Hilbert-Schmidt Weighted Composition Operator on the Fock space

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Abstract. In this paper we consider the weighted composition operator \( uC_\phi \) on the Fock space. We give necessary and sufficient conditions on \( u \) and \( \phi \) which will ensure that the weighted composition operator belongs to the Hilbert-Schmidt class. As a consequence of our main result, we completely determine the Hilbert-Schmidt composition operator on the Fock space.

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1. INTRODUCTION

Throughout this paper, let \( dm \) denote the usual Lebesgue measure on \( \mathbb{C} \). For \( \alpha > 0 \), the Fock space \( \mathcal{F}_\alpha^2 \) is the space of all entire functions \( f \) on \( \mathbb{C} \) for which

\[
\|f\|_{\mathcal{F}_\alpha^2}^2 = \frac{\alpha}{\pi} \int_{\mathbb{C}} |f(z)|^2 e^{-\alpha|z|^2} dm(z) < \infty.
\]

\( \mathcal{F}_\alpha^2 \) is a Hilbert space with an inner product

\[
\langle f, g \rangle_{\mathcal{F}_\alpha^2} = \frac{\alpha}{\pi} \int_{\mathbb{C}} f(z)\overline{g(z)}e^{-\alpha|z|^2} dm(z).
\]

Let \( e_n(z) = \sqrt{\frac{\alpha^n}{n!}} z^n \) for a positive integer \( n \). Then the set \( \{e_n\} \) forms an orthonormal basis for \( \mathcal{F}_\alpha^2 \). Since each point evaluation is a bounded linear functional on \( \mathcal{F}_\alpha^2 \), thus for \( z \in \mathbb{C} \) there exists a unique function \( K_z \in \mathcal{F}_\alpha^2 \) such that \( f(z) = \langle f, K_z \rangle_{\mathcal{F}_\alpha^2} \) for \( f \in \mathcal{F}_\alpha^2 \). That is \( K_z \) is the reproducing kernel

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function for $\mathcal{F}_2^\alpha$. Using the orthonormal basis $\{e_n\}$ for $\mathcal{F}_2^\alpha$, we can compute the explicit formula for $K_z$:

$$K_z(w) = \exp\{\alpha\langle w, z \rangle\}, \quad w \in \mathbb{C},$$

(see [6]). Here the notation $\langle \cdot, \cdot \rangle$ denotes the usual inner product on $\mathbb{C}$. Note that $\|K_z\|_{\mathcal{F}_2^\alpha} = \exp(\frac{\alpha}{2} |z|^2)$. We also use the normalized kernel function $k_z(w) = \exp\{\alpha\langle w, z \rangle - \frac{\alpha}{2} |z|^2\}$.

The problems for the Fock space, such as an interpolation sequence or a sampling set, have been studied by many authors. See, for example, [7, 9]. Moreover, several studies have been made on concrete operators, such as Toeplitz operators and Hankel operators. Some results on Toeplitz operators have been found by Grudsky, Vasilevski, Sangadji and Stroethoff ([4, 8, 10]). In [1, 5, 6, 10], these authors have dealt with Hankel operators and Hankel forms.

In this paper, our object is a weighted composition operator on $\mathcal{F}_2^\alpha$. Let $u$ and $\varphi$ be entire functions. The weighted composition operator $uC_\varphi$ is defined by $uC_\varphi f = u \cdot (f \circ \varphi)$ for an entire function $f$. Recently, the author have given a necessary and sufficient condition for the boundedness or compactness of the operator $uC_\varphi$ ([11]). The normalized kernel function $k_z$ played an important role in our characterization. To give a necessary and sufficient condition for $uC_\varphi$ belongs to the Hilbert-Schmidt class, we will also use the function $k_z$.

The following theorem is our main result.

**Theorem 1.** Suppose that $\varphi$ and $u$ are entire functions on $\mathbb{C}$ such that $uC_\varphi$ is bounded on $\mathcal{F}_2^\alpha$. Then the following conditions are equivalent:

(a) $uC_\varphi$ is a Hilbert-Schmidt operator,

(b) $\int |u(z)|^2 \exp\{\alpha\langle |\varphi(z)|^2 - |z|^2\}dm(z) < \infty$,

(c) $\int \left( \int |u(w)|^2 \exp\{\alpha\langle \varphi(w), z \rangle^2 - \frac{\alpha}{2} |w|^2\}dm(w) \right) dm(z) < \infty$.

2. **Proofs of main theorem**

In order to prove our result, we give a trace formula for positive operators on $\mathcal{F}_2^\alpha$. In the case that the usual Bergman space defined on classical domains of $\mathbb{C}$, this type formula is well-known. See, for example, Proposition 6.3.2 in [12].

**Lemma 1.** If $T$ is a positive or trace class operator on $\mathcal{F}_2^\alpha$, then the trace $tr(T)$ of $T$ is given by

$$tr(T) = \frac{\alpha}{\pi} \int_{\mathbb{C}} \langle Tk_z, k_z \rangle_{\mathcal{F}_2^\alpha} dm(z).$$
Proof. Fix an orthonormal set \( \{ \varphi_j \} \) for \( \mathcal{F}_a^2 \). Then we have

\[
tr(T) = \sum_{j=0}^{\infty} \langle T \varphi_j, \varphi_j \rangle_{\mathcal{F}_a^2} = \sum_{j=0}^{\infty} \frac{\alpha}{\pi} \int_{\mathbb{C}} T\varphi_j(z)\overline{\varphi_j(z)}e^{-\alpha|z|^2} dm(z)
\]

\[
= \sum_{j=0}^{\infty} \frac{\alpha}{\pi} \int_{\mathbb{C}} T\varphi_j, K_z \rangle_{\mathcal{F}_a^2} \overline{\varphi_j(z)}e^{-\alpha|z|^2} dm(z)
\]

\[
= \frac{\alpha}{\pi} \int_{\mathbb{C}} T \sum_{j=0}^{\infty} \varphi_j(z)\varphi_j, K_z \rangle_{\mathcal{F}_a^2} e^{-\alpha|z|^2} dm(z)
\]

\[
= \frac{\alpha}{\pi} \int_{\mathbb{C}} \langle TK_z, K_z \rangle_{\mathcal{F}_a^2} e^{-\alpha|z|^2} dm(z) = \frac{\alpha}{\pi} \int_{\mathbb{C}} \langle Tk_z, k_z \rangle_{\mathcal{F}_a^2} dm(z).
\]

Thus the lemma was proved.

Proofs of Theorem 1. First we show that (a) is equivalent to (b). Since the set \( \{ e_n \} \) is an orthonormal set for \( \mathcal{F}_a^2 \), we have

\[
\sum_{n=0}^{\infty} \| uC \varphi e_n \|_{\mathcal{F}_a^2}^2 = \sum_{n=0}^{\infty} \frac{\alpha}{\pi} \int_{\mathbb{C}} |u(z)|^2 \frac{\alpha^n}{n!} |\varphi(z)|^{2n} e^{-\alpha|z|^2} dm(z)
\]

\[
= \frac{\alpha}{\pi} \int_{\mathbb{C}} |u(z)|^2 e^{-\alpha|z|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} |\varphi(z)|^{2n} dm(z)
\]

\[
= \frac{\alpha}{\pi} \int_{\mathbb{C}} |u(z)|^2 e^{-\alpha|z|^2} e^{\alpha|\varphi(z)|^2} dm(z)
\]

\[
= \frac{\alpha}{\pi} \int_{\mathbb{C}} |u(z)|^2 \exp\{ |\varphi(z)|^2 - |z|^2 \} dm(z).
\]

Thus \( uC \varphi \) is Hilbert-Schmidt if and only if

\[
\frac{\alpha}{\pi} \int_{\mathbb{C}} |u(z)|^2 \exp\{ |\varphi(z)|^2 - |z|^2 \} dm(z) < \infty,
\]

and so we see that (a) and (b) are equivalent.

Next we prove that (a) and (c) are equivalent. According to a general theory of operators on the Hilbert space, we see that \( uC \varphi \) is a Hilbert-Schmidt operator if and only if \( (uC \varphi)^*uC \varphi \) is a trace class operator. Since \( (uC \varphi)^*uC \varphi \) is a positive operator on \( \mathcal{F}_a^2 \), then Lemma 1 implies that \( uC \varphi \) is Hilbert-Schmidt if and only if

\[
(1) \quad \frac{\alpha}{\pi} \int_{\mathbb{C}} \langle (uC \varphi)^*uC \varphi k_z, k_z \rangle_{\mathcal{F}_a^2} dm(z) < \infty.
\]

On the other hand, we obtain that for each \( z \in \mathbb{C} \)

\[
\langle (uC \varphi)^*uC \varphi k_z, k_z \rangle_{\mathcal{F}_a^2} = \langle uC \varphi k_z, uC \varphi k_z \rangle_{\mathcal{F}_a^2}
\]

\[
= \frac{\alpha}{\pi} \int_{\mathbb{C}} |u(w)|^2 \exp\alpha \{ |\varphi(w), z|^2 e^{\alpha(-|z|^2-|w|^2)} dm(w).
\]
Combining this with (1), we can conclude that $uC\phi$ belongs to the Hilbert-Schmidt class if and only if
\[
\frac{\alpha}{\pi} \int_{\mathbb{C}} \left( \frac{\alpha}{\pi} \int_{\mathbb{C}} |u(w)|^2 |\exp \alpha (\phi(w), z)^2 e^{\alpha(-|z|^2-|w|^2)} dm(w) \right) dm(z) < \infty.
\]
This completes the proof.

3. Applications

In this section, we concern two spatial cases of the operator $uC\phi$ on $\mathcal{F}_2^2$.

First we consider the case that $u$ is the constant function 1, that is the composition operator $C\phi$. Many mathematicians have studied about composition operators on various analytic function spaces. Recently Carswell, MacCluer and Schuster [3] completely determined the bounded or compact composition operator on $\mathcal{F}_2^2$ as follows:

**Proposition 1** (Carswell, MacCluer and Schuster [3]). Suppose that $\phi$ is an entire function.

(a) If $C\phi$ is bounded on $\mathcal{F}_2^2$, then $\phi(z) = az + b$, where $|a| \leq 1$, and if $|a| = 1$, then $b = 0$.

(b) If $C\phi$ is compact on $\mathcal{F}_2^2$, then $\phi(z) = az + b$, where $|a| < 1$.

Conversely, suppose that $\phi(z) = az + b$.

(c) If $|a| = 1$ and $b = 0$, then $C\phi$ is bounded on $\mathcal{F}_2^2$.

(d) If $|a| < 1$, then $C\phi$ is compact on $\mathcal{F}_2^2$.

As an easy corollary of Theorem 1 and Proposition 1 we get the following result.

**Corollary 1.** Suppose that $\phi(z) = az + b$ with $|a| \leq 1$ and $b \in \mathbb{C}$. Then the following conditions are equivalent:

(a) $C\phi$ is a compact operator,

(b) $C\phi$ is a Hilbert-Schmidt operator,

(c) $|a| < 1$.

**Proof.** (b) $\Rightarrow$ (a) is obvious. Proposition 1 have shown that (a) implies (c). So we only prove that (c) implies (b). Let $|a| < \beta < 1$ and $R = \frac{|b|}{\beta - |a|}$. For $z \in \mathbb{C}$ with $|z| > R$, we have $\beta |z| > |az + b| = |\phi(z)|$, and so
\[
\int_{|z| > R} \exp \alpha \{|\phi(z)|^2 - |z|^2\} dm(z) \leq \int_{|z| > R} \exp \alpha \{(|\beta^2 - 1)|z|^2\} dm(z) < \infty.
\]
On the other hand, we see that
\[
\int_{|z| \leq R} \exp \alpha \{|\phi(z)|^2 - |z|^2\} dm(z) \leq \max_{|z| \leq R} (\exp \alpha |\phi(z)|^2) \int_{|z| \leq R} \exp \alpha \{-|z|^2\} dm(z) < \infty.
\]
These inequalities and Theorem 1 show that $C_\phi$ belongs to the Hilbert-Schmidt class. This completes the proof.

Next we consider the case $\phi(z) = z$. In this case, the operator $uC_\phi$ is called the multiplication operator and denoted by $M_u$. Let $L^2(\mathbb{C}, d\mu)$ denote the Lebesgue space with respect to the measure

$$d\mu(z) = \frac{\alpha}{\pi} \exp\{-\alpha|z|^2\} dm(z).$$

Then the Fock space $\mathcal{F}_\alpha^2$ is a closed subspace of $L^2(\mathbb{C}, d\mu)$. For each $u \in L^\infty(\mathbb{C})$, the multiplication operator $M_u : \mathcal{F}_\alpha^2 \to L^2(\mathbb{C}, d\mu)$ is defined by $M_u f = uf$ for $f \in \mathcal{F}_\alpha^2$. This operator is closely related to the Toeplitz and Hankel operators on $\mathcal{F}_\alpha^2$. For these studies, refer to [1, 2, 5, 10]. For $g \in L^\infty(\mathbb{C})$ the Berezin symbol $\tilde{g}$ of $g$ is defined by

$$\tilde{g}(w) = \frac{\alpha}{\pi} \int_\mathbb{C} g(z) \exp\{-\alpha|w - z|^2\} dm(z), \quad w \in \mathbb{C}.$$ 

The Berezin symbol is used for analyzing the properties of these operators. For instance, K. Stroethoff characterized the compactness of $M_u$ in terms of the Berezin symbol $\tilde{|u|^2}$ as follows:

**Proposition 2** (K. Stroethoff [10]). Let $u \in L^\infty(\mathbb{C})$. The following statements are equivalent:

(a) $M_u$ is compact,
(b) $\tilde{|u|^2}(w) \to 0$ as $|w| \to \infty$.

As another application of Lemma 1 we get a necessary and sufficient condition for $M_u : \mathcal{F}_\alpha^2 \to L^2(\mathbb{C}, d\mu)$ is a Hilbert-Schmidt operator.

**Corollary 2.** Let $u \in L^\infty(\mathbb{C})$. The following statements are equivalent:

(a) $M_u$ is a Hilbert-Schmidt operator,
(b) $\tilde{|u|^2} \in L^1(\mathbb{C})$.

**Proof.** For $w \in \mathbb{C}$ we have

$$\tilde{|u|^2}(w) = \frac{\alpha}{\pi} \int_\mathbb{C} |u(z)|^2 \exp (2\alpha \Re(z, w)) \exp \alpha\{-|z|^2 - |w|^2\} dm(z)$$

$$= \frac{\alpha}{\pi} \int_\mathbb{C} |u(z)|^2 |k_w(z)|^2 e^{-\alpha|z|^2} dm(z) = \langle M_{|u|^2} k_w, k_w \rangle_{\mathcal{F}_\alpha^2}.$$ 

(2)

This implies that $\tilde{|u|^2}$ belongs to $L^1(\mathbb{C})$ if and only if

$$\int_\mathbb{C} \langle M_{|u|^2} k_w, k_w \rangle_{\mathcal{F}_\alpha^2} dm(w) < \infty.$$

The above equation (2) implies that the Berezin symbol $\tilde{|u|^2}$ is the integrand which appear in the trace formula for $M_{|u|^2}$. Since $M_{|u|^2} = M_u^* M_u$ on $\mathcal{F}_\alpha^2$ and
these are positive operators, thus Lemma 1 shows that

\[ M_u \text{ is in the Hilbert-Schmidt class } \iff M_u^*M_u \text{ is in the trace class } \iff |u|^2 \in L^1(\mathbb{C}), \]

which was to be proved. \(\square\)

REFERENCES


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