Int. Journal of Math. Analysis, Vol. 1, 2007, no. 16, 769 - 774

Hilbert-Schmidt Weighted Composition Operator on the Fock space

Sei-ichiro Ueki[1]

Department of Mathematics
Faculty of Science Division II
Tokyo University of Science, Japan

Abstract. In this paper we consider the weighted composition operator uC_φ on the Fock space. We give necessary and sufficient conditions on u and φ which will ensure that the weighted composition operator belongs to the Hilbert-Schmidt class. As a consequence of our main result, we completely determine the Hilbert-Schmidt compositon operator on the Fock space.

Mathematics Subject Classification: Primary 47B33, 47B38; Secondary 30D15

Keywords: Hilbert-shcmidt operator, weighted composition operator, Fock space, entire function

1. Introduction

Throughout this paper, let dm denote the usual Lebesgue measure on $\mathbb{C}$. For $\alpha > 0$, the *Fock space* $\mathcal{F}_\alpha^2$ is the space of all entire functions f on $\mathbb{C}$ for which

$$\|f\|_{\mathcal{F}_\alpha^2}^2 = \frac{\alpha}{\pi}\int_{\mathbb{C}} |f(z)|^2 e^{-\alpha|z|^2} dm(z) < \infty.$$

$\mathcal{F}_\alpha^2$ is a Hilbert space with an inner product

$$\langle f, g\rangle_{\mathcal{F}_\alpha^2} = \frac{\alpha}{\pi}\int_{\mathbb{C}} f(z)\overline{g(z)} e^{-\alpha|z|^2} dm(z).$$

Let $e_n(z) = \sqrt{\frac{\alpha^n}{n!}} z^n$ for a positive integer n. Then the set $\{e_n\}$ forms an orthonormal basis for $\mathcal{F}_\alpha^2$. Since each point evaluation is a bounded linear functional on $\mathcal{F}_\alpha^2$, thus for $z \in \mathbb{C}$ there exists a unique function $K_z \in \mathcal{F}_\alpha^2$ such that $f(z) = \langle f, K_z\rangle_{\mathcal{F}_\alpha^2}$ for $f \in \mathcal{F}_\alpha^2$. That is K_z is the reproducing kernel

[1]Postal address: 2-18-4, Miyashiro Chuo, Minamisaitama, Saitama 345-0821 Japan
ueki@ms.kuki.tus.ac.jp
sueki@camel.plala.or.jp

function for $\mathcal{F}^2_\alpha$. Using the orthonormal basis $\{e_n\}$ for $\mathcal{F}^2_\alpha$, we can compute the explicit formula for K_z:

$$K_z(w) = \exp\{\alpha\langle w, z\rangle\}, \quad w \in \mathbb{C},$$

(see [6]). Here the notation $\langle\cdot,\cdot\rangle$ denotes the usual inner product on $\mathbb{C}$. Note that $\|K_z\|_{\mathcal{F}^2_\alpha} = \exp(\frac{\alpha}{2}|z|^2)$. We also use the normalized kernel function $k_z(w) = \exp\{\alpha\langle w, z\rangle - \frac{\alpha}{2}|z|^2\}$.

The problems for the Fock space, such as an interpolation sequence or a sampling set, have been studied by many authors. See, for example, [7, 9]. Moreover, sevral studies have been made on concrete operators, such as Toeplitz operators and Hankel operators. Some results on Toeplitz operators have been found by Grudsky, Vasilevski, Sangadji and Stroethoff ([4, 8, 10]). In [1, 5, 6, 10], these authors have dealt with Hankel operators and Hankel forms.

In this paper, our object is a weighted composition operator on $\mathcal{F}^2_\alpha$. Let u and φ be entire functions. The *weighted composition operator* uC_φ is defined by $uC_\varphi f = u\cdot(f\circ\varphi)$ for an entire function f. Recently, the author have given a necessary and sufficient condition for the boundedness or compactness of the operator uC_φ ([11]). The normalized kernel function k_z played an important role in our characterization. To give a necessary and sufficient condition for uC_φ belongs to the Hilbert-Schmidt class, we will also use the function k_z.

The following theorem is our main result.

Theorem 1. *Suppose that φ and u are entire functions on $\mathbb{C}$ such that uC_φ is bounded on $\mathcal{F}^2_\alpha$. Then the following conditions are equivalent:*

(a) *uC_φ is a Hilbert-Schmidt operator,*

(b) $\displaystyle\int_{\mathbb{C}} |u(z)|^2 \exp\alpha\{|\varphi(z)|^2 - |z|^2\} dm(z) < \infty,$

(c) $\displaystyle\int_{\mathbb{C}} \left(\int_{\mathbb{C}} |u(w)|^2 |\exp\alpha\langle\varphi(w), z\rangle|^2 e^{\alpha\{-|z|^2-|w|^2\}} dm(w)\right) dm(z) < \infty.$

2. Proofs of main theorem

In order to prove our result, we give a trace formula for positive operators on $\mathcal{F}^2_\alpha$. In the case that the usual Bergman space defined on classical domains of $\mathbb{C}$, this type formula is well-known. See, for example, Proposition 6.3.2 in [12].

Lemma 1. *If T is a positive or trace class operator on $\mathcal{F}^2_\alpha$, then the trace $tr(T)$ of T is given by*

$$tr(T) = \frac{\alpha}{\pi}\int_{\mathbb{C}} \langle Tk_z, k_z\rangle_{\mathcal{F}^2_\alpha} dm(z).$$

Proof. Fix an orthonormal set $\{\varphi_j\}$ for $\mathcal{F}_\alpha^2$. Then we have

$$\begin{aligned}
tr(T) &= \sum_{j=0}^{\infty}\langle T\varphi_j, \varphi_j\rangle_{\mathcal{F}_\alpha^2} = \sum_{j=0}^{\infty}\frac{\alpha}{\pi}\int_{\mathbb{C}} T\varphi_j(z)\overline{\varphi_j(z)}e^{-\alpha|z|^2}dm(z) \\
&= \sum_{j=0}^{\infty}\frac{\alpha}{\pi}\int_{\mathbb{C}}\langle T\varphi_j, K_z\rangle_{\mathcal{F}_\alpha^2}\overline{\varphi_j(z)}e^{-\alpha|z|^2}dm(z) \\
&= \frac{\alpha}{\pi}\int_{\mathbb{C}}\langle T\sum_{j=0}^{\infty}\overline{\varphi_j(z)}\varphi_j, K_z\rangle_{\mathcal{F}_\alpha^2}e^{-\alpha|z|^2}dm(z) \\
&= \frac{\alpha}{\pi}\int_{\mathbb{C}}\langle TK_z, K_z\rangle_{\mathcal{F}_\alpha^2}e^{-\alpha|z|^2}dm(z) = \frac{\alpha}{\pi}\int_{\mathbb{C}}\langle Tk_z, k_z\rangle_{\mathcal{F}_\alpha^2}dm(z).
\end{aligned}$$

Thus the lemma was proved. □

Proofs of Theorem 1. First we show that (a) is equivalent to (b). Since the set $\{e_n\}$ is an orthonormal set for $\mathcal{F}_\alpha^2$, we have

$$\begin{aligned}
\sum_{n=0}^{\infty}\|uC_\varphi e_n\|_{\mathcal{F}_\alpha^2}^2 &= \sum_{n=0}^{\infty}\frac{\alpha}{\pi}\int_{\mathbb{C}}|u(z)|^2\frac{\alpha^n}{n!}|\varphi(z)|^{2n}e^{-\alpha|z|^2}dm(z) \\
&= \frac{\alpha}{\pi}\int_{\mathbb{C}}|u(z)|^2e^{-\alpha|z|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{n!}|\varphi(z)|^{2n}dm(z) \\
&= \frac{\alpha}{\pi}\int_{\mathbb{C}}|u(z)|^2e^{-\alpha|z|^2}e^{\alpha|\varphi(z)|^2}dm(z) \\
&= \frac{\alpha}{\pi}\int_{\mathbb{C}}|u(z)|^2\exp\alpha\{|\varphi(z)|^2-|z|^2\}dm(z).
\end{aligned}$$

Thus uC_φ is Hilbert-Schmidt if and only if

$$\frac{\alpha}{\pi}\int_{\mathbb{C}}|u(z)|^2\exp\alpha\{|\varphi(z)|^2-|z|^2\}dm(z) < \infty,$$

and so we see that (a) and (b) are equivalent.

Next we prove that (a) and (c) are equivalent. According to a general theory of operators on the Hilbert space, we see that uC_φ is a Hilbert-Schmidt operator if and only if $(uC_\varphi)^*uC_\varphi$ is a trace class operator. Since $(uC_\varphi)^*uC_\varphi$ is a positive operator on $\mathcal{F}_\alpha^2$, then Lemma 1 implies that uC_φ is Hilbert-Schmidt if and only if

$$\frac{\alpha}{\pi}\int_{\mathbb{C}}\langle(uC_\varphi)^*uC_\varphi k_z, k_z\rangle_{\mathcal{F}_\alpha^2}dm(z) < \infty. \tag{1}$$

On the other hand, we obtain that for each $z \in \mathbb{C}$

$$\begin{aligned}
&\langle(uC_\varphi)^*uC_\varphi k_z, k_z\rangle_{\mathcal{F}_\alpha^2} = \langle uC_\varphi k_z, uC_\varphi k_z\rangle_{\mathcal{F}_\alpha^2} \\
&= \frac{\alpha}{\pi}\int_{\mathbb{C}}|u(w)|^2|\exp\alpha\langle\varphi(w), z\rangle|^2e^{\alpha\{-|z|^2-|w|^2\}}dm(w).
\end{aligned}$$

Combining this with (1), we can conclude that uC_φ belongs to the Hilbert-Schmidt class if and only if

$$\frac{\alpha}{\pi}\int_{\mathbb{C}}\left(\frac{\alpha}{\pi}\int_{\mathbb{C}}|u(w)|^2|\exp\alpha\langle\varphi(w),z\rangle|^2e^{\alpha\{-|z|^2-|w|^2\}}dm(w)\right)dm(z)<\infty.$$

This completes the proof. □

3. Applications

In this section, we concern two spacial cases of the operator uC_φ on $\mathcal{F}^2_\alpha$.

First we consider the case that u is the constant function 1, that is the composition operator C_φ. Many mathematicians have studied about composition operators on various analytic function spaces. Recently Carswell, MacCluer and Schuster [3] completely determined the bounded or compact composition operator on $\mathcal{F}^2_\alpha$ as follows:

Proposition 1 (Carswell, MacCluer and Schuster [3])**.** *Suppose that φ is an entire function.*

(a) *If C_φ is bounded on $\mathcal{F}^2_\alpha$, then $\varphi(z)=az+b$, where $|a|\leq 1$, and if $|a|=1$, then $b=0$.*
(b) *If C_φ is compact on $\mathcal{F}^2_\alpha$, then $\varphi(z)=az+b$, where $|a|<1$.*

Conversely, suppose that $\varphi(z)=az+b$.

(c) *If $|a|=1$ and $b=0$, then C_φ is bounded on $\mathcal{F}^2_\alpha$.*
(d) *If $|a|<1$, then C_φ is compact on $\mathcal{F}^2_\alpha$.*

As an easy corollary of Theorem 1 and Proposition 1 we get the following result.

Corollary 1. *Suppose that $\varphi(z)=az+b$ with $|a|\leq 1$ and $b\in\mathbb{C}$. Then the following conditions are equivalent:*

(a) *C_φ is a compact operator,*
(b) *C_φ is a Hilbert-Schmidt operator,*
(c) *$|a|<1$.*

Proof. (b) ⇒ (a) is obvious. Proposition 1 have shown that (a) implies (c). So we only prove that (c) implies (b). Let $|a|<\beta<1$ and $R=\frac{|b|}{\beta-|a|}$. For $z\in\mathbb{C}$ with $|z|>R$, we have $\beta|z|>|az+b|=|\varphi(z)|$, and so

$$\int_{|z|>R}\exp\alpha\{|\varphi(z)|^2-|z|^2\}dm(z)\leq\int_{|z|>R}\exp\alpha\{(\beta^2-1)|z|^2\}dm(z)<\infty.$$

On the other hand, we see that

$$\begin{aligned}&\int_{|z|\leq R}\exp\alpha\{|\varphi(z)|^2-|z|^2\}dm(z)\\&\leq\max_{|z|\leq R}\left(\exp\alpha|\varphi(z)|^2\right)\int_{|z|\leq R}\exp\alpha\{-|z|^2\}dm(z)<\infty.\end{aligned}$$

These inequalities and Theorem 1 show that C_φ belongs to the Hilbert-Schmidt class. This completes the proof. $\square$

Next we consider the case $\varphi(z) = z$. In this case, the operator uC_φ is called the *multiplication operator* and denoted by M_u. Let $L^2(\mathbb{C}, d\mu)$ denote the Lebesgue space with respect to the measure

$$d\mu(z) = \frac{\alpha}{\pi} \exp\{-\alpha|z|^2\} dm(z).$$

Then the Fock space $\mathcal{F}^2_\alpha$ is a closed subspace of $L^2(\mathbb{C}, d\mu)$. For each $u \in L^\infty(\mathbb{C})$, the *multiplication operator* $M_u : \mathcal{F}^2_\alpha \to L^2(\mathbb{C}, d\mu)$ is defined by $M_u f = uf$ for $f \in \mathcal{F}^2_\alpha$. This operator is closely related to the Toeplitz and Hankel operators on $\mathcal{F}^2_\alpha$. For these studies, refer to [1, 2, 5, 10]. For $g \in L^\infty(\mathbb{C})$ the *Berezin symbol* $\widetilde{g}$ of g is defined by

$$\widetilde{g}(w) = \frac{\alpha}{\pi} \int_{\mathbb{C}} g(z) \exp\{-\alpha|w - z|^2\} dm(z), \quad w \in \mathbb{C}.$$

The Berezin symbol is used for analyzing the properties of these operators. For instance, K. Stroethoff characterized the compactness of M_u in terms of the Berezin symbol $\widetilde{|u|^2}$ as follows:

Proposition 2 (K. Stroethoff [10])**.** *Let $u \in L^\infty(\mathbb{C})$. The following statements are equivalent:*

(a) *M_u is compact,*
(b) *$\widetilde{|u|^2}(w) \to 0$ as $|w| \to \infty$.*

As another application of Lemma 1 we get a necessary and sufficient condition for $M_u : \mathcal{F}^2_\alpha \to L^2(\mathbb{C}, d\mu)$ is a Hilbert-Schmidt operator.

Corollary 2. *Let $u \in L^\infty(\mathbb{C})$. The following statements are equivalent:*

(a) *M_u is a Hilbert-Schmidt operator,*
(b) *$\widetilde{|u|^2} \in L^1(\mathbb{C})$.*

Proof. For $w \in \mathbb{C}$ we have

$$\begin{aligned} \widetilde{|u|^2}(w) &= \frac{\alpha}{\pi} \int_{\mathbb{C}} |u(z)|^2 \exp\left(2\alpha \mathrm{Re}\langle z, w\rangle\right) \exp \alpha\{-|z|^2 - |w|^2\} dm(z) \\ &= \frac{\alpha}{\pi} \int_{\mathbb{C}} |u(z)|^2 |k_w(z)|^2 e^{-\alpha|z|^2} dm(z) = \langle M_{|u|^2} k_w, k_w\rangle_{\mathcal{F}^2_\alpha}. \end{aligned} \tag{2}$$

This implies that $\widetilde{|u|^2}$ belongs to $L^1(\mathbb{C})$ if and only if

$$\int_{\mathbb{C}} \langle M_{|u|^2} k_w, k_w\rangle_{\mathcal{F}^2_\alpha} dm(w) < \infty.$$

The above equation (2) implies that the Berezin symbol $\widetilde{|u|^2}$ is the integrand which appear in the trace formula for $M_{|u|^2}$. Since $M_{|u|^2} = M_u{}^* M_u$ on $\mathcal{F}^2_\alpha$ and these are positive operators, thus Lemma 1 shows that

$$\begin{aligned} M_u \text{ is in the Hilbert-Schmidt class} &\Longleftrightarrow {M_u}^* M_u \text{ is in the trace class} \\ &\Longleftrightarrow \widetilde{|u|^2} \in L^1(\mathbb{C}), \end{aligned}$$

which was to be proved. □

References

[1] W. Bauer, *Hilbert-Schmidt Hankel operators on the Segal-Bargmann space*, Proc. Amer. Math. Soc., **132** (2004), 2989–2996.

[2] C.A. Berger and L.A. Coburn, *Toeplitz operators on the Segal-Bargmann space*, Trans. Amer. Math. Soc., **301** (1987), 813–829.

[3] B.J. Carswell, B.D. MacCluer and A. Schuster, *Composition operators on the Fock space*, Acta Sci. Math. (Szeged), **69** (2003), 871–887.

[4] S.M. Grudsky and N.L. Vasilevski, *Toeplitz operators on the Fock space: radial component effects*, Integral Equations Operator Theory, **44** (2002), 10–37.

[5] J. Janas, *Toeplitz and Hankel operators on Bargmann spaces*, Glasgow Math. J., **30** (1988), 315–323.

[6] S. Janson, J. Peetre and R. Rochberg, *Hankel forms and the Fock space*, Rev. Math. Iberoamericana, **3** (1987), 61–129.

[7] J. Ortega-Cerdà, *Sampling Measures*, Publ. Mat., **42** (1998), 559–566.

[8] Sangadji and K. Stroethoff, *Compact Toeplitz operators on generalized Fock spaces*, Acta Sci. Math. (Szeged), **64** (1998), 657–669.

[9] K. Seip, *Density theorems for sampling and interpolation in the Bargmann–Fock space I*, J. Reine Angew. Math., **429** (1992), 91–106.

[10] K. Stroethoff, *Hankel and Toeplitz operators on the Fock space*, Michigan Math. J., **39** (1992), 3–16.

[11] S. Ueki, *Weighted composition operator on the Fock space*, Proc. Amer. Math. Sci., **135** (2007), 1405–1410.

[12] K. Zhu, Operator theory in function spaces, Marcel Dekker, New York, 1990.

Received: March 15, 2007