]r_0 - \varepsilon, r_0[ $ (for some $\varepsilon > 0$) where $u'$ is strictly increasing and strictly negative, that is $(|u'|^{p-2}u')'(r) > 0$ for any $r \in ]r_0 - \varepsilon, r_0[ $; hence by letting $r \to r_0$ we get $(|u'|^{p-2}u')'(r_0) \geq 0$, a contradiction.

**Remark 3.4** As a consequence of the above theorem, any strictly positive solution $u$ of problem $(P)$ satisfies $\lim_{r \to +\infty} u(r) = 0$ and $\lim_{r \to +\infty} u'(r) = 0$ if $\alpha \neq 0$.

Now for any $c > 0$, define the function

$$E_c(r) = cu(r) + ru'(r), \quad r > 0.$$  \hfill (3.9)

Hence, using $(E_1)$, we have for any $r > 0$ such that $u'(r) \neq 0$,

$$(p - 1)|u'|^{p-2}(r)E'_c(r) = (p - 1)(c - \frac{N - p}{p - 1})|u'|^{p-2}u'(r) - \alpha r^2 u'(r) - \beta qr^2 |u|^{q-1}u'(r) - ru. \hfill (3.10)$$

Consequently, if $E_c(r_0) = 0$ for some $r_0 > 0$, equation $(E_1)$ gives

$$(p - 1)|u'|^{p-2}(r_0)E'_c(r_0) = r_0 u(r_0) \left[ \alpha c - 1 + c\beta q |u|^{q-1}(r_0) + c^{p-1}(p - 1)\left(\frac{N - p}{p - 1} - c\right)\frac{|u'|^{p-2}(r_0)}{r_0^{p-1}} \right], \hfill (3.11)$$

from which the sign of $E_c(r)$ for large $r$ can be obtained.

**Theorem 3.5** Let $u$ be a strictly positive solution of $(P)$ and $c > 0$. Then, for large $r$, $E_c(r)$ has a constant sign in the following cases.

(i) $\alpha \leq 0$ or $\alpha > 0$ and $c \neq \frac{1}{\alpha}$;
(ii) \( c = \frac{1}{\alpha} > 0 \) and \( \beta < 0 \) or \( c = \frac{1}{\alpha} \neq \frac{N - p}{p - 1} \) and \( \beta = 0 \).

**Proof:** Assume that there exists a large \( r_0 \) such that \( E_c(r_0) = 0 \).

Since \( \lim_{r \to +\infty} u(r) = 0 \) and according to (3.11), we have for \( \alpha c > 1 \) (respectively \( \alpha c < 1 \)), \( E'_c(r_0) > 0 \) (respectively \( E'_c(r_0) < 0 \)) and thereby \( E_c(r) \) has a constant sign for large \( r \); what gives exactly (i).

To prove (ii) we note that if \( c = \frac{1}{\alpha} \) equation (3.11) can be written in the following form

\[
(p - 1) |u'|^{p-2} (r_0) E'_c(r_0) = r_0 u^q (r_0) \left[ \frac{\beta q}{\alpha} + \left( \frac{1}{\alpha} \right)^{p-1} (p - 1) \left( \frac{N - p}{p - 1} - \frac{1}{\alpha} \right) \frac{u^{p-1-q}(r_0)}{r_0^p} \right].
\]

Therefore, if \( \beta < 0 \), using the fact that \( \lim_{r \to +\infty} u(r) = 0 \) and \( p - 1 - q > 0 \), the leading term in the last equality is \( \frac{\beta q}{\alpha} r_0 u^q (r_0) \).

On the other hand if \( \beta = 0 \) and \( \frac{1}{\alpha} \neq \frac{N - p}{p - 1} \), \( E'_c(r_0) \) has the same sign as

\[
(\frac{1}{\alpha})^{p-1} (p - 1) \left( \frac{N - p}{p - 1} - \frac{1}{\alpha} \right) r_0^{1-p} u^{p-1}(r_0).
\]

Consequently, \( E'_c(r) \) has a constant sign for large \( r \). This complete the proof.

**Theorem 3.6** Assume \( \alpha N > 1 \). Let \( u \) be a strictly positive solution of problem (P). Then, \( \lim_{r \to +\infty} r^{1/\alpha} u(r) \) exists and is strictly positive.

The proof of the theorem will be done in several lemmas

**Lemma 3.7** Assume \( \alpha N \neq 1 \) or \( \alpha N = 1 \) and \( \beta < 0 \). Let \( u \) be a strictly positive solution of problem (P). Then, the function \( f(r) \) is positive for large \( r \).

**Proof:** According to to (3.4) and (3.7) we have

\[
f'(r) = r^{N-1} u \left[ \alpha N - 1 + \beta N |u|^{q-1} \right]. \tag{3.12}
\]

If \( \alpha N \neq 1 \). Since \( \lim_{r \to +\infty} |u|^{q-1}(r) = \lim_{r \to +\infty} u^{q-1}(r) = 0 \) (because \( u > 0 \) and \( q > 1 \)), then we have for large \( r \), \( f'(r) > 0 \) for \( \alpha N > 1 \) or \( f'(r) < 0 \) for \( \alpha N < 1 \). If \( \alpha N = 1 \), then

\[
f'(r) = \beta N r^{N-1} u^q(r) < 0.
\]

Hence the function \( f(r) \) is monotone for large \( r \) and thereby she has a constant sign for large \( r \).
Suppose that there exists $r_1$ large such that $f(r) \leq 0$ for $r \geq r_1$. Then, using the fact that $u'(r) < 0$, we get

$$|u'|^{p-1} \geq ru \left[ \alpha + \beta u^{q-1} \right]$$

for $r \geq r_1$. This means since $\lim_{r \to +\infty} u(r) = 0$, that

$$|u'|^{p-1} \geq \frac{\alpha}{2} ru$$

for $r \geq r_1$. Integrating this last inequality on $(r_1, r)$, we get

$$\frac{r^{p-2}}{p} u^{p-1}(r) - \frac{r^{p-2}}{p} u^{p-1}(r_1) = \frac{p-2}{p} \left( \frac{\alpha}{2} \right) r^{p-1}$$

By letting $r$ to $+\infty$, we get a contradiction. Consequently, $f(r)$ is positive for large $r$.

**Lemma 3.8** Assume $\alpha N > 1$. Let $u$ be a strictly positive solution of problem $(P)$. Then

$$\lim_{r \to +\infty} r^N u(r) = +\infty.$$

**Proof:** Since $\alpha N > 1$, then $f'(r) > 0$ and $f(r) > 0$ for large $r$. Hence, there exists some constant $C > 0$ such that $f(r) > C$ for large $r$. Moreover, using the fact that $u'(r) < 0$ and $\beta \leq 0$, we obtain

$$f(r) < \alpha r^N u(r).$$

Then

$$r^N u(r) > C_1 = C/\alpha$$

for large $r$. On the other hand, we have by (3.12)

$$rf'(r) = r^N u \left[ \alpha N - 1 + \beta N |u|^{q-1} \right].$$

As $\alpha N > 1$ and $\lim_{r \to +\infty} u(r) = 0$, we get

$$rf'(r) > \frac{\alpha N - 1}{2} r^N u(r)$$

for large $r$, which gives

$$rf'(r) > \frac{\alpha N - 1}{2} C_1$$

for large $r$. Integrating this last inequality on $(r_1, r)$ for large $r_1$, we get

$$\lim_{r \to +\infty} f(r) = +\infty.\text{ This means that } \lim_{r \to +\infty} r^N u(r) = +\infty.$$
Lemma 3.9 Assume $\alpha > 0$. Let $u$ be a strictly positive solution of problem (P). Then, for any $0 < k < \frac{1}{\alpha}$, \[ \lim_{r \to +\infty} r^k u(r) \in [0, +\infty[. \]

Proof: Let $0 < k < \frac{1}{\alpha}$. We know by Theorem 3.5 that $E_k(r)$ has a constant sign for large $r$. Suppose that $E_k(r)$ is positive for large $r$. Then, by (3.9) and the fact that $u' < 0$, we get
\[ r |u'(r)| \leq ku(r) \]
for large $r$. On the other hand, using the fact that $\alpha > 0, \beta \leq 0$ and $u' < 0$, we have by $E_1$
\[ (|u'|^{p-2}u')'(r) \leq \frac{N - 1}{r} |u'|^{p-1}(r) + \alpha r |u'(r)| - u(r). \]
Hence
\[ (|u'|^{p-2}u')'(r) \leq (N - 1)k^{p-1} \frac{u^{p-1}(r)}{r^p} + \alpha ku(r) - u(r), \]
thus
\[ (|u'|^{p-2}u')'(r) \leq u(r) \left[ \alpha k - 1 + (N - 1)k^{p-1} \frac{u^{p-2}(r)}{r^p} \right]. \]
Using the fact that $u > 0$, $\alpha k - 1 < 0$ and $\lim_{r \to +\infty} u(r) = 0$, we obtain $(|u'|^{p-2}u')'(r) < 0$ for large $r$. As $u' < 0$, this implies that $\lim_{r \to +\infty} u'(r) \in [-\infty, 0]$, which is impossible. Hence, $E_k(r)$ is negative for large $r$. On the other hand, it easy to see that for any $c > 0$
\[ (r^c u(r))' = r^{c-1} E_c(r). \]
Then, the function $r^k u(r)$ is decreasing and thereby she has a finite limit. Now we turn to the proof of Theorem 3.6.

Proof: (of Theorem 3.6). Let $u$ be a strictly positive solution of problem (P). We consider the following function
\[ I(r) = r^{1/\alpha} \left[ u + \frac{1}{\alpha r^2} |u'|^{p-2} u' \right]. \]
By a simple computation we get
\[ I'(r) = -\frac{1}{\alpha} r^{1/\alpha} \left[ (N - 1/\alpha) \frac{|u'|^{p-2} u'(r)}{r^2} + q\beta u^{q-1} u'(r) \right]. \]
We claim that $I(r) \sim r^{1/\alpha} u(r)$ near infinity. Indeed, since $u > 0$ and $u' < 0$, the function $I$ can be written as
\[ I(r) = r^{1/\alpha} u \left[ 1 - \frac{1}{\alpha} \frac{|u'|^{p-1}}{ru} \right]. \]
Since $\alpha N > 1$, Theorem 3.5 means that $E_N(r)$ has the same sign for large $r$. Hence, according to (3.13) and Lemma 3.8, we get $E_N(r)$ is positive for large $r$. This means that

$$|u'(r)| \leq N \frac{u(r)}{r}.$$  \hfill (3.17)

Hence

$$\frac{|u'|^{p-1}(r)}{ru} \leq N^{p-1} \frac{u^{p-2}(r)}{r^p}.$$  

As $\lim_{r \to +\infty} u(r) = 0$, we get $\lim_{r \to +\infty} \frac{|u'|^{p-1}(r)}{ru} = 0$. Consequently, $I(r) \sim r^{1/\alpha}u(r)$ near infinity and thereby, it’s enough to show that $\lim_{r \to +\infty} I(r) \in [0, +\infty[$. Now, we distinguish two cases:

**Case 1:** $\beta = 0$.

As $u' < 0$, it easy to see that

$$I'(r) = \frac{1}{\alpha} (N - 1/\alpha) r^{1/\alpha - 2} |u'|^{p-1}(r).$$  \hfill (3.18)

Hence, $I'(r) > 0$ for any $r > 0$. Moreover, as $I(0) = 0$, then $I(r) > 0$ for any $r > 0$. This implies that $\lim_{r \to +\infty} I(r) \in [0, +\infty]$. Suppose that $\lim_{r \to +\infty} I(r) = +\infty$.

We claim that $r^2 I'(r)$ is bounded for large $r$. Indeed, using (3.18) and (3.17), we get

$$r^2 I'(r) \leq \frac{1}{\alpha} (N - 1/\alpha) N^{p-1} \frac{r^{1/\alpha} u^{p-1}(r)}{r^{p-1}}.$$  

Put $\sigma = \frac{1}{p-1} \left( \frac{1}{\alpha} + 1 - p \right)$. Then

$$r^2 I'(r) \leq \frac{1}{\alpha} (N - 1/\alpha) N^{p-1} (r^\sigma u(r))^{p-1}.$$  

if $\sigma \leq 0$, we have $\lim_{r \to +\infty} r^\sigma u(r) = 0$ (because $\lim_{r \to +\infty} u(r) = 0$) and it obvious that $r^2 I'(r)$ is bounded for large $r$. If $\sigma > 0$, using the fact that $\sigma < 1/\alpha$ and lemma 3.9, we deduce that $r^\sigma u(r)$ has a finite limit and thereby $r^2 I'(r)$ is bounded for large $r$. This implies that there exists some constant $C > 0$ such that for large $r$

$$I'(r) \leq Cr^{-2}.$$  

Integrating this last inequality on $(r_1, r)$ for large $r_1$, we get

$$I(r) - I(r_1) \leq -Cr^{-1} + Cr_1^{-1}.$$  

By letting $r \to +\infty$, we obtain a contradiction. Hence, $\lim_{r \to +\infty} I(r) \in [0, +\infty]$, which means that $\lim_{r \to +\infty} r^{1/\alpha} u(r) \in [0, +\infty]$.
cas 2: \( \beta < 0 \).

Recalling the expression of \( I'(r) \) in (3.15), we have

\[
I'(r) = -\frac{1}{\alpha} r^{1/\alpha} u^{q-1} u'(r) \left[ q\beta + \frac{(N - 1/\alpha)}{r^2 u^{q-1}} \right].
\]

Using (3.17), we get

\[
\frac{|u'|^{p-2}(r)}{r^2 u^{q-1}} \leq N^{p-2} u^{p-1}.
\]

As \( \lim_{r \to +\infty} u(r) = 0 \) and \( p - q - 1 > 0 \), then \( \lim_{r \to +\infty} \frac{|u'|^{p-2}(r)}{r^2 u^{q-1}} = 0 \). Hence, near infinity we have

\[
I'(r) \sim -\frac{q\beta}{\alpha} r^{1/\alpha} u^{q-1} u'(r).
\]

Consequently, as \( \beta < 0 \) and \( u' < 0 \), \( I'(r) < 0 \) for large \( r \). On the other hand, by lemma 3.7 and the fact that \( \beta < 0 \), we deduce that for large \( r \)

\[
\alpha u + \frac{|u'|^{p-2} u'(r)}{r} > 0.
\]

Therefore, by (3.14), \( I(r) > 0 \) for large \( r \). This means that \( \lim_{r \to +\infty} I(r) \in [0, +\infty[ \).

Suppose that \( \lim_{r \to +\infty} I(r) = 0 \). Then, \( \lim_{r \to +\infty} r^{1/\alpha} u(r) = 0 \). Hence, for large \( r \)

\[
u(r) \leq r^{-1/\alpha}
\]

and by (3.17)

\[
|u'(r)| \leq Nr^{-1/\alpha - 1}.
\]

Hence, the functions \( r \to r^{1/\alpha} u^{q-1}(r)|u'(r)| \) and \( r \to r^{1/\alpha - 2} |u'(r)|^{p-1} \) belong to \( L^1([r_0, \infty]) \) for any \( r_0 > 0 \); therefore \( I'(r) \in L^1([r_0, \infty]) \) and then \( I(r) = -\int_{r_0}^{+\infty} I(t) dt \). This yields

\[
u(r) \leq \frac{1}{\alpha r} |u'|^{p-1} + q \frac{|\beta|}{\alpha} r^{-1/\alpha} \int_{r}^{+\infty} s^{1/\alpha} u^{q-1}(s) |u'(s)| ds.
\]

In view of (3.19) and (3.20), we obtain for large \( r \)

\[
u(r) \leq C(r^{-p-(p-1)/\alpha} + r^{-q/\alpha}),
\]

for some \( C > 0 \). As \( p - 1 - q > 0 \), we get for large \( r \)

\[
u(r) \leq Cr^{-q/\alpha}.
\]

If we define the sequence \( \{m_k\}_{k \in \mathbb{N}} \) by

\[
m_0 = \frac{1}{\alpha}, \quad m_k = qm_{k-1}; \quad k \geq 1,
\]
we see that \( \lim_{r \to +\infty} m_k = +\infty \), and it follows by induction starting with \( m_0 = 1/\alpha \) that the function \( r^m u(r) \) is bounded for all positive integers \( m \). But this contradicts lemma 3.8. Consequently, \( \lim_{r \to +\infty} I(r) \in ]0, +\infty[ \) and thereby \( \lim_{r \to +\infty} r^{1/\alpha} u(r) \in ]0, +\infty[ \). This completes the proof.

4 Classification of Solutions

In this section we give a classification of solutions of problem (P). We start with the following result.

**Theorem 4.1** Assume \( \alpha \leq 0 \). Then any solution \( u \) of problem (P) changes sign.

**Proof:** The proof will be done in two steps.

**STEP 1.** \( u \) is not strictly positive. Suppose for the contrary that \( u > 0 \). Using the fact that \( \alpha \leq 0 \), \( \beta \leq 0 \) and \( u'(r) < 0 \), we get for \( c = N \) in equation (3.10), \( E_N'(r) < 0 \) for any \( r > 0 \). As \( E_N(0) > 0 \), suppose that there exists \( r_0 > 0 \) the first zero such that \( E_N(r_0) = 0 \), then \( E_N(r) - E_N(r_0) = 0 \) for any \( r > r_0 \). Hence \( \lim_{r \to +\infty} E_N(r) \in [-\infty, 0[ \). This means that \( \lim_{r \to +\infty} ru'(r) \in [-\infty, 0[ \). But, this contradicts the fact that \( u \) is strictly positive. Hence, \( E_N(r) > 0 \) for any \( r > 0 \) and (3.17) follows. On the other hand, using once again the fact that \( \alpha \leq 0 \), \( \beta \leq 0 \) and \( u' < 0 \), we have by \( E_1 \)

\[
(|u'|^{p-2}u')'(r) \leq \frac{N-1}{r} |u'|^{p-1}(r) - u(r).
\]

Hence, by (3.17)

\[
(|u'|^{p-2}u')'(r) \leq (N-1)N^{p-1} \frac{u^{p-1}(r)}{r^p} - u(r).
\]

Equivalently

\[
(|u'|^{p-2}u')'(r) \leq -u(r) \left[ 1 - (N-1)N^{p-1} \frac{u^{p-2}(r)}{r^p} \right].
\]

Using the fact that \( u > 0 \) and \( \lim_{r \to +\infty} u(r) = 0 \), we see that \( (|u'|^{p-2}u')'(r) < 0 \) for large \( r \). This implies that \( \lim_{r \to +\infty} u'(r) \in [-\infty, 0[ \). But, this contradicts the fact that \( u \) is strictly positive.

**STEP 2.** \( u \) changes sign. According to Step 1, let \( r_0 > 0 \) the first zero of \( u \). By (3.7) and (3.12), we have \( f(0) = 0 \) and \( f'(r) < 0 \) for any \( r \in (0, r_0) \), then \( f(r_0) = r_0^{N-1} |u'|^{p-2} u'(r_0) < 0 \) and thereby \( u'(r_0) < 0 \). Hence \( u \) changes sign.
Theorem 4.2 Suppose $\alpha N > 1$ and $\beta = 0$. Then, any solution of problem (P) is strictly positive.

Proof: Assume by contradiction that $u(r_0) = 0$ (where $r_0 > 0$ is the first zero of $u$). Then, $u'(r_0) \leq 0$. On the other hand, by (3.12), we have $f'(r) > 0$ for any $r \in (0, r_0)$. Hence, using the fact that $f(0) = 0$, we get $f(r_0) > 0$, in contradiction with the fact that $f(r_0) \leq 0$. The theorem is proved.

Theorem 4.3 Suppose $\alpha N > 1$ and $\beta < 0$. Then, for any $A < A_0 = \left(\frac{\alpha N - 1}{-\beta N}\right)^{1/(q-1)}$, the solution $u(\cdot, A)$ is strictly positive.

Proof: We argue by contradiction. Thus, assume that $u(r_0) = 0$ (where $r_0 > 0$ is the first zero of $u$). Then, $u'(r_0) \leq 0$. On the other hand, multiplying the equation $(E_1)$ by $r^{N-1}$ and integrating on $(0, r_0)$, we get

$$r_0^{N-1} |u'(r_0)|^{p-2} u'(r_0) = \int_0^{r_0} s^{N-1} u(s) \left[\alpha N - 1 + \beta N u^{q-1}(s)\right] ds. \quad (4.1)$$

Let

$$g(r) = (\alpha N - 1) + \beta N u^{q-1}(r)$$

for $r \in (0, r_0)$. Then

$$g'(r) = \beta N (q - 1) u^{q-2}(r) u'(r)$$

for $r \in (0, r_0)$. Hence, using the fact that $u > 0$, $u' < 0$ and $\beta < 0$, we see that $g'(r) > 0$ for $r \in (0, r_0)$. Which gives $g(r) > g(0) > 0$ for $r \in (0, r_0)$. Consequently,

$$\int_0^{r_0} s^{N-1} u(s) g(s) ds > 0,$$

The obtained sign contradiction with (4.1) proves our assertion.

Theorem 4.4 Assume $0 < \alpha N < 1$ or $\alpha N = 1$ and $\beta < 0$. Then any solution $u$ of problem (P) changes sign.

Proof: First, we claim that $u$ is not strictly positive. In fact, suppose for contrary that $u > 0$. Then $\lim_{r \to +\infty} u(r) = 0$. Now, we consider the function

$$J(r) = \left[u(r) + \frac{1}{\alpha r^{N-1}} |u'|^{p-2} u'(r)\right] r^N. \quad (4.2)$$

Then, $J$ satisfies

$$J'(r) = \frac{1}{\alpha} r^{N-1} \left[\alpha N - 1 - q\beta r u^{q-2} u'\right] u(r). \quad (4.3)$$
Hence, \( J'(r) < 0 \), for any \( r > 0 \). Since \( J(0) = 0 \), then \( J(r) < 0 \) for any \( r > 0 \). As \( \beta \leq 0 \), this implies that \( f(r) \leq \alpha J(r) < 0 \) for any \( r > 0 \). But this contradicts lemma 3.7. Thereby there exists \( r_0 \) the first zero of \( u \). Then \( u'(r_0) \leq 0 \). If \( u'(r_0) = 0 \); according to (4.1) we have

\[
\int_0^{r_0} s^{N-1} u(s) \left[ \alpha N - 1 + \beta N u^{q-1}(s) \right] ds = 0;
\]

which is impossible by virtue of our hypotheses. Then \( u'(r_0) < 0 \) and \( u \) changes sign.

References


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