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Some Results on b-Orthogonality in 2-Normed Linear Spaces

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Abstract

We shall consider b-orthogonality in the 2-normed linear spaces, and shall give some results in this filed. Also, we shall give some characterizations about 2-normed in approximation theory.

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1 Introduction

The concept of linear 2-normed spaces has been investigated by S. Gähler and has been developed extensively in different subjects by many authors (see [1-6]).

Let X be a linear space of dimension greater than 1. Suppose $\|.,.\|$ is a real-valued function on $X \times X$ satisfying the following conditions:
a) $\|x, y\| = 0$ if and only if x and y are linearly dependent vectors.
b) $\|x, y\| = \|y, x\|$ for all $x, y \in X$.
c) $\|\lambda x, y\| = |\lambda|\|x, y\|$ for all $\lambda \in \mathbf{R}$ and all $x, y \in X$.
d) $\|x + y, z\| \leq \|x, z\| + \|y, z\|$ for all $x, y, z \in X$.
Then $\|.,.\|$ is called a 2-norm on X and $(X, \|.,.\|)$ is called a linear 2-normed space. Some of the basic properties of 2-norms, that they are non-negative and $\|x, y + \alpha x\| = \|x, y\|$ for all $x, y \in X$ and all $\alpha \in \mathbf{R}$.

Every 2-normed space is a locally convex topological vector space. In fact for a fixed $b \in X$, $p_b(x) = \|x, b\|$, $x \in X$, is a seminorm and the family $P = \{p_b : \ b \in X\}$ of seminorms generates a locally convex topology on X.

Let $(X, \|.,.\|)$ be a 2-normed space and let W_1 and W_2 be two subspaces of X. A map $f : W_1 \times W_2 \to \mathbf{R}$ is called a bilinear 2-functional on $W_1 \times W_2$ whenever for all $x_1, x_2 \in W_1, y_1, y_2 \in W_2$ and all $\lambda_1, \lambda_2 \in \mathbf{R}$;
(i) $f(x_1 + x_2, y_1 + y_2) = f(x_1, y_1) + f(x_1, y_2) + f(x_2, y_1) + f(x_2, y_2)$,
(ii) $f(\lambda_1 x_1, \lambda_2 y_1) = \lambda_1 \lambda_2 f(x_1, y_1)$.

A bilinear 2-functional $f : W_1 \times W_2 \to \mathbf{R}$ is called bounded if there exists a non-negative real number M (called a Lipschitz constant for f) such that $|f(x, y)| \leq M\|x, y\|$ for all $x \in W_1$ and all $y \in W_2$. Also, the norm of a bilinear 2-functional f is defined by

$$\|f\| = \inf\{M \geq 0 : \ M \textit{ is a Lipschitz constant for } f\}.$$

It is known that ([4])

$$\begin{aligned} \|f\| &= \sup\{|f(x,y)| : \ (x,y) \in W_1 \times W_2, \ \|x,y\| \leq 1\} \\ &= \sup\{|f(x,y)| : \ (x,y) \in W_1 \times W_2, \ \|x,y\| = 1\} \\ &= \sup\{|f(x,y)|/\|x,y\| : \ (x,y) \in W_1 \times W_2, \ \|x,y\| > 0\}. \end{aligned}$$

For a 2-normed space $(X, \|.,.\|)$ and $0 \neq b \in X$, we denote by X_b^* the Banach space of all bounded bilinear 2-functionals on $X\times < b >$, where $< b >$ be the subspace of X generated by b.

Definition 1.1. *Let $(X, \|.,.\|)$ be a 2-normed space, $x, y \in X$. If there exists $b \in X$ such that $\|x, b\| \neq 0$ and $\|x, b\| \leq \|x + \alpha y, b\|$ for all scalar $\alpha \in \boldsymbol{R}$. Then x is b-orthogonal to y and denoted by $x \perp^b y$.*

If W_1 and W_2 are subsets of X, if there exists $b \in X$ such that for all $y_1 \in W_1$, $y_2 \in W_2$, $y_1 \perp^b y_2$, then we say that $W_1 \perp^b W_2$.

Let $(X, \|.,.\|)$ be a 2-normed space, W be a linear subspace of X and $b \in X$. $w_0 \in W$ is b-best approximation for $x \in X$, if $x - w_0 \perp^b W$. The set of all b-best approximations of x in W is denoted by $P_W^b(x)$. Also W is called b-proximinal if for every $x \in X \backslash (W+ < b >)$, there exists $w_0 \in W$ such that $w_0 \in P_W^b(x)$. (see [10])

The following basic lemma is important in the proof of main results.

Proposition 1.2 (3]) *Let $(X, \|.,.\|)$ be a 2-normed space,W be a subspace of X, $b \in X$ and let $< b >$ be the subspace of X generated by b. If $x_0 \in X$ is such that*

$$\delta = \inf\{\|x_0 - w, b\| : \ w \in W\} > 0,$$

then there exists a bounded bilinear functional $F : \ X\times < b > \longrightarrow \boldsymbol{R}$ such that $F|_W\times < b >= 0$, $F(x_0, b) = 1$ and $\|F\| = \frac{1}{\delta}$.

2 Main results

In this section we shall obtain characterization of b-orthogonality in 2-normed spaces.
Let X be a 2-normed linear space. For $F \subseteq X$, put

$$M_F^b = \{f \in X_b^* : \ \|f\| = 1, \ f(x,b) = \|x,b\|, \ \forall x \in F\}.$$

Theorem 2.1. *Let X be a 2-normed linear space, $b \in X$, $y \in X$ and $x \in X \backslash (< b >)$. Then the following statements are equivalent:*
1) $x \perp^b y$.
2) There exists $f \in M_x^b$ such that $f(y,b) = 0$.

Proof. 2) $\longrightarrow$ 1). First suppose there exists $f \in M_x^b$ such that $f(y,b) = 0$. Then

$$\|x,b\| = f(x,b) = f(x+\alpha y, b) \leq \|f\|\|x+\alpha y, b\| = \|x+\alpha y, b\|.$$

Therefore $x \perp^b y$.
1) $\longrightarrow$ 2). Suppose $x \perp^b y$ and $W =< y >$ is the subspace of X generated by y. Then

$$\inf\{\|x-\alpha y, b\| : \ \alpha y \in W\} \geq \|x,b\| > 0.$$

From Proposition 1.2, there exists $g \in X_b^*$ such that $g(y,b) = 0$, $g(x,b) = 1$ and $\|g\| = \frac{1}{\delta}$. Put $f = \delta g$, then $f(y,b) = 0$, $f(x,b) = \|x,b\|$ and $\|f\| = 1$. ■

Corollary 2.2. *Let X be a 2-normed linear space. Let W be a linear subspace of X, $b \in X$ and $F \subseteq X \backslash (W + < b >)$. Then the following statements are equivalent:*
1) $F \perp^b W$.
2) There exists $f \in M_F^b$ such that $f|_{W \times <b>} = 0$.

Corollary 2.3. *Let X be a 2-normed linear space. Let W be a linear subspace of X and $b \in X \backslash W$. Then the following statements are equivalent:*
1) $W \perp^b x$.
2) For all $g \in W$ there exists $f \in M_g^b$ such that $f(x,b) = 0$.

Let X be a 2-normed space, W be a subspace of X and $b \in X \backslash W$. put

$$\breve{W}_b = \{x \in X : \ \ W \perp^b x\}.$$

Theorem 2.4. *Let X be a 2-normed linear space. Let W be a linear subspace of X, $b \in X \backslash W$. If $\breve{W}_b$ is a subspace. Then the following statements are equivalent:*
1) $W \perp^b F$.
2) For all $w \in W$, there exists $f \in M_w^b$ such that $f|_{F \times <b>} = 0$.

Proof. 2) $\longrightarrow$ 1). Since for all $w \in W$, $w\perp^b \breve{W}_b$, Therefore $0 \in P^b_{\breve{W}_b}(w)$. From Theorem 2.1, there exists $f \in M^b_w$ such that $f|_{\breve{W}_b\times<b>} = 0$. Since $F \subseteq \breve{W}_b$, it follows that $f|_{F\times<b>} = 0$.
2) $\longrightarrow$ 1). Suppose for all $w \in W$, there exists $f \in M^b_w$ such that $f|_{F\times<b>} = 0$. For each $w \in W$ and each $x \in F$, we have

$$\|w, b\| = f(w, b) = f(w + \alpha x, b) \leq \|f\|\|w + \alpha x, b\| = \|w + \alpha x, b\|,$$

for all scalar α. Therefore $W\perp^b F$. ■

Example 2.5. Let $X = \mathbf{R}^3$, $W = \{(0, x, x) : \ x \in \mathbf{R}$ and

$$\begin{aligned} &\| \ \ (x_1, x_2, x_3), (y_1, y_2, y_3) \ \ \| \\ = \ \ &\max\{|x_1y_2 - x_2y_1| + |x_1y_3 - x_3y_1|, |x_1y_2 - x_2y_1| + |x_2y_3 - x_3y_2|\}, \end{aligned}$$

For all $(x_1, x_2, x_3), (y_1, y_2, y_3) \in X$. Then $\|., .\|$ is a 2-norm on X. If $x = (1, 0, 1)$ and $b = (2, 2, 0)$, It is clear that $x\perp^b W$.

At first we state the following lemma of Hahn- Banach which is needed in the proof of the main results.

A mapping $p : \ X \longrightarrow \mathbf{R}$, where X is a vector space, is called a sublinear mapping if it satisfies the following two properties:
a. $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$; and
b. $p(\alpha x) = \alpha p(x)$ for all $x \in X$ and $\alpha \geq 0$.

Lemma 2.6. *Let p be a sublinear mapping on a vector space X, and let Y be a vector subspace of X. If f is a linear functional on M such that;*

$$f(x) \leq p(x) \quad (x \in M).$$

Then f can be extended to a linear functional g on all of X satisfying

$$g(x) \leq p(x) \quad (x \in X).$$

Theorem 2.7. *Let X be a 2-normed linear space, $b \in X$ and $x, y \in X\backslash(< b >)$. Then there exists a number a such that $x\perp^b ax + y$.*

Proof. Suppose $M =< x >$ linear subspace of X spanned by x. Define $f_b : \ M \longrightarrow \mathbf{R}$ by $f_b(\alpha x) = \alpha\|x, b\|$. Since $p_b(x) = \|x, b\|$ is a sublinear and $f_b(z) \leq p_b(z)$. From Lemma 2.6, there is a linear functional $\Lambda_b : \ X \longrightarrow \mathbf{R}$ such that $\Lambda_b(x) = p_b(x)$ and for all $z \in X$, $\Lambda_b(z) \leq \|z, b\|$. By definition of bilinear 2-functional $\|\Lambda_b\| = 1$. For y define $a = -\frac{\Lambda_b(y)}{\Lambda_b(x)}$, therefore $\Lambda_b(ax + y) = 0$ and $x\perp^b ax + y$. ■

Theorem 2.8. *Let X be a 2-normed linear space, $b \in X$ and $x, y \in X \backslash (< b >)$. Then there exists a number a such that $ax + y \perp^b x$.*

Proof. By definition, $ax + y \perp x$ if and only if $\|ax + y + kx, b\| \geq \|ax + y, b\|$ for all k, or if and only if $\|ax + y\|$ is the smallest value of $\|kx + y, b\|$. Since $\|kx + y, b\|$ is continuous in k, it must take on its minimum. This number a is a value of k for which $\|kx + y, b\|$ takes on its absolute minimum. ■

Definition 2.9. *Let X be a 2-normed linear space. We say that X is a 2-smooth if for any $x \neq 0$ and every $b \in X$ such that $\|x, b\| \neq 0$, there is a unique linear functional Λ_b such that*

$$\Lambda_b(x) = \|x, b\|, \quad \|\Lambda_b\| = 1$$

Theorem 2.10. Let X be a 2-normed linear space. The space X is 2-smooth if and only if for all $x, y \in X$ and every $b \in X$ such that $\|x, b\| \neq 0$, there is a unique number a such that $x \perp^b ax + y$.

Proof. Suppose X is a 2-smooth, that is, for $x \in X$ and every $b \in X$ such that $\|x, b\| \neq 0$, there is a unique linear functional f_b with $\|f_b\| = 1$ and $f_b(x) = \|x, b\|$. If for $x, y \in X$ and numbers a, c, $x \perp^b ax+y$ and $x \perp^b cx+y$, from Lemma 2.7, there are linear functionals f_b, g_b such that $f_b(x) = g_b(x) = \|x, b\|$, $\|f_b\| = \|g_b\| = 1$ and $f_b(ax+y) = g_b(cx+y) = 0$. Since $f_b = g_b$ therefore $a = c$.

Conversely, suppose for all $x, y \in X$, there is a unique number a such that $x \perp^b ax + y$. Suppose for $x \in X$ and $b \in X$ such that $\|x, b\| \neq 0$, there are linear functionals f_b and g_b such that $\|f_b\| = 1$, $\|g_b\| = 1$, $f_b(x) = \|x, b\|$, $g_b(x) = \|x, b\|$. If for some $y \in X$, $f_b(y) \neq g_b(y)$. Put $a = -\frac{f_b(y)}{\|x,b\|}$ and $c = -\frac{g_b(y)}{\|x,b\|}$, From Theorem 2.7, $x \perp^b ax + y$ and $x \perp^b cx + y$. this is a contradict. Therefore X is 2-smooth. ■

Definition 2.11. Let X be a 2-normed linear space, $x, y, b \in X$ and $\|x, b\| \neq 0$. The orthogonality is called additivity, if $x \perp^b y$ and $x \perp^b z$, then $x \perp^b y + z$.

Theorem 2.12. *Let X be a 2-normed linear space, $b \in X$ and $x, y \in X \backslash (< b >)$. The orthogonality is additive if and only if for all $x, y \in X$ and every $b \in X$ such that $\|x, b\| \neq 0$, there is a unique number a such that $x \perp^b ax + y$.*

Proof. Suppose for all $x, y \in X$, and every $b \in X$ such that $\|x, b\| \neq 0$, there is a unique number a such that $x \perp^b ax + y$. From Theorem 2.10, X is 2-smooth. If $x \perp^b y$ and $x \perp^b z$, there is a unique linear functional f_b such that $f_b(x) = \|x, b\|$ and $\|f_b\| = 1$. Consider the hyperplane

$$H_b = \{u \in X : \ f_b(u) = 0\}.$$

It is clear that $y, z \in H_b$ and H_b is a subspace. Therefore $y + z \in H_b$ and $x \perp^b y + z$.

Conversely, suppose orthogonality is additive and $x \perp^b ax + y$ and $x \perp^b cx + y$. Then $x \perp^b -(cx + y)$ and additivity gives $x \perp^b (a - c)x$. But this means that $\|x + k(a - c)x\| \geq \|x\|$, which is true for all k only if $a = c$. Therefore there is a unique number a such that $x \perp^b ax + y$. ■

Let X be a linear space of dimension greater then 1 over the filed $\mathbf{K} = \mathbf{R}$ of real numbers or the filed $\mathbf{K} = \mathbf{C}$ of complex numbers. Suppose that $(.,.|.)$ is a $\mathbf{K}$-valued function defined on $X \times X \times X$ satisfying the following conditions:
a) $(x, x|z) \geq 0$ and $(x, x|z) = 0$ if and only if x and z are linearly dependent;
b) $(x, x|z) = \overline{(z, z|x)}$;
c) $(y, x|z) = \overline{(x, y|z)}$;
d) $(\alpha x, y|z) = \alpha(x, y|z)$ for any scalar $\alpha \in \mathbf{K}$;
e) $(x + x', y|z) = (x, y|z) + (x', y|z)$.
$(.,.|.)$ is called a 2-inner product on X and $(X, (.,.|.))$ is called a 2-inner product space. some basic properties of 2-inner products $(.,.|.)$ can be immediately obtained in [1-3].

Let $(X, (.,.|.))$ be a 2-inner product space. We can define a 2-norm on $X \times X$ by

$$\|x, y\| = \sqrt{(x, x|y)}.$$

Lemma 2.13. *Let $(X, (.,.|.))$ be a 2-inner product space, $b \in X$ and $x, y \in X \backslash (< b >)$. Then*

$$x \perp^b y \Leftrightarrow (x, y|b) = 0.$$

Proof. If $x \perp^b y$ and $(x, y|b) \neq 0$, we define $\alpha = -\frac{(x,y|b)}{(y,y|b)}$, we have

$$\begin{aligned}
\|x + \alpha y, b\|^2 &= (x + \alpha y, x + \alpha y|b) \\
&= (x, x + \alpha y|b) + \alpha(x, x + \alpha y|b) \\
&= \overline{(x + \alpha y, y|b)} + \overline{(x + \alpha y, y|b)} \\
&= \overline{(x, x|b) + \alpha(y, x|b)} + \overline{\alpha(x, y|b) + \alpha\overline{\alpha}(y, y|b)} \\
&= (x, x|b) + \overline{\alpha}(x, y|b) + \alpha\overline{(x, y|b)} + |\alpha|^2(x, y|b) \\
&= \|x, b\|^2 - \frac{|(x, y|b)|^2}{\|x, b\|^2} \\
&< \|x, b\|^2.
\end{aligned}$$

Therefore x is not b-orthogonal to y. Therefore $(x, y|b) = 0$.
Conversely, $(x, y|b) = 0$, then for any scalar α we have

$$\begin{aligned}
\|x + \alpha y, b\|^2 &= (x + \alpha y, x + \alpha y|b) \\
&= \|x, b\| + |\alpha|^2\|y, b\|^2 \\
&\geq \|x, b\|^2.
\end{aligned}$$

Therefore $\|x, b\| \leq \|x + \alpha y, b\|$. ■

Let $(X, (.,.|.))$ be a 2-inner product space, W be a subspace of X and $b \in X$. Put

$$W_b^{\perp} = \{x \in X : \ (x, g|b) = 0, \forall g \in G\}.$$

Corollary 2.14. *Let $(X, (.,.|.))$ be a 2-inner product space, W be a subspace of X and $b \in X$. Then*

$$W_b^{\perp} = \breve{W}_b.$$

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