

A Third-Order Three-Point Boundary Value Problem with Nonlinear Terms Depending on the Higher Order Derivative

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Abstract

In this paper, we are concerned with the following third-order three-point boundary value problem

$$u'''(t) + f(t, u(t), u'(t), u''(t)) = 0, \quad t \in (0, 1),$$

$$u(0) = 0, \quad u'(0) = 0, \quad u'(1) = \xi u'(\eta),$$

where $f : [0, 1] \times R^3 \rightarrow R$ is continuous, $\xi > 0, 0 < \eta < 1$ such that $\xi\eta < 1$. Under some appropriate conditions, By using two pairs of lower and upper solutions method of Henderson and Thompson and Leray-Schauder degree theory, the existence result of at least three solutions for the given problem is given.

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1 Introduction

This paper deals with the multiplicity of solutions for the following third order three-point boundary value problem (BVP, for short)

$$u'''(t) + f(t, u(t), u'(t), u''(t)) = 0, \quad t \in (0, 1), \quad (1)$$

$$u(0) = 0, u'(0) = 0, u'(1) = \xi u'(\eta). \quad (2)$$

Throughout this paper, we suppose that $\xi > 0, 0 < \eta < 1$, such that $\xi\eta < 1$, and $f : [0, 1] \times R^3 \rightarrow R$ is continuous.

Third-order boundary value problems (BVPs) were discussed in many papers in recent years, for instance, see [1,2,3,5,9,10] and references therein. However, in the above mentioned papers, the authors all considered the existence of solution. Recently, Henderson and Thompson [8] obtained the existence of three solutions to the following second-order two-point boundary value problem

$$y''(t) + f(t, y(t), y'(t)) = 0, \quad \text{for all } t \in [0, 1], \quad (3)$$

$$y(0) = 0 = y(1). \quad (4)$$

They assumed there exist two pairs of lower and upper solutions α_1, β_1 and α_2, β_2 for problem (3),(4), satisfying $\alpha_1 \leq \alpha_2, \beta_1 \leq \beta_2, \alpha_2 \not\leq \beta_1$, the restriction condition $\beta_1 \leq \alpha_2$ on $[0, 1]$ was firstly weakened to $\alpha_2 \not\leq \beta_1$ on $[0, 1]$.

By so far, very few multiplicity results were established for third order nonlinear multi-point boundary value problems. Motivated by the above works, the purpose of this article is to study the multiplicity of solutions to BVP (1),(2).

In this paper, we apply two pairs of lower and upper solutions method of Henderson and Thompson [8] to study BVP (1),(2). Under the condition that $f(t, u, v, w)$ satisfies a Nagumo condition, we obtain the existence of three solutions by use of Leray-Schauder degree theory. We refer the reader to [2,3,4,6,7,11,12] for the recent results of nonlinear multi-point boundary value problems.

2 Background Notation and Definitions

In the following, we shall use the classical spaces $C[0, 1]$, $C^2[0, 1]$ and $L^1[0, 1]$. For $x \in C^2[0, 1]$, we use the norm $\|x\|_\infty = \max\{|x(t)| : t \in [0, 1]\}$, and $\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty, \|x''\|_\infty\}$. We will use the Sobolev space $W^{3,1}(0, 1)$ which defined by

$$W^{3,1}(0, 1) = \{x : [0, 1] \rightarrow R \mid x, x', x'' \text{ are absolutely continuous on } [0, 1] \text{ with } x''' \in L^1[0, 1]\}.$$

Definition 2.1 A function $\alpha(t) \in W^{3,1}(0, 1)$ is called a lower solution for problem (1),(2), if

$$\alpha'''(t) + f(t, \alpha(t), \alpha'(t), \alpha''(t)) \geq 0, \quad 0 < t < 1, \quad (5)$$

and

$$\alpha(0) \leq 0, \alpha'(0) \leq 0, \alpha'(1) \leq \xi \alpha'(\eta). \quad (6)$$

Similarly, a function $\beta(t) \in W^{3,1}(0, 1)$ is called an upper solution for problem (1),(2), if

$$\beta'''(t) + f(t, \beta(t), \beta'(t), \beta''(t)) \leq 0, \quad 0 < t < 1, \quad (7)$$

and

$$\beta(0) \geq 0, \beta'(0) \geq 0, \beta'(1) \geq \xi \beta'(\eta). \quad (8)$$

Remark 2.2 We will say α (β) is a strict lower solution (a strict upper solution) for problem (1),(2), if the inequality (5) ((7)) is strict for $t \in (0, 1)$.

Remark 2.3 Let $f : [0, 1] \times \mathbb{R}^3 \rightarrow \mathbb{R}$ be continuous, and let u be a solution of (1),(2). If α is a strict lower solution for (1),(2) with $\alpha' \leq u'$, then $\alpha' < u'$ on $(0, 1)$. Similarly, If β is a strict upper solution for (1),(2) with $u' \leq \beta'$, then $u' < \beta'$ on $(0, 1)$.

Definition 2.4 Let α be a lower solution and β an upper solution for problem (1),(2) satisfying $\alpha \leq \beta$ and $\alpha' \leq \beta'$ on $[0, 1]$. We say that f satisfies a Nagumo condition with respect to α and β , if there exists a function $\Phi \in C([0, \infty); (0, +\infty))$ such that

$$|f(t, u, v, w)| \leq \Phi(|w|), \quad (9)$$

for all $(t, u, v, w) \in [0, 1] \times [\alpha(t), \beta(t)] \times [\alpha'(t), \beta'(t)] \times \mathbb{R}$, and

$$\int_0^\infty \frac{s}{\Phi(s)} ds = \infty. \quad (10)$$

It is clear that we can give the Green's function $G(t,s)$ of the problem (1),(2) since the boundary conditions satisfies $1 - \xi\eta > 0$. (i.e. the non-resonance case).

3 Existence of Triple Solutions

Theorem 3.1 Assume that

(A1) There exist two strict lower solutions α_1 and α_2 and strict upper solutions β_1 and β_2 of (1),(2), satisfying $\alpha'_1 \leq \alpha'_2 \leq \beta'_2$, $\alpha'_1 \leq \beta'_1 \leq \beta'_2$, $\alpha'_2 \not\leq \beta'_1$ on $[0, 1]$;

(A2) Let $f(t, u, v, w) : [0, 1] \times R^3 \rightarrow R$ be a continuous function and nondecreasing with respect to u , for $(t, u, v, w) \in [0, 1] \times [\alpha_1(t), \beta_2(t)] \times R^2$;

(A3) f satisfies Nagumo condition with respect to α_1 and β_2 .

Then boundary value problem (1),(2) has at least three solutions u_1, u_2 and u_3 satisfying

$$\begin{aligned} \alpha_1 &\leq u_1 \leq \beta_1, & \alpha_1' &\leq u_1' \leq \beta_1', & \text{on } [0, 1], \\ \alpha_2 &\leq u_2 \leq \beta_2, & \alpha_2' &\leq u_2' \leq \beta_2', & \text{on } [0, 1], \\ u_3 &\not\leq \beta_1, & u_3' &\not\leq \beta_1' & \text{and } u_3 \not\leq \alpha_2, u_3' \not\leq \alpha_2', & \text{on } [0, 1]. \end{aligned}$$

Proof From assumption (A3), we can choose $C > 0$, such that

$$\int_{\lambda}^C \frac{s}{\Phi(s)} ds > \lambda, \quad (11)$$

where $\lambda = \max_{t \in [0, 1]} \beta_2'(t) - \min_{t \in [0, 1]} \alpha_1'(t)$. Let

$$L = \max\{\|\alpha_1''\|_{\infty}, \|\beta_2''\|_{\infty}, C, 2\lambda\}.$$

We define three auxiliary functions f_1, f_2 and $F : [0, 1] \times R^3 \rightarrow R$ as

$$f_1(t, u, v, w) = \begin{cases} f(t, \beta_2, v, w), & u > \beta_2(t), t \in [0, 1]; \\ f(t, u, v, w), & \alpha_1(t) \leq u \leq \beta_2(t), t \in [0, 1]; \\ f(t, \alpha_1, v, w), & u < \alpha_1(t), t \in [0, 1], \end{cases} \quad (12)$$

$$f_2(t, u, v, w) = \begin{cases} f_1(t, u, \beta_2', w), & v > \beta_2'(t), t \in [0, 1]; \\ f_1(t, u, v, w), & \alpha_1'(t) \leq v \leq \beta_2'(t), t \in [0, 1]; \\ f_1(t, u, \alpha_1', w), & v < \alpha_1'(t), t \in [0, 1], \end{cases} \quad (13)$$

and

$$F(t, u, v, w) = \begin{cases} f_2(t, u, v, L), & w > L, t \in [0, 1]; \\ f_2(t, u, v, w) & |w| \leq L, t \in [0, 1]; \\ f_2(t, u, v, -L), & w < -L, t \in [0, 1]. \end{cases} \quad (14)$$

Thus F is a continuous function on $[0, 1] \times R^3$, satisfying

$$|F(t, u, v, w)| \leq M, \text{ for } (t, u, v, w) \in [0, 1] \times R^3, \quad (15)$$

where constant M also satisfies $M > \max\{\|\alpha_1\|_{\infty}, \|\beta_2\|_{\infty}\}$.

Consider the modified problem

$$u'''(t) + F(t, u, u', u'') = 0, t \in (0, 1), \quad (16)$$

with boundary conditions (2).

To finish the proof from the definition of F , it suffices to show that problem (16) with (2) has at least three solutions u_1, u_2 and u_3 satisfying

$$\alpha_1(t) \leq u_i(t) \leq \beta_2(t), \alpha'_1(t) \leq u'_i(t) \leq \beta'_2(t), |u''_i(t)| \leq L, t \in [0, 1], i = 1, 2, 3, \tag{17}$$

since $F = f$ in the region. We divide the proof into two steps.

Step 1. Suppose that problem (16) with (2) has a solution u , then u satisfies (17), moreover, u is a solution of problem (1),(2) .

We first show that $\alpha'_1 \leq u' \leq \beta'_2$ on $[0, 1]$. We only need to show $u' \leq \beta'_2$ on $[0, 1]$. Similarly, we can prove $\alpha'_1 \leq u'$ on $[0, 1]$, hence we omit it. If $u' \leq \beta'_2$ on $[0, 1]$ is not true, then there exists $t \in [0, 1]$ with $u' > \beta'_2$. Set $\omega(t) := u'(t) - \beta'_2(t)$, then $\omega(t_0) = \max\{u'(t) - \beta'_2(t) : t \in [0, 1]\} > 0$ for some $t_0 \in [0, 1]$.

Case (I). If $t_0 = 0$, then $u'(0) > \beta'_2(0)$, from (8), we have the contradiction $\beta'_2(0) \geq 0 = u'(0)$.

Case (II). If $t_0 \in (0, 1)$, we have $\omega(t_0) > 0, \omega'(t_0) = 0$, and $\omega''(t_0) \leq 0$. But on the other hand,

$$\begin{aligned} \omega''(t_0) &= -F(t_0, u(t_0), u'(t_0), u''(t_0)) - \beta''_2(t_0) \\ &= -f_2(t_0, u(t_0), u'(t_0), \beta''_2(t_0)) - \beta''_2(t_0) \\ &= -f_1(t_0, u(t_0), \beta'_2(t_0), \beta''_2(t_0)) - \beta''_2(t_0). \end{aligned}$$

Subcase (i). If $u(t_0) > \beta_2(t_0)$, from the above inequality, one has

$$\begin{aligned} \omega''(t_0) &= -f_1(t_0, u(t_0), \beta'_2(t_0), \beta''_2(t_0)) - \beta''_2(t_0) \\ &= -f(t_0, \beta_2(t_0), \beta'_2(t_0), \beta''_2(t_0)) - \beta''_2(t_0) \\ &> 0. \end{aligned}$$

Subcase (ii). If $u(t_0) \leq \beta_2(t_0)$, from the above inequality and (A_2) ,

$$\begin{aligned} \omega''(t_0) &= -f_1(t_0, u(t_0), \beta'_2(t_0), \beta''_2(t_0)) - \beta''_2(t_0) \\ &= -f(t_0, u(t_0), \beta'_2(t_0), \beta''_2(t_0)) - \beta''_2(t_0) \\ &> -f(t_0, \beta_2(t_0), \beta'_2(t_0), \beta''_2(t_0)) - \beta''_2(t_0) \\ &> 0. \end{aligned}$$

Which is a contraction.

Case (III). If $t_0 = 1$, then

$$\omega(1) > 0. \tag{18}$$

From (8), we have $\omega(0) \leq 0$, thus there exists $\sigma \in [0, 1)$ such that

$$\omega(\sigma) = 0, \text{ and } \omega(t) > 0, \text{ for all } t \in (\sigma, 1]. \tag{19}$$

If $\sigma \in (\eta, 1)$, then there exists $t_1 \in (0, \sigma)$ such that $\omega(t_1) = \max\{\omega(t) : t \in [0, \sigma]\}$. From (2), (8) and (18), we have

$$\begin{aligned}\omega(t_1) &\geq \omega(\eta) = u'(\eta) - \beta'_2(\eta) \\ &\geq \frac{1}{\xi}[u'(1) - \beta'_2(1)] \\ &= \frac{1}{\xi}\omega(1) > 0.\end{aligned}$$

Moreover, $\omega'(t_1) = 0$ and $\omega''(t_1) \leq 0$. Similar to the case (II), we have contradiction.

If $\sigma \in (0, \eta)$, then for all $t \in [\sigma, 1]$, we have that $\omega(t) \geq 0$. We consider the following two subcases: Subcase (i). $\omega'(t) \geq 0$, $t \in [\sigma, 1]$; Subcase (ii). there exists some $t_2 \in (\sigma, 1)$, such that $\omega(t_2) > 0$, $\omega'(t_2) = 0$, $\omega''(t_2) \leq 0$.

For Subcase (i), similar to case (II), we have

$$\omega''(t) > 0$$

or

$$\omega(t) > 0, \omega''(t) > 0, \text{ for all } t \in (\sigma, 1],$$

which implies that the graph of ω is concave upward on $(\sigma, 1]$, and so

$$\frac{\omega(\eta)}{\eta} < \frac{\omega(1)}{1}.$$

On the other hand, we have

$$\omega(1) = u'(1) - \beta'_2(1) \leq \xi[u'(\eta) - \beta'_2(\eta)] = \xi\omega(\eta),$$

from $0 < \xi < \frac{1}{\eta}$, we obtain

$$\frac{\omega(\eta)}{\eta} \geq \frac{\omega(1)}{1}.$$

Which is a contradiction.

For Subcase (i), similar to the argument of case (II), we have contradiction.

Thus $u' \leq \beta'_2$ on $[0, 1]$, then $\alpha'_1 \leq u' \leq \beta'_2$ on $[0, 1]$.

Since $\alpha(0) \leq 0$, $\beta(0) \geq 0$, by integrating the above inequalities on $[0, t]$, we obtain $\alpha_1 \leq u \leq \beta_2$ on $[0, 1]$.

Now we show that $|u''| \leq L$ on $[0, 1]$. If the assertion is not true, without loss of the generality, we suppose that there exists $t \in [0, 1]$, satisfying $u''(t) > L$. Let t_3 be the point where $u''(t) - L$ attains its positive maximum over $[0, 1]$. From mean value theorem and $\alpha'_1 \leq u' \leq \beta'_2$ on $[0, 1]$, there exists $\theta \in (0, 1)$, such that

$$u''(\theta) = u'(1) - u'(0) \leq \lambda < L.$$

Since $u''(t) \in C[0, 1]$, then there exists interval $[t_4, t_5] \subseteq [0, 1]$ (or $[t_5, t_4] \subseteq [0, 1]$), such that

$$u''(t_4) = \lambda, u''(t_5) = L, \quad \lambda < u''(t) < L, \quad t \in (t_4, t_5). \quad (20)$$

Thus from (9), we obtain

$$|u'''(t)| = |F(t, u, u', u'')| = |f(t, u, u', u'')| \leq \Phi(|u''|), t \in (t_4, t_5),$$

then

$$\left| \int_{t_4}^{t_5} \frac{u''(t)u'''(t)}{\Phi(u''(t))} dt \right| \leq \left| \int_{t_4}^{t_5} u''(t) dt \right| \leq \lambda. \quad (21)$$

From (11) and (20), we have

$$\left| \int_{t_4}^{t_5} \frac{u''(t)u'''(t)}{\Phi(u''(t))} dt \right| = \left| \int_{\lambda}^L \frac{s}{\Phi(s)} ds \right| > \lambda. \quad (22)$$

Then (21) contradicts (22). So that $|u''| \leq L$ on $[0, 1]$. Thus x is the required solution.

Step 2. We show that problem (16) with (2) has at three solutions u_1, u_2 and u_3 .

Let

$$\Omega = \{u \in C^2[0, 1] : \|u\| < PM + L\}$$

where $P > \max\{\max_{t \in [0, 1]} \int_0^1 |G(t, s)| ds, 1\}$, $G(t, s)$ is Greet's function the problem (1),(2).

Define $S : C[0, 1] \longrightarrow C^2[0, 1]$ by

$$(S\phi)(t) = \int_0^1 G(t, s)\phi(s)ds,$$

for all $\phi \in C[0, 1]$ and $t \in [0, 1]$. It is clear that S is completely continuous.

Define $H : C^2[0, 1] \longrightarrow C[0, 1]$ as

$$H(\phi)(t) = F(t, \phi(t), \phi'(t), \phi''(t)).$$

Then $u \in C^2[0, 1]$ is a solution of (16) with (2) if and only if $(I - SH)(u) = 0$. For $u \in \overline{\Omega}$, we have

$$\begin{aligned} SH(x) &= \int_0^1 G(t, s)F(s, u(s), u'(s), u''(s))ds \\ &\leq M \int_0^1 G(t, s)ds \\ &< PM < PM + L \end{aligned}$$

Clearly $SH(\overline{\Omega}) \subset \Omega$ and SH is completely continuous. Then we have

$$\deg(I - SH, \Omega, 0) = \deg(I, \Omega, 0) = 1. \tag{23}$$

Let

$$\Omega_{\alpha_2} = \{u \in \Omega : u' > \alpha'_2 \text{ on } (0, 1)\}, \quad \Omega^{\beta_1} = \{u \in \Omega : u' < \beta'_1 \text{ on } (0, 1)\}.$$

Since $\alpha'_2 \not\leq \beta'_1, \alpha'_2 \geq \alpha'_1 > -L$, and $\beta'_1 \leq \beta'_2 < L$, it follows that

$$\Omega_{\alpha_2} \neq \emptyset \neq \Omega^{\beta_1}, \quad \overline{\Omega_{\alpha_2}} \cap \overline{\Omega^{\beta_1}} = \emptyset, \quad \Omega \setminus \overline{\{\Omega_{\alpha_2} \cup \Omega^{\beta_1}\}} \neq \emptyset.$$

By assumptions (A1) and Remark 2, there is no solution on $\partial\Omega_{\alpha_2} \cup \partial\Omega^{\beta_1}$. Thus

$$\begin{aligned} \deg(I - SH, \Omega, 0) &= \deg(I - SH, \Omega \setminus \overline{\{\Omega_{\alpha_2} \cup \Omega^{\beta_1}\}}, 0) \\ &\quad + \deg(I - SH, \Omega^{\beta_1}, 0) + \deg(I - SH, \Omega_{\alpha_2}, 0). \end{aligned}$$

If we prove that

$$\deg(I - SH, \Omega^{\beta_1}, 0) = \deg(I - SH, \Omega_{\alpha_2}, 0) = 1,$$

then

$$\deg(I - SH, \Omega \setminus \overline{\{\Omega_{\alpha_2} \cup \Omega^{\beta_1}\}}, 0) = -1,$$

and hence there are solutions in $\Omega_{\alpha_2}, \Omega^{\beta_1}$ and $\Omega \setminus \overline{\{\Omega_{\alpha_2} \cup \Omega^{\beta_1}\}}$ respectively.

We show that $\deg(I - SH, \Omega_{\alpha_2}, 0) = 1$. The proof that $\deg(I - SH, \Omega^{\beta_1}, 0) = 1$ is the same and hence omitted. Similar to the definitions of f_1 , we define

$$\begin{aligned} f_1^*(t, u, v, w) &= \begin{cases} f(t, \beta_2, v, w), & u > \beta_2(t), t \in [0, 1]; \\ f(t, u, v, w), & \alpha_2(t) \leq u \leq \beta_2(t), t \in [0, 1]; \\ f(t, \alpha_2, v, w), & u < \alpha_2(t), t \in [0, 1]. \end{cases} \\ f_2^*(t, u, v, w) &= \begin{cases} f_1^*(t, u, \beta'_2, w), & v > \beta'_2(t), t \in [0, 1]; \\ f_1^*(t, u, v, w), & \alpha'_2(t) \leq v \leq \beta'_2(t), t \in [0, 1]; \\ f_1^*(t, u, \alpha'_2, w), & v < \alpha'_2(t), t \in [0, 1]. \end{cases} \end{aligned}$$

Now from $I - SH|_{\overline{\Omega_{\alpha_2}}}$, we define its extension $I - SH^* : \overline{\Omega} \rightarrow C^2[0, 1]$, as follows.

$$F^*(t, u, v, w) = \begin{cases} f_2^*(t, u, v, L), & w > L, t \in [0, 1]; \\ f_2^*(t, u, v, w), & |w| \leq L, t \in [0, 1]; \\ f_2^*(t, u, v, -L), & w < -L, t \in [0, 1]. \end{cases} \tag{24}$$

Thus F^* is a continuous function on $[0, 1] \times \mathbb{R}^3$ and satisfies

$$|F^*(t, u, v, w)| \leq M, \tag{25}$$

for all $(t, u, v, w) \in [0, 1] \times R^3$, where M is given in (15).

Define $H^* : C^2[0, 1] \rightarrow C[0, 1]$ as follows

$$H^*(\phi)(t) = F^*(t, \phi(t), \phi'(t), \phi''(t)).$$

Then $u \in C^2[0, 1]$ is a solution of $(I - SH^*)(u) = 0$ if and only if u is a solution of

$$u'''(t) + F^*(t, u, u', u'') = 0, t \in (0, 1), \quad (26)$$

with (2). Similar to the above argument, it follows that u is a solution of (26) with (2) only if $u \in \Omega_{\alpha_2}$. Thus

$$\deg(I - SH^*, \Omega \setminus \overline{\Omega}_{\alpha_2}) = 0.$$

Similarly, we show that $SH^*(\overline{\Omega}) \subset \Omega$. Then we have

$$\deg(I - SH^*, \Omega, 0) = 1.$$

Thus

$$\begin{aligned} \deg(I - SH, \Omega_{\alpha_2}, 0) &= \deg(I - SH^*, \Omega_{\alpha_2}, 0) \\ &= \deg(I - SH^*, \Omega \setminus \overline{\Omega}_{\alpha_2}, 0) + \deg(I - SH^*, \Omega_{\alpha_2}, 0) \\ &= \deg(I - SH^*, \Omega, 0) = 1. \end{aligned}$$

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