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ϵ-Orthogonality in Vector Spaces

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Abstract

The purpose of this paper is to introduce and to discuss the concept of ϵ-orthogonality in the vector spaces, and obtaining some results on ϵ-orthogonality in vector spaces similar to ϵ-orthogonality of normed spaces. We shall obtain some characterizations of the ϵ-best approximation and the ϵ-best coapproximation in vector spaces.

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1. Introduction

Suppose that X is a vector space over $\phi \in \{\mathbf{C}, \mathbf{R}\}$. A seminorm is a function $p: \ X \to [0, \infty)$ having the following properties:
a) $p(x+y) \leq p(x) + p(y) \ \ \forall x, y \in X$;
b) $p(\alpha x) = |\alpha| p(x), \ \forall \alpha \in \phi$ and $\forall x \in X$.
It follows from b) that $p(0) = 0$. A norm is a seminorm p such that
c) $x = 0 \ if \ p(x) = 0$.
Many authors have introduced the concept of orthogonality in different ways (see [1-3], [5-6]). In [1] Brikhoff modified the concept of orthogonality in the normed spaces. Let X be a normed linear space and $x, y \in X$, $\epsilon > 0$ be given, x is said to be ϵ-orthogonal to y and is denoted by $x \perp_\epsilon y$ if and only if $\|x\| \leq \|x + \alpha y\| + \epsilon$ for all scalar α (see [6]). In this note, we shall define ϵ-orthogonality in the vector spaces.

Definition 1.1. Let X be a vector space, and $0 \neq x, y \in X$. We say that x is ϵ-orthogonal to y if there exists a seminorm p on X such that

$$p(x) \neq 0, \ \ p(x) \leq \inf{}_{\alpha} p(x + \alpha y) + \epsilon,$$

in this case we write $x\perp_\epsilon^p y$. If M_1 and M_2 are subsets of X, we say that M_1 is ϵ-orthogonal to M_2 if there exists a seminorm p on X such that $g_1\perp_\epsilon^p g_2$ for all $g_1 \in M_1,\ g_2 \in M_2$. If M_1 is ϵ-orthogonal to M_2, we write $M_1\perp_\epsilon M_2$ with respect to p (abbreviated by w.r.t. p).
Suppose p is a seminorm on X. If $x \in X$, put

$$M_{x,\epsilon}^p = \{\Lambda :\ X \overset{linear}{\rightarrow} \phi :\ |\Lambda(z)| \leq p(z), \forall z \in X,\ \Lambda(x) \geq p(x) - \epsilon\}.$$

At first we state the following lemma of Hahn- Banach which is needed in the proof of the main results.

Lemma 1.2. *Let M be a subspace of a vector space X, p be a seminorm on X, and let f be a linear functional on M such that;*

$$|f(x)| \leq p(x) \quad (x \in M).$$

Then f extends to a linear functional Λ on X that satisfies

$$|\Lambda(x)| \leq p(x) \quad (x \in X).$$

2. Main Results

In this section we state and prove our main results.

Theorem 2.1. *Let X be a vector space, G be a subspace of X, $\epsilon > 0$ be given, p be a seminorm on X, $x \in X\backslash\bar{G}$ and $p(x) \neq 0$. Then the following statements are equivalent:*
(a) $x\perp_\epsilon G$ w.r.t. p
(b) There exists a linear functional Λ on X such that $\Lambda \in M_{x,\epsilon}^p$ and $\Lambda|_G = 0$.

Proof. $(a) \Rightarrow (b)$. Suppose $x\perp_\epsilon G$ w.r.t. p. Consider $M =< x > \bigoplus G$. We define a linear functional f on M by $f(\alpha x + g) = \alpha p(x)$. It is clear that $f(g) = 0$ for every $g \in G$ and $f(x) = p(x)$. Now suppose $z = \alpha x + g \in M$. Therefore

$$\begin{aligned} |f(z)| &= |f(\alpha x + g)| \\ &= |\alpha| p(x) \\ &\leq |\alpha| p(x + \frac{1}{\alpha} g) \\ &= p(\alpha x + g) \\ &= p(z). \end{aligned}$$

From Lemma 1.2, there exists a linear functional Λ on X with the conditions

$$\Lambda(x) = p(x),\ \Lambda|_G = 0,\ |\Lambda(z)| \leq p(z)\ for\ all\ z \in X.$$

Therefore

$$\Lambda \in M^p_{x,\epsilon},\ \Lambda|_G = 0.$$

$(b) \Rightarrow (a)$. Suppose that there exists a linear functional Λ on X such that $\Lambda \in M^P_{x,\epsilon}$ and $\Lambda|_G = 0$. For every $\alpha \in \phi$ and $g \in G$, we have

$$\epsilon + p(x+\alpha g) \geq \epsilon + |\Lambda(x+\alpha g)| = \epsilon + |\Lambda(x)| \geq p(x).$$

Therefore $\epsilon + \inf{}_\alpha p(x+\alpha g) \geq p(x)$, then $x \perp^p_\epsilon g$. Therefore $x \perp_\epsilon G$ w.r.t. p. $\blacksquare$

Corollary 2.2. *Let X be a vector space, $\epsilon > 0$ be given, p is a seminorm on X and $x, y \in X$ and $p(x) \neq 0$. Then the following two conditions are equivalent:*
(a) $x \perp^p_\epsilon y$
(b)There exists a linear functional Λ on X such that, $\Lambda \in M^p_{x,\epsilon}$ and $\Lambda(y) = 0$.

Lemma 2.3. *Let X be a vector space and p be a seminorm on X. Then the following statements are true:*
(a) If $x, y \in X$ and for all $\epsilon > 0$ we have $x \perp^p_\epsilon y$ then $< x > \cap < y >= \{0\}$.
(b) If $\epsilon > 0$, $x, y \in X$ and $x \perp^p_\epsilon y$, then $x \perp^p_\delta y$ for all $\delta > \epsilon$.
(c) If $x \perp^p_\epsilon y$ and $|\beta| \leq 1$ then $\beta x \perp^p_\epsilon \beta y$

Proof. The parts (b) and (c) are easily proved, therefore we prove only (a). Suppose $z \in < x > \cap < y >$. Then $z = \alpha x = \beta y$ for some scalars $\alpha, \beta \in \phi$. If $\alpha = 0$, then $z = 0$. In the otherwise, suppose $\alpha \neq 0$. From Corollary 2.2, there exists a linear functional Λ on X such that, $\Lambda \in {M^p}_{x,\epsilon}$ and $\Lambda(y) = 0$. Therefore $\Lambda(z) = 0$, it follows that for all $\epsilon > 0$, $p(x) \leq \epsilon$. This is a contradiction with $p(x) \neq 0$. $\blacksquare$

Corollary 2.4. *Let X be a vector space, A be a nonempty subset of X, $\epsilon > 0$ be given, p be a seminorm on X such that $p(g) \neq 0$ for all $g \in A$ and $x \in X \backslash < A >$. Then the following two conditions are equivalent:*
(a) $A \perp_\epsilon x$ w.r.t. p
(b) For every $g \in A$ there exists a linear functional Λ on X with $\Lambda \in {M^p}_{g,\epsilon}$ and $\Lambda(x) = 0$.

In the normed linear spaces, the concepts of ϵ-best approximation and ϵ-best coapproximation have been defined. (see [5-6]) we shall define these concepts for the vector spaces.

Let G be a subspace of the vector space X and $\epsilon > 0$ be given. For any seminorm p, we will define

$$\begin{aligned} \hat{G}^p_\epsilon &= \{x \in X : \ \ p(x) \neq 0,\ \ p(x) \leq \inf{}_{g \in G} p(x-g) + \epsilon\} \\ &= \{x \in X : \ \ x \perp_\epsilon G \ w.r.t.\ p\}, \end{aligned}$$

and

$$\breve{G}^p_\epsilon = \{x \in X : \ \ G \perp_\epsilon x \ w.r.t. \ p\},$$

also a point $g_0 \in G$ is said to be a ϵ-best approximation (resp. ϵ-best coapproximation) for $x \in X$ if and only if $x - g_0 \in \hat{G}^p_\epsilon$ $(resp. \ x - g_0 \in \breve{G}^p_\epsilon)$.
The set of all ϵ-best approximations (resp. ϵ-best coapproximations) of $x \in X$ in G is shown by $P^p_{G,\epsilon}(x)$ $(resp. \ R^p_{G,\epsilon}(x))$. In other words

$$P^p_{G,\epsilon}(x) = \{g_0 \in G \ : \ x - g_0 \in \hat{G}^p_\epsilon\}$$

and

$$R^p_{G,\epsilon}(x) = \{g_0 \in G \ : \ x - g_0 \in \breve{G}^p_\epsilon\}.$$

Corollary 2.5. *Let X be a vector space, G be a subspace of X, ϵ be given, p be a seminorm on X, $x \in X \backslash \bar{G}$ and $p(x) \neq 0$. Then the following statements are equivalent:*
(a) $g_0 \in P^p_{G,\epsilon}(x)$
(b) There exists a linear functional Λ on X such that $\Lambda \in M^p{}_{x-g_0,\epsilon}$ and $\Lambda|_G = 0$.

Corollary 2.6. *Let X be a vector space, G be a subspace of X, ϵ be given, p be a seminorm on X such that $p(g) \neq 0$ for all $g \in G$ and $x \in X \backslash \bar{G}$. Then the following statements are equivalent:*
(a) $g_0 \in R^p_{G,\epsilon}(x)$
(b) For all $g \in G$ there exists a linear functional Λ on X such that $\Lambda \in M^p{}_{g,\epsilon}$ and $\Lambda(x - g_0) = 0$.

Lemma 2.7. *Let X be a vector space, G be a subspace of X, ϵ be given and p be a seminorm on X. Then*
(a) $G \cap \hat{G}^p_\epsilon = \{g \in G : \ p(g) \leq \epsilon\}$
(b) $G \cap \breve{G}^p_\epsilon = \{g \in G : \ p(g) \leq \epsilon\}$.
(c) For all $x \in X$, $P^p_{G,\epsilon}(x) = G \cap (x - \hat{G}^p_\epsilon)$.
(d) For all $x \in X$, $R^p_{G,\epsilon}(x) = G \cap (x - \breve{G}^p_\epsilon)$.

Proof. The parts (c) and (d) are consequences of definition of ϵ-orthogonality. Suppose $x \in G \cap \hat{G}^p_\epsilon$ (resp. $x \in G \cap \breve{G}^p_\epsilon$), then $x \perp_\epsilon G$ w.r.t. p (resp. $G \perp_\epsilon x$ w.r.t. p) and $x \in G$. Therefore $x \perp^p_\epsilon x$, it follows that $p(x) \leq \epsilon$. ■

Corollary 2.8. *Let X be a vector space, G be a subspace of X, ϵ be given, p be a seminorm on X, $x \in X \backslash \bar{G}$ and $p(x) \neq 0$. Then*

$$R^p_{G,\epsilon}(x) = \bigcup_{y \in G} \bigcap_{g \in G} (P^p_{[x,y],\epsilon}(x) \cap G)$$

Whenever $[x, y] = \{\alpha x + (1 - \alpha) y : \ |\alpha| \leq 1\}$.

Proof. Since

$$\begin{aligned} y \in R^p_{G,\epsilon}(x) &\iff \forall g \in G:\ g \perp^p_\epsilon x - y \\ &\iff \forall g \in G:\ g - y \perp^p_\epsilon x - y,\ y \in G \end{aligned}$$

Also

$$g - y \perp^p_\epsilon x - y \iff p(g - y) \leq inf_\alpha p(g - (\alpha x + (1-\alpha)y)) + \epsilon$$

Therefore

$$y \in R^p_{G,\epsilon}(x) \iff y \in \bigcup_{y \in G} \bigcap_{g \in G} (P^p_{[x,y],\epsilon}(x) \cap G). \blacksquare$$

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