Oscillation of Nonlinear
Hyperbolic Equation with Impulses

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Abstract

In this paper, oscillatory properties of the solutions for certain nonlinear impulsive hyperbolic equations are investigated and new sufficient condition for oscillation of the solutions are established.

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1 Introduction

Theory of impulsive partial differential equations can be applied to many fields, such as to biology, population growth, engineering, generic repression, control theory and climate model. In the last few years, the fundamental theory of partial differential equations with impulses has undergone intensive development. The qualitative theory of this class of equations, however, is still in an initial stage of development. We may easily visualize situations in nature where abrupt change such as shock and disasters may occur. These phenomena are short-time perturbations whose duration is negligible in comparison with the duration of the whole evolution process. Consequently, it is natural to assume, in modeling these problems, that these perturbations act instantaneously, that is, in the form of impulses. In 1991, the first paper [1] on this class of equations was published. But for instance, only a few papers have been published on oscillation theory of impulsive partial differential equations. Recently, Bainov, Minchev, Luo and Liu[2-7] investigated the oscillation of solutions of impulsive partial differential equations.

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In this paper, we’ll discuss the oscillatory properties of solutions for a class of nonlinear impulsive hyperbolic equation (1) under the boundary condition (4).

\[
\frac{\partial^2 u}{\partial t^2} = a(t)h(u)\Delta u - q(t,x)f(u(t,x)), \quad t \neq t_k, \quad (t,x) \in R_+ \times \Omega = G \quad (1)
\]

\[
u(t_{k+},x) - u(t_{k+},x) = q_k u(t_k,x), \quad (2)
\]

\[
u(t_{k+},x) - u(t_{k+},x) = b_k u(t_k,x), \quad t = t_k, \quad k = 1,2, \ldots \quad (3)
\]

with the boundary condition

\[
\frac{\partial u}{\partial n} = g(t,x,u), \quad (t,x) \in R_+ \times \partial \Omega \quad (4)
\]

Here \( \Omega \subset R^N \) is a bounded domain with boundary \( \partial \Omega \) smooth enough and \( n \) is a unit exterior normal vector of \( \partial \Omega \).

Assume that the following conditions are fulfilled:

\( H_1 \) \( a(t) \in PC(R_+,R_+), q(t,x) \in C(R_+ \times \bar{\Omega}, (0,\infty)) \); where PC denote the class of functions which are piecewise continuous in \( t \) with discontinuities of first kind only at \( t = t_k \) and left continuous at \( t = t_k \), \( k = 1,2, \ldots \).

\( H_2 \) \( h'(u), f(u) \in C(R,R); f(u)/u \geq C = \text{const.} > 0, \) for \( u \neq 0; uh'(u) \geq 0, g(t,x,u) \) is continuous and \( uh(u)g(t,x,u) < 0, q_k > -1, b_k > -1, 0 < t_1 < t_2 < \cdots < t_k < \cdots, \lim_{t \to \infty} t_k = \infty. \)

We introduce the notations: \( \Gamma_k = \{ (t,x) : t \in (t_k,t_{k+1}), x \in \Omega \}, \Gamma_k = \bigcup_{k=0}^{\infty} \Gamma_k, \bar{\Gamma} = \bigcup_{k=0}^{\infty} \bar{\Gamma}_k, \bar{G} = \bigcup_{k=0}^{\infty} \bar{\Gamma}_k, v(t) = \int_{\Omega} u(t,x)dx \) and \( p(t) = \min \{ q(t,x), x \in \Omega \}. \)

**Definition 1.** The solution \( u \in C^2(\Gamma) \cap C^1(\bar{\Gamma}) \) of problem (1), (4) is called nonoscillatory in the domain \( G \) if it is either eventually positive or eventually negative. Otherwise, it is called oscillatory.

## 2 Oscillation properties of the problem

The proof of theorem needs the following lemmas.

**Lemma 1.** Let \( u \in C^2(\Gamma) \cap C^1(\bar{\Gamma}) \) be a positive solution of the problem (1), (4) in \( G \), then function \( v(t) \) satisfies the following impulsive differential inequality

\[
v''(t) + Cp(t)v(t) \leq 0, \quad (t \neq t_k) \quad (5)
\]

\[
v(t_{k+}) = (1 + q_k)v(t_k), \quad k = 1,2, \ldots \quad (6)
\]

\[
v'(t_{k+}) = (1 + b_k)v'(t_k), \quad k = 1,2, \ldots \quad (7)
\]
Proof. Let \( u(t, x) \) be a positive solution of the problem (1),(4) in \( G \). Without loss of generality, we may assume that \( u(t, x) > 0 \), for any \((t, x) \in [t_0, \infty) \times \Omega \).

For \( t \geq t_0, k = 1, 2, \cdots \), integrating (1) with respect to \( x \) over \( \Omega \) yields

\[
\frac{d^2}{dt^2} \left[ \int_{\Omega} u dx \right] = a(t) \int_{\Omega} h(u) \Delta u dx - \int_{\Omega} q(t, x) f(u(t, x)) dx.
\]

By Green’s formula and boundary condition we have

\[
\int_{\Omega} h(u) \Delta u dx = \int_{\partial \Omega} h(u) \frac{\partial u}{\partial n} ds - \int_{\Omega} h'(u) |\nabla u|^2 dx \leq - \int_{\Omega} h'(u) |\nabla u|^2 dx \leq 0.
\]

From condition \( H_2 \), we can easily obtain

\[
\int_{\Omega} q(t, x) f(u(t, x)) dx \geq C p(t) \int_{\Omega} u(t, x) dx,
\]

It follows that

\[
v'' + C p(t) v(t) \leq 0, \quad (t \geq t_0, \ t \neq t_k)
\]

(8)

Where \( v(t) > 0 \).

For \( t > t_0, \ t = t_k, k = 1, 2, \cdots \), we have

\[
\int_{\Omega} u(t^+_k, x) dx - \int_{\Omega} u(t^-_k, x) dx = q_k \int_{\Omega} u(t_k, x) dx,
\]

\[
\int_{\Omega} u_t(t^+_k, x) dx - \int_{\Omega} u_t(t^-_k, x) dx = b_k \int_{\Omega} u_t(t_k, x) dx.
\]

This implies

\[
v(t^+_k) = (1 + q_k) v(t_k).
\]

(9)

\[
v'(t^+_k) = (1 + b_k) v'(t_k) \quad k = 1, 2, \cdots
\]

(10)

Hence we obtain that \( v(t) > 0 \) is a positive solution of differential inequality (5)-(7). This ends the proof of the lemma.

Lemma 2. [11,Theorem 1.4.1]. Assume that

(i) \( m(t) \in PC^1[R^+, R] \) is left continuous at \( t_k \) for \( k = 1, 2, \cdots \),

(ii) for \( k = 1, 2, \cdots, t \geq t_0, \)

\[
m'(t) \leq p(t) m(t) + q(t) \quad (t \neq t_k).
\]

\[
m(t^+_k) \leq d_k m(t_k) + e_k.
\]


where \( p(t), q(t) \in C(R^+, R), d_k \geq 0 \) and \( e_k \) are real constants, \( PC^1[R^+, R] = \{x : R^+ \to R; x(t) \) is continuous and continuously differentiable everywhere except some \( t_k \) at which \( x(t_k^+), x(t_k^-), x'(t_k^-) \) and \( x'(t_k^-) \) exist and \( x(t_k) = x(t_k^-), x'(t_k) = x'(t_k^-) \} \).

Then

\[
m(t) \leq m(t_0) \prod_{t_0 < t_k < t} d_k \exp\left(\int_{t_0}^{t_k} p(s)ds\right) + \int_{t_0}^{t} \prod_{s < t_k < t} d_k \exp\left(\int_{s}^{t_k} p(r)dr\right)q(s)ds
+ \sum_{t_0 < t_k < t} \prod_{t_k < t_k < t} \sum_{s < t_k < t} d_j \exp\left(\int_{s}^{t_k} p(s)ds\right)e_k.
\]

From Lemma 2, we can obtain the following lemma 3. See also [9].

**Lemma 3.** Let \( v(t) \) be eventually positive(negative) solution of differential inequality (5)-(7). Assume that there exists \( T \geq t_0 \) such that \( v(t) > 0(v(t) < 0) \) for \( t \geq T \). If the following condition hold,

\[
\lim_{t \to +\infty} \int_{t_0}^{t} \prod_{t_0 < t_k < s} \frac{1+b_k}{1+q_k}ds = +\infty,
\]

then \( v'(t) \geq 0(v'(t) \leq 0) \) for \( t \in [T, t_1] \bigcup (\bigcup_{k=1}^{+\infty}(t_k, t_{k+1})), \) where \( l = \min\{k : t_k \geq T\} \).

**Theorem 1.** Let condition (11) and the following condition (12) hold for some \( j_0 \),

\[
\lim_{t \to +\infty} \int_{t_0}^{t} \prod_{t_0 < t_k < s} \frac{1+q_k}{1+b_k}p(s)ds = +\infty,
\]

then every solution of the problem (1),(4) oscillates in \( G \).

**Proof.** Let \( u(t, x) \) be a nonoscillatory solution of (1),(4). Without loss of generality, we can assume that \( u(t, x) > 0 \) for any \( (t, x) \in [t_0, \infty) \times \Omega. \) From Lemma 1, we know that \( v(t) \) is a positive solution of (5)-(7). Thus from Lemma 3, we can find that \( v'(t) \geq 0 \) for \( t \geq t_0 \).

For \( t \geq t_0, t \neq t_k, k = 1, 2, \cdots, \) define

\[
w(t) = \frac{v'(t)}{v(t)}, \quad t \geq t_0.
\]

Then we have \( w(t) > 0, \ t \geq t_0, v'(t) - w(t)v(t) = 0. \) We may assume that \( v(t_0) = 1, \) thus we have that for \( t \geq t_0 \)

\[
v(t) = \exp\left(\int_{t_0}^{t} w(s)ds\right)
\]
\begin{align}
  v'(t) &= w(t) \exp\left( \int_{t_0}^{t} w(s) ds \right) \\
  v''(t) &= w^2(t) \exp\left( \int_{t_0}^{t} w(s) ds \right) + w'(t) \exp\left( \int_{t_0}^{t} w(s) ds \right)
\end{align}

We substitute (13)-(15) into (5) and can obtain the following inequality

\[ w^2(t) + w'(t) + Cp(t) \leq 0. \]

which implies

\[ w'(t) + Cp(t) \leq 0. \quad (t \geq t_0) \]

From (6),(7) we can obtain

\[ w(t_k^+) = \frac{v'(t_k^+)}{v(t_k^+)} = \frac{1 + b_k}{1 + q_k} w(t_k), \quad k = 1, 2, \ldots. \]

It follows that

\[ w'(t) \leq -Cp(t) \quad (t \neq t_k). \quad (16) \]

\[ w(t_k^+) = \frac{1 + b_k}{1 + q_k} w(t_k) \quad (t = t_k). \quad (17) \]

By using Lemma 2, we obtain

\[ w(t) \leq w(t_0) \prod_{t_0 < t_k < t} \frac{1 + b_k}{1 + q_k} + \int_{t_0}^{t} \prod_{s < t_k < t} \frac{1 + b_k}{1 + q_k} (-Cp(s)) ds \]

\[ = \prod_{t_0 < t_k < t} \frac{1 + b_k}{1 + q_k} \left\{ w(t_0) - \int_{t_0}^{t} \prod_{t_0 < t_k < s} \frac{1 + q_k}{1 + b_k} Cp(s) ds \right\}. \]

Since \( w(t) > 0 \), the last inequality contradicts (14). The proof of Theorem is completed.

**References**


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