Int. Journal of Math. Analysis, Vol. 1, 2007, no. 16, 791 - 804

Oscillation and Nonoscillation of Higher Order Neutral Difference Equations with "Maxima"

Xiaopei Li[1]

Department of Mathematics
Zhanjiang Normal University
Zhanjiang, Guangdong 524048, P. R. China
Lixp27333@sina.com

Xiaoliang Zhou

Department of Mathematics
Guangdong Ocean University
Zhanjiang, Guangdong 524088, P.R. China
zjhdzxl@yahoo.com.cn

Abstract

In this paper, we study the oscillation and nonoscillation of higher order neutral difference equations with "maxima" of the form

$$\Delta^m (x_n - p_n x_{n-k}) + q_n \max_{s \in [n-l,n]} x_n = 0$$

where $[n-l,n] = \{n-l, n-l+1, \cdots, n\}$, l, k are positive integer, $\{p_n\}$ and $\{q_n\}$ are nonnegative real sequences. We obtain some sufficient conditions for all solutions to be oscillatory and for all nonoscillatory solutions to be asymptotic.

Mathematics Subject Classification: 34C10

[1] Supported by Guangdong provincial natural science Foundation.

Keywords: Neutral difference equation, Oscillation, Asymptotic behavior, Maxima

1 Introduction

Consider the higher order neutral difference equations of the form

$$\Delta^m(x_n - p_n x_{n-k}) + q_n \max_{s\in[n-l,n]} x_n = 0 \tag{1}$$

where $[n-l,n] = \{n-l, n-l+1, \cdots, n\}$, $n \in N(n_0) = \{n_0, n_0+1, \cdots\}$ (n_0 is a fixed positive integer), l, k are positive integer, $\{p_n\}$ and $\{q_n\}$ are nonnegative real sequences, and m is a positive integer. The oscillatory and asymptotic behaviors of equation (1) as $m = 1$ and $m = 2$ had been studied in [1] and [2]. But, almost the results are based on the hypothesis

$$\sum^{\infty} q_n = \infty. \tag{2}$$

As we know, there are seldom articles concerning the property of the equation (1) under the reverse hypothesis as follows

$$\sum^{\infty} q_n < \infty. \tag{3}$$

The purpose of this paper is to study the equation (1) under the hypothesis (3). Some sufficient conditions for all solutions of equation (1) to be oscillatory and for all nonoscillatory solutions of equation (1) to be asymptotic are obtained.

Throughout the paper, the term "solution" of (1) is always used for such real sequences $\{x_n\}$ that satisfy (1) for all $n \in N(n_0)$ and for which $sup\{|x_n| : n \geq s\} > 0$ for any $s \in N(n_0)$. A solution of (1) is called nonoscillatory if it is eventually positive or negative. Otherwise it is called oscillatory. Equation (1) is called oscillatory if all its solutions are oscillatory.

In order to discuss the properties of equation (1) conveniently, we always assume all inequalities are correct for all sufficiently large n.

2 The Main Results

To prove our main results we need the following lemmas which are in [4] and [5].

LEMMA 2.1 (*see*[4]) *Let y_n be defined for $n \geq n_0$ and $y_n > 0$ with $\triangle^m y_n$ of constant sign for $n \geq n_0$ and not identically zero. Then, there exists an integer i, $0 \leq i \leq m$ with $(i+m)$ odd for $\triangle^m y_n \leq 0$ and $(i+m)$ even for $\triangle^m y_n \geq 0$ such that*

(i) $i \leq m-1$ implies $(-1)^{i+j} \triangle^j y_n > 0$ for all $n \geq n_0$, $i \leq j \leq m-1$,

(ii) $i \geq 1$ implies $\triangle^j y_n > 0$ for all $n \geq n_0$, $0 \leq j \leq i-1$.

LEMMA 2.2 (*see*[5]) *Let y_n be defined for $n \geq n_0$ and $y_n > 0$ with $\Delta^m y_n \leq 0$ for $n \geq n_0$ and not identically zero. Then, there exists a large integer $n_1 \geq n_0$ such that*

$$y_n \geq \frac{1}{(m-1)!}(n-n_1)^{m-1}\Delta^{m-1} y_{2^{m-i-1}n}, \qquad n \geq n_1$$

where i is defined as in Lemma 2.1. Further, if y_n is increasing, then

$$y_n \geq \frac{2^{2(1-m)}}{(m-1)!} n^{m-1}\Delta^{m-1} y_n, \qquad n \geq 2^{m-1}n_1$$

Let $\{x_n\}$ be a solution of equation (1) and define a sequence $\{y_n\}$ as follows

$$y_n = x_n - p_n x_{n-k}.$$

THEOREM 2.1 *Suppose that m is even, $0 \leq p_n \leq p$ and*

$$\lim_{d \to \infty} \sum_{i=1}^{d} \prod_{j=1}^{i} P_{n+jk}^{-1} = \infty \quad for\ any\ n, \tag{4}$$

and

$$\sum^{\infty} \min_{s \in [n-l,n]} (s^{m-1} Q_s) = \infty \tag{5}$$

where $Q_s = \sum_{n=s}^{\infty} q_n$. If the inequality equation

$$\Delta^2 w_n - q_n^* w_{n+k} \geq 0 \tag{6}$$

where $q_n^* = q_n \max_{s \in [n-l,n]} p_{s+k}^{-1}$, has no positive solution, then equation (1) is oscillatory.

Proof. For the sake of contradiction, we assume equation (1) is not oscillatory and let $\{x_n\}$ be a nonoscillatory solution. Then, there are two cases to consider

(A) $\{x_n\}$ is a positive solution,

(B) $\{x_n\}$ is a negative solution.

Case (A). From the equation (1), we have $\Delta^m y_n < 0$ eventually. So, $\Delta^i y_n$ $(i = 0, 1, \cdots, m)$ are eventually of one sign and one of the following four possible cases may be hold.

(a) $\Delta^{m-1} y_n < 0$, $\Delta y_n < 0$ and $y_n < 0$,

(b) $\Delta^{m-1} y_n > 0$, $\Delta y_n < 0$ and $y_n < 0$,

(c) $\Delta^{m-1} y_n > 0$, $\Delta y_n > 0$ and $y_n > 0$ and

(d) $\Delta^{m-1} y_n > 0$, $\Delta y_n > 0$ and $y_n < 0$.

Case (a). In this case, we easily see that $y_n \to -\infty$ as $n \to \infty$. Thus, there exists a $\alpha > 0$ such that

$$x_n - p_n x_{n-k} = y_n \le -\alpha.$$

Then, for d is some positive integer,

$$\begin{aligned} x_{n+dk} &\le p_{n+dk} x_{n+dk-k} - \alpha \le \cdots \\ &\le p_{n+dk} p_{n+(d-1)k} \cdots p_{n+k} x_n - p_{n+dk} \cdots p_{n+2k} \alpha - \cdots - \alpha \\ &= \prod_{j=1}^{d} p_{n+jk} (x_n - \alpha \sum_{i=1}^{d} \prod_{j=1}^{i} p_{n+jk}^{-1}) < 0. \end{aligned}$$

This contradicts the hypothesis $x_n > 0$.

Case (b). In this case, we also easily see that there exists a $\alpha > 0$ such that $y_n \le -\alpha$ for all large n. Thus, the proof is similar to case(a) and will be omitted.

Case (c). In this case, we can see that there exists a $\alpha > 0$ such that $y_n > \alpha$ for all large n. Therefore, $x_n - p_n x_{n-k} > \alpha$. So $x_n > \alpha$ and $\max\limits_{s \in [n-l, n]} > \alpha$ for all large n and

$$\Delta^{m-1} y_t - \Delta^{m-1} y_n < -\sum_{i=n}^{t-1} q_i \alpha.$$

Because $\Delta^{m-1}y_t > 0$, we can have $\Delta^{m-1}y_n > \sum_{i=n}^{t-1} q_i\alpha$. Letting $t \to \infty$, we obtain that

$$\Delta^{m-1}y_n \geq \alpha \sum_{i=n}^{\infty} q_i = \alpha Q_n. \tag{7}$$

On the other hand, by applying Lemma 2.2, we can get

$$y_n \geq \beta n^{(m-1)}\Delta^{m-1}y_n, \tag{8}$$

where $\beta = 2^{2-2m}/(m-1)!$. By (7) and (8), we have

$$x_n \geq y_n \geq \gamma n^{(m-1)}Q_n,$$

where $\gamma = \alpha\beta$. In view of (1), we see that

$$\Delta^m y_n \leq -\gamma q_n \max_{s\in[n-l,n]}(s^{m-1}Q_s). \tag{9}$$

Summing (9) from large n_0 to n,

$$\Delta^{m-1}y_n \leq \Delta^{m-1}y_{n_0} - \gamma \sum_{i=n_0}^{n-1} q_i \max_{s\in[i-l,i]}(s^{m-1}Q_s).$$

By the hypothesis (5), we have $\Delta^{m-1}y_n \to -\infty$, as $n \to \infty$. This is a contradiction.

Case (d). In this case, we easily see that $(-1)^i\Delta^i y_n < 0$ for $i = 0, 1, \cdots, m$. From $y_n = x_n - p_n x_{n-k} > -p_n x_{n-k}$, we have

$$\max_{s\in[n-l,n]} x_s > \max_{s\in[n-l,n]}\left(-\frac{y_{s+k}}{p_{s+k}}\right) \geq -y_{n+k}\max_{s\in[n-l,n]} p_{s+k}^{-1} \tag{10}$$

and

$$\Delta^m y_n = -q_n \max_{s\in[n-l,n]} x_s \leq y_{n+k}q_n \max_{s\in[n-l,n]} p_{s+k}^{-1}. \tag{11}$$

On the other hand, by Taylor formula (see [4]), we can see that, for $j \leq t \leq s-1$,

$$\begin{aligned} y_t &= \sum_{i=0}^{m-3} \frac{(s+i-1-t)^{(i)}}{i!}(-1)^i\Delta^i y_s \\ &+ \sum_{j=t}^{s-1} \frac{(j+m-3-t)^{(m-3)}}{(m-3)!}(-1)^{m-2}\Delta^{m-2}y_j. \end{aligned} \tag{12}$$

Thus, we have

$$y_t \leq \sum_{j=t}^{s-1} \frac{(j+m-3-t)^{(m-3)}}{(m-3)!} \Delta^{m-2} y_j.$$

Replacing both t and $(s-1)$ with n, we have

$$y_n \leq \frac{(m-3)^{(m-3)}}{(m-3)!} \Delta^{m-2} y_n = \Delta^{m-2} y_n. \tag{13}$$

From (11) and (13), we have

$$\Delta^m y_n \leq q_n \max_{s \in [n-l,n]} p_{s+k}^{-1} \Delta^{m-2} y_{n+k}.$$

Denoting $w_n = \Delta^{m-2} y_n$, we can see that $w_n < 0$ for all large n and $\{w_n\}$ is a solution of the following equation

$$\Delta^2 w_n - q_n^* w_{n+k} \leq 0.$$

This contradicts the hypothesis of the Theorem. So far, we prove the case (A) doesn't hold.

Case (B). From the equation (1), we have $\Delta^m y_n > 0$ eventually. So, $\Delta^i y_n$ ($i = 0, 1, \cdots, m$) are eventually of one sign and one of the following four possible cases may be hold.

(e) $\Delta^{m-1} y_n > 0$, $\Delta y_n > 0$ and $y_n > 0$,

(f) $\Delta^{m-1} y_n < 0$, $\Delta y_n > 0$ and $y_n > 0$,

(g) $\Delta^{m-1} y_n < 0$, $\Delta y_n < 0$ and $y_n < 0$ and

(h) $\Delta^{m-1} y_n < 0$, $\Delta y_n < 0$ and $y_n > 0$.

Cases (e)and (f). In these cases, we can obtain contradictions by using similar proofs of (a) and (b).

Case (g). In this case, we easily see that there exists a positive constant α such that $y_n < -\alpha$ for large n. Similar to case (c), we can prove that

$$\Delta^{m-1} y_n \leq -\alpha Q_n. \tag{14}$$

By applying Lemma 2.2 to $\{-y_n\}$,

$$-y_n \geq \beta n^{(m-1)} (-\Delta^{m-1} y_n) \tag{15}$$

where $\beta = \frac{2^{2-2m}}{(m-1)!}$. In view of $x_n \leq x_n - p_n x_{n-k} = y_n$, we have

$$x_n \leq \beta n^{(m-1)} \Delta^{m-1} y_n \leq -\alpha\beta n^{(m-1)} Q_n,$$

and

$$\Delta^m y_n \geq -q_n \max_{s\in[n-l,n]} (-\gamma s^{(m-1)} Q_s) = \gamma q_n \min_{s\in[n-l,n]} (s^{(m-1)} Q_s)$$

where $\gamma = \alpha\beta$. Similarly, we can obtain a contradiction.

Case (h). In this case, The proof is similar to that of (d), we can get

$$\Delta^m y_n \geq q_n \max_{s\in[n-l,n]} p_{s+k}^{-1} \Delta^{m-2} y_{n+k}.$$

Therefore the inequality

$$\Delta^2 w_n - q_n^* w_{n+k} \geq 0$$

has an eventually positive solution $\{w_n\} = \{\Delta^{m-2} y_n\}$. This contradicts the hypothesis of the Theorem.

The proof of Theorem is complete.

THEOREM 2.2 *Suppose that m is odd, $0 \leq p_n \leq p$ and conditions (4) and (5) hold. If*

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n} q_i \max_{s\in[i-l,i]} p_s > 1, \tag{16}$$

then equation (1) is oscillatory.

Proof. Similar to the proof of Theorem 2.1, we can assume that $\{x_n\}$ satisfies the following two cases

(A) $\{x_n\}$ is a positive solution,

(B) $\{x_n\}$ is a negative solution.

Case (A). Similarly, we have the following four possible cases to consider.

(a) $\Delta^{m-1} y_n < 0$, $\Delta y_n < 0$ and $y_n < 0$,

(b) $\Delta^{m-1} y_n > 0$, $\Delta y_n < 0$ and $y_n < 0$,

(c) $\Delta^{m-1} y_n > 0$, $\Delta y_n > 0$ and $y_n > 0$,

(d) $\Delta^{m-1} y_n > 0$, $\Delta y_n < 0$ and $y_n > 0$.

Case (B). Similarly, we have the following four possible cases to consider.

(e) $\Delta^{m-1} y_n > 0$, $\Delta y_n > 0$ and $y_n > 0$,

(f) $\Delta^{m-1}y_n < 0$, $\Delta y_n > 0$ and $y_n > 0$,

(g) $\Delta^{m-1}y_n < 0$, $\Delta y_n < 0$ and $y_n < 0$,

(h) $\Delta^{m-1}y_n < 0$, $\Delta y_n > 0$ and $y_n < 0$.

The proofs of cases (a), (b), (c), (e), (f) and (g) are similar to that of Theorem 2.1. The detail of their proofs will be omitted. The following proofs are about the cases (d) and (h).

First, we consider the case (d). As m is odd, we easily see that $(-1)^i\Delta^i y_n > 0$ for $i = 0, 1, \cdots, m$. From $0 < y_n = x_n - p_n x_{n-k} \leq x_n$, we have $x_n \geq p_n x_{n-k} \geq p_n y_{n-k}$ and $\max\limits_{s\in[n-l,n]} x_s \geq \max\limits_{s\in[n-l,n]}(p_s y_{s-k}) \geq y_{n-k} \max\limits_{s\in[n-l,n]} p_s$. Therefore

$$\Delta^m y_n = -q_n \max_{s\in[n-l,n]} x_s \leq -(q_n \max_{s\in[n-l,n]} p_s) y_{n-k}.$$

By Taylor formula (see [4]), we see that, for $j \leq t \leq s-1$,

$$\begin{aligned} y_t &= \sum_{i=0}^{m-2} \frac{(s+i-1-t)^{(i)}}{i!}(-1)^i\Delta^i y_s \\ &+ \sum_{j=t}^{s-1} \frac{(j+m-2-t)^{(m-2)}}{(m-2)!}(-1)^{m-1}\Delta^{m-1}y_j \\ &\geq \sum_{j=t}^{s-1} \frac{(j+m-2-t)^{(m-2)}}{(m-2)!}\Delta^{m-1}y_j. \end{aligned}$$

Replacing both t and $(s-1)$ with n,

$$y_n \geq \Delta^{m-1}y_n.$$

So, we have

$$\Delta^m y_n + (q_n \max_{s\in[n-l,n]} p_s)\Delta^{m-1}y_{n-k} \leq 0.$$

That is the inequality

$$\Delta w_n + (q_n \max_{s\in[n-l,n]} p_s) w_{n-k} \leq 0$$

has an eventually positive solution $\{w_n\} = \{\Delta^{m-1}y_n\}$. Then by Lemma 1.1 in [6], we know that the equation

$$\Delta w_n + (q_n \max_{s\in[n-l,n]} p_s) w_{n-k} = 0$$

also has an eventually positive solution. However, by the condition (16) and the oscillation theorem in [7], the equation is oscillatory. This is a contradiction.

Next, we consider case (h). We easily see that $(-1)^i\Delta^i y_n < 0$ for $i = 0, 1, \cdots, m$. Similar to the proof of case (d), we can also know that the equation

$$\Delta w_n + (q_n \max_{s\in[n-l,n]} p_s)w_{n-k} = 0$$

has an eventually positive solution $\{w_n\} = \{-\Delta^{m-1}y_n\}$. So a contradiction can be obtained. The detail of the proof will be omitted.

So far, the proof of the Theorem 2.2 is complete.

THEOREM 2.3 *Suppose that m is even, $0 \le p_n \le p$ and (3) holds. For any positive integer sequence $\{t_i\}$ with $\lim_{i\to\infty} t_i = \infty$, there is*

$$\sum_{n=n_0}^{\infty} n^{m-2} \sum_{i=j,t_j\ge n}^{\infty} q_{t_i} = \infty, \tag{17}$$

the following equation is oscillatory

$$\Delta w_n + q_n^* w_{n-(k+l)} = 0. \tag{18}$$

where $q_n^* = \frac{2^{2(1-m)}}{(m-1)!}(n-(k+l))^{m-1} q_n \max_{s\in[n-l,n]} p_s$.
(A) If $\{x_n\}$ is a positive solution of equation (1), then $\lim_{n\to\infty} x_n = 0$.
(B) If $\{x_n\}$ is a negative solution of equation (1), then $\limsup_{n\to\infty} x_n = 0$.

Proof. (A) Since $\{x_n\}$ is a positive solution of equation (1), the possible cases (a)–(d) in the proof of Theorem 2.1 would be considered. Similarly, the cases (a) and (b) don't hold and their proofs will be omitted.

Now, we give the proofs about cases (c) and (d).

Case (c). From $\Delta y_n > 0$ and $y_n > 0$, we know that $x_n > y_n$ and $x_n = p_n x_{n-k} + y_n > p_n x_{n-k} > p_n y_{n-k}$. So, there is

$$\max_{s\in[n-l,n]} x_s > \max_{s\in[n-l,n]} (p_s y_{s-k}) > y_{n-(k+l)} \max_{s\in[n-l,n]} p_s.$$

By the equation (1), we have

$$\Delta^m y_n + y_{n-(k+l)} q_n \max_{s\in[n-l,n]} p_s \le 0. \tag{19}$$

On the other hand, from Lemma 2.2, we have

$$y_n \geq \frac{2^{2(1-m)}}{(m-1)!} n^{(m-1)} \Delta^{m-1} y_n. \tag{20}$$

Combining (19) and (20) and denoting $w_n = \Delta^{m-1} y_n$, we can see that $\{w_n\}$ is an eventually positive solution of the following inequality

$$\Delta w_n + q_n^* w_{n-(k+l)} \leq 0, \tag{21}$$

where $q_n^* = \frac{2^{2(1-m)}}{(m-1)!}(n-(k+l))^{m-1} q_n \max_{s\in[n-l,n]} p_s$. Therefore, by Lemma 1.1 in [6], we know that the equation (18) has an eventually positive solution. This is a contradiction. So, case (c) doesn't hold.

From above proofs, we can see that the case (d) is an only possible for $\{x_n\}$ is an eventually positive solution of equation (1). Thus, we can conclude that $\lim\limits_{n\to\infty} x_n = 0$. Otherwise, we can suppose that there exist a number $\delta > 0$ and a real integer sequence $\{t_i\}$ with $\lim\limits_{i\to\infty} t_i = \infty$ such that $x_{t_i} \geq \delta > 0$.

Since that $(-1)^i \Delta^i y_n < 0$, for $i = 0, 1, \cdots, m$, we easily see that $\Delta^i y_n \to 0$ as $n \to \infty$, for $i = 0, 1, \cdots, m-1$. So, we have

$$\Delta^m y_n + q_n x_n \leq \Delta^m y_n + q_n \max_{s\in[n-l,n]} x_s = 0.$$

By summing above inequality from n_1 to ∞, we have

$$-\Delta^{m-1} y_{n_1} \leq -\sum_{n_0=n_1}^{\infty} q_{n_0} x_{n_0}.$$

Further, summing above inequality $(m-1)$ times, we can have

$$\begin{aligned}
-y_n &\geq \sum_{n_{m-1}=n}^{\infty} \cdots \sum_{n_1=n_2}^{\infty} \sum_{n_0=n_1}^{\infty} q_{n_0} x_{n_0} \\
&\geq \sum_{n_1=n}^{\infty} \frac{(n_1 - n + m - 2)^{(m-2)}}{(m-2)!} \sum_{n_0=n_1}^{\infty} q_{n_0} x_{n_0} \\
&\geq \frac{1}{2^{m-2}(m-2)!} \sum_{s=2n}^{\infty} s^{m-2} \sum_{i=j, t_i \geq s}^{\infty} q_{t_i} x_{t_i} \\
&\geq \frac{\delta}{2^{m-2}(m-2)!} \sum_{s=2n}^{\infty} s^{m-2} \sum_{i=j, t_i \geq s}^{\infty} q_{t_i}.
\end{aligned}$$

Therefore, we can obtain that

$$\sum_{s=2n}^{\infty} s^{m-2} \sum_{i=j, t_i \geq s}^{\infty} q_{t_i} < \infty.$$

This contradicts the hypothesis (17).

So, we have proven the case (A).

(B). Since $\{x_n\}$ is a negative solution of equation (1), similar to the analysis of Theorem 2.1, we can see that there exist four possible cases (e)–(h) to be considered. Cases (e) and (f) can be proved to be impossible and the proofs will be omitted.

Next, we give the proofs about cases (g) and (h).

Case (g). From $\Delta y_n < 0$ and $y_n < 0$ and using similar deducing process of case (c) above, we obtain that

$$\Delta^m y_n + y_{n-(k+l)} q_n \max_{s\in[n-l,n]} p_s \geq 0 \tag{22}$$

and

$$-y_n \geq \frac{2^{2(1-m)}}{(m-1)!} n^{(m-1)} (-\Delta^{m-1} y_n). \tag{23}$$

Combining (22) and (23) and denoting $w_n = -\Delta^{m-1} y_n$, we can see that $\{w_n\}$ is an eventually positive solution of (21). Similarly, a contradiction is obtained. So, case (g) doesn't hold.

From above proofs, we can see that the case (h) is an only possible for $\{x_n\}$ is an eventually negative solution of equation (1). Thus, we can conclude that $\limsup_{n\to\infty} x_n = 0$. Otherwise, we can suppose that there exist a number $\delta > 0$ and $x_n \leq -\delta < 0$ eventually. So, we have

$$0 = \Delta^m y_n + q_n \max_{s\in[n-l,n]} x_s \leq \Delta^m y_n + q_n(-\delta).$$

Hence, summing above inequality m times, we can have

$$\begin{aligned} y_n &\geq \delta \sum_{n_{m-1}=n}^{\infty} \cdots \sum_{n_1=n_2}^{\infty} \sum_{n_0=n_1}^{\infty} q_{n_0} \\ &\geq \frac{\delta}{2^{m-2}(m-2)!} \sum_{s=2n}^{\infty} s^{m-2} \sum_{n_0=s}^{\infty} q_{n_0}. \end{aligned}$$

Therefore, we can obtain that

$$\sum_{s=2n}^{\infty} s^{m-2} \sum_{n_0=s}^{\infty} q_{n_0} < \infty.$$

This contradicts the hypothesis (17).

So far , we have proven the theorem.

REMARK 2.1 *When m is odd, we can have similar conclusion to Theorem 2.3 if the hypotheses of Theorem 2.3 hold.*

The following theorems will consider the properties that the nonoscillatory solutions of equation (1) tend to infinity.

THEOREM 2.4 *Suppose that m is even, $0 \le p_n \le p$ and*

$$\sum_{n=n_0}^{\infty} n q_n = \infty, \tag{24}$$

and the inequality equation (6) has no positive solution.

(A) If $\{x_n\}$ is a positive solution of equation (1), then $\lim_{n\to\infty} x_n = \infty$.

(B) If $\{x_n\}$ is a negative solution of equation (1), then $\lim_{n\to\infty} x_n = -\infty$.

Proof. (A) Similar to the proof of Theorem 2.1, we have the same four cases (a)–(d) to consider.

(a). We can easily see that $y_n \to -\infty$ as $n \to \infty$. From $y_n = x_n - p_n x_{n-k} > -p_n x_{n-k} > p x_{n-k}$, we can obtain $x_n \to \infty$ as $n \to \infty$.

(b). Since that $\Delta y_n < 0$ and $y_n < 0$, we have the following two possible cases

(1). $y_n \to -\infty$ as $n \to \infty$,

(2). $y_n \to -L$ as $n \to \infty$, where L is some positive number.

If (1) holds, we also have the conclusion $x_n \to \infty$ as $n \to \infty$.

If (2) holds, we have $x_n - p_n x_{n-k} = y_n < -L/2$, further, $p x_{n-k} > p_n x_{n-k} > L/2$ and $x_{n-k} > L/(2p)$. So, by equation (1) we have

$$\Delta^m y_n + q_n \frac{L}{2p} \le \Delta^m y_n + q_n \max_{s\in[n-l,n]} x_s = 0.$$

By summing the above inequality and considering $\Delta^{m-1} y_n > 0$, we have

$$\Delta^{m-1} y_n \ge \sum_{s=n}^{\infty} q_s \frac{L}{2p}.$$

Summing the above inequality again from n_0 to N, we have

$$\Delta^{m-2} y_{N+1} - \Delta^{m-2} y_{n_0} \ge \sum_{s_1=n_0}^{N} \sum_{s=s_1}^{\infty} q_s \frac{L}{2p}.$$

Let $N \to \infty$, the left of above inequality can be

$$\sum_{s_1=n_0}^{N} \sum_{s=s_1}^{\infty} q_s \frac{L}{2p} \to \frac{L}{2p} \sum_{s_1=n_0}^{\infty} \sum_{s=s_1}^{\infty} q_s = \frac{L}{2p} \sum_{s=n_0}^{\infty} (s-n_0) q_s = \infty.$$

So, $\Delta^{m-2} y_{N+1} \to \infty$ as $N \to \infty$. This will lead $\Delta^i y_n > 0$ $(i = 0, 1, \cdots, m-3)$ and a contradiction to occur.

(c) Since that $\Delta y_n > 0$ and $y_n > 0$, we have the following two possible cases

(1). $y_n \to \infty$ as $n \to \infty$,

(2). $y_n \to L$ as $n \to \infty$, where L is some positive number.

If (1) holds, we also have the conclusion $x_n \to \infty$ as $n \to \infty$.

If (2) holds, we have $x_n > x_n - p_n x_{n-k} = y_n > L/2$, further,

$$\Delta^m y_n + q_n \frac{L}{2} \le \Delta^m y_n + q_n \max_{s \in [n-l, n]} x_s = 0.$$

By summing the above inequality and considering $\Delta^{m-1} y_n > 0$, we have

$$\Delta^{m-2} y_{N+1} - \Delta^{m-2} y_{n_0} \ge \sum_{s_1=n_0}^{N} \sum_{s=s_1}^{\infty} q_s \frac{L}{2}.$$

Let $N \to \infty$, we also see that $\Delta^{m-2} y_{N+1} \to \infty$ as $N \to \infty$. This lead to $y_n \to \infty$ as $n \to \infty$. This is a contradiction.

(d). In this case, the contradiction will be obtained by similar proof method to that of case (d) in theorem 2.1. we omit it.

(B). In this case, there are four same cases (e)–(h) to consider. The proofs are similar to (A) and will be omitted.

So far, the proof of theorem is completed.

REMARK 2.2 *When m is odd, we can have similar conclusion to Theorem 2.4 if only changing the hypothesis that equation* (6) *has no positive solution into that* (16) *holds.*

References

[1] J.W.Luo, Z.R.Liu and Y.H.Yu, Oscillation and nonoscillation of neutral differece equation with ”maxima”, *Acta Mathematicae Applicatae Sinica*, **25(3)** (2002), 385-391(in Chinese).

[2] G.Zhang and S.S.Cheng, Asymptotic dichotomy for nonoscillatory solutions of a nonlinear difference equation, *Applications Mathematicae*, **25**(4)(1999),393-399.

[3] X.L.Zhou and J.R.Yan, Oscillatory and asymptotic behavior of higher order neutral difference equations with "maxima", to appear.

[4] R.P.Agarwal, *Diffference equations and inequalities*, Marcel Dekker, New York, 1992.

[5] R.P.Agarwal and S.R.Grace, Oscillation of higher-order nonlinear difference equations of neutral type, *Appl.Math.Lett.*, **12**(1999),77-83.

[6] G.Ladas and C.Qian, Comparison results and linearized oscillations for higher order difference equations, *Internat.J.Math. and Math.Sci*, **15**(1992),129-142.

Received: March 20, 2007