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Strong Convergence Theorems for a Finite Family of Asymptotically Quasi-Nonexpansive Mappings

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Abstract

In this paper, we are concerned with the study of a multi-step iterative scheme with errors insolving a finite family of asymptotically quasi-nonexpansive self-mappings. We approximate the common fixed points of a finite family of asymptotically quasi-nonexpansive self-mappings by convergence of the scheme in a uniformly convex Banach space. Our results extend and improve some recent results, Q. Liu [Iterative sequences for asymptotically quasi-nonexpansive mappings with error member, J. Math. Anal. Appl. 259(2001), 18-24], S Plubtieng, R. Wangkeeree, R. Punpaeng, [On the convergence of modified Noor iterations with errors for asymptotically nonexpansive mappings, J. Math. Anal. Appl. 322(2006), 1018 - 1029.]

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1 Introduction

Let C be a subset of normed space X, and let T be a self-mapping on C. T is said to be nonexpansive provided that $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$,

T is called asymptotically nonexpansive if there exists sequence $\{u_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} u_n = 0$ such that $\|T^n x - T^n y\| \leq (1 + u_n)\|x - y\|$, for all $x, y \in C$. A map T is called asymptotically quasi-nonexpansive mapping if there exists $u_n \in [0, +\infty)$, with $\lim_{n\to\infty} u_n = 0$, such that $\|T^n x - p\| \leq (1 + u_n)\|x - p\|$, for all $x \in C$ and for all $p \in F(T)$, and $n \in \mathbf{N}$. From the above definitions, it follows that a nonexpansive mapping must be asymptotically nonexpansive, and if $F(T) \neq \phi$ then asymptotically nonexpansive mapping must be asymptotically quasi-nonexpansive mappings. Fixed point iterations process for nonexpansive mappings and assymptotically nonexpansive mappings in Banach spaces including Mann and Ishikawa iterations process have been studied extensively by many authors to solve the nonlinear operator equations as well as variational inequations; see [8, 9, 10, 11, 14, 15, 17, 18, 19]. In 1973, Petryshyn and Williamson [11] proved a sufficient and necessary condition for Picard iterative sequence and Mann iterative sequences to converge to fixed points for quasi-nonexpansive mappings. In 1997, Ghosh and Debnath [3] extended the result of [11] and gave the sufficient and necessary condition for Ishikawa iterative sequence to converge to fixed points for quasi-nonexpansive mappings. Recentry, Liu [5], extended the above result and obtained some sufficient and necessary condition for Ishikawa iterative sequences to converge to fixed points for asymptotically quasi-nonexpansive mappings. In 2001, Liu [6, 7] proved some sufficient and necessary conditions for Ishikawa iterative sequences of asymptotically quasi-nonexpansive mappings with errors member to converge to fixed points. On the other hand, Das and Debata [2] and Takahashi and Tamura [16] introduce and studied a generalization of Ishikawa iterative schemes with errors for two mappings in hillbert space and Banach space, respectively. Recentry, Khan and Fukhar-ud-din [4] extend their scheme to the modified Ishikawa iterative schemes with errors for two mappings and gave weak and strong convergence theorems. Moreover, Plubtieng R. Wangkeeree, R. Punpaeng[12] established several weak and strong convergence theorems are established for a modified three-step iterative scheme with errors for three asymptotically nonexpansive mappings. Inspired and motivated by these fact, we introduce and study a multi-step iterative schemes with errors for a finite family of asymptotically quasi-nonexpansive mappings. Our schemes can be viewed as an extension for iterative schemes of Liu [6, 7], Xu and Noor [19] and Cho, Zhou and Guo [1] . The scheme is defined as follows.

2 Preliminary

Let X be uniformly convex Banach space and let $T_1, T_2, ..., T_N : X \longrightarrow X$ be mapping. For any given $x_0 \in X$ the sequence $\{x_n\}$ defined by

$$\begin{cases} x_1 = x \in X, \\ x_n^{(1)} = \alpha_n^{(1)} x_n + \beta_n^{(1)} T_1^n x_n + \gamma_n^{(1)} v_n^{(1)}, \\ x_n^{(2)} = \alpha_n^{(2)} x_n + \beta_n^{(2)} T_2^n x_n^{(1)} + \gamma_n^{(2)} v_n^{(2)}, \\ \vdots \\ x_{n+1} = x_n^{(N)} = \alpha_n^{(N)} x_n + \beta_n^{(N)} T_N^n x_n^{(N-1)} + \gamma_n^{(N)} v_n^{(N)}, n \geq 1, \end{cases} \tag{2.1}$$

where $\{\alpha_n^{(1)}\}, ..., \{\alpha_n^{(N)}\}, \{\beta_n^{(1)}\}, ..., \{\beta_n^{(N)}\}, \{\gamma_n^{(1)}\}, ..., \{\gamma_n^{(N)}\}$ are sequences in $[0, 1]$ with $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$ for all $i = 1, 2, 3, ..., N$ and $\{v_n^{(1)}\}, \{v_n^{(2)}\}, ..., \{v_n^{(N)}\}$ are bounded sequences in X.

This iteration scheme (2.1) are called the multi-step iteration with errors. These iteration introduce the Mann-Ishikawa-Three-step iteration as spacial case.

If $T_1 = T_2 = T_3 = ... = T_N := T$, then the sequence $\{x_n\}$ of (1.1) defined by

$$\begin{cases} x_1 = x \in X, \\ x_n^{(1)} = \alpha_n^{(1)} x_n + \beta_n^{(1)} T^n x_n + \gamma_n^{(1)} v_n^{(1)}, \\ x_n^{(2)} = \alpha_n^{(2)} x_n + \beta_n^{(2)} T^n x_n^{(1)} + \gamma_n^{(2)} v_n^{(2)}, \\ \vdots \\ x_{n+1} = x_n^{(N)} = \alpha_n^{(N)} x_n + \beta_n^{(N)} T^n x_n^{(N-1)} + \gamma_n^{(N)} v_n^{(N)}, n \geq 1, \end{cases} \tag{2.2}$$

where $\{\alpha_n^{(1)}\}, ..., \{\alpha_n^{(N)}\}, \{\beta_n^{(1)}\}, ..., \{\beta_n^{(N)}\}, \{\gamma_n^{(1)}\}, ..., \{\gamma_n^{(N)}\}$ are sequences in $[0, 1]$ with $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$ for all $i = 1, 2, 3, ..., N$ and $\{v_n^{(1)}\}, \{v_n^{(2)}\}, ..., \{v_n^{(N)}\}$ are bounded sequences in X.

For $N = 2$, $T_1 = T_2 \equiv T$ $\alpha_n^1 = \alpha_n^{'}, \beta_n^1 = \beta_n^{'}, \gamma_n^1 = \gamma_n^{'}, \alpha_n^2 = \alpha_n, \beta_n^2 = \beta_n$ and $\gamma_n^2 = \gamma_n$, then (2.1) reduces to the modified Ishikawa iterative scheme with errors defined by Liu [5];

$$\begin{cases} x_{n+1} = \alpha_n x_n + \beta_n T^n y_n + \gamma_n u_n, \\ y_n = \alpha_n^{'} x_n + \beta_n^{'} T^n x_n + \gamma_n^{'} v_n, n \geq 1. \end{cases} \tag{2.3}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha_n^{'}\}, \{\beta_n^{'}\}, \{\gamma_n^{'}\}$ are sequence in [0,1] with

$$\alpha_n + \beta_n + \gamma_n = \alpha_n^{'} + \beta_n^{'} + \gamma_n^{'} = 1$$

and $\{u_n\}, \{v_n\}$ are two bounded sequence in X.

In the sequal, the following lemmas are need to prove our main results.

Lemma 2.1 [13, *J. Schu's Lemma.*] *Let X be a uniform convex Banach space,* $0 < \alpha \leq t_n \leq \beta < 1$, $x_n, y_n \in X$, $\limsup_{n\to\infty} \|x_n\| \leq a, \limsup_{n\to\infty} \|y_n\| \leq a$, *and* $\limsup_{n\to\infty} \|t_n x_n - (1-t_n) y_n\| = a$, $a \geq 0$. *Then* $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.2 [2, *Lemma 2.*] *Let nonnegative series* $(\alpha_n), (\beta_n), (\gamma_n)$ *satisfy* $\alpha_{n+1} \leq (1+\beta_n)\alpha_n + \gamma_n,\ \forall n \in \mathbf{N}$, *and* $\Sigma_{n=1}^{\infty}\beta_n < \infty, \Sigma_{n=1}^{\infty}\gamma_n < \infty$; *then* $\lim_{n\to\infty}\alpha_n$ *exists.*

3 Main Results

In this section we first prove some sufficient and neccesary condition of the multi-step iterative scheme with errors for a finite family of asymptotically quasi-nonexpansive mappings to converge to common fixed point. In order to prove our main results, the following lemma are needed.

Lemma 3.1 *Let* X *be a nonempty convex subset of uniformly convex Banach space. Let* $T_1, T_2, ..., T_N$ *be asymptotically quasi-nonexpansive mappings of* X, *with sequences* $\{u_n^{(1)}\}$, ..., $\{u_n^{(N)}\}$ *and* $F = \cap_{i=1}^{N} F(T_i) \neq \phi$. *Let* $\{x_n\}$ *be the sequence as defined by (2.1) with* $\Sigma_{n=1}^{\infty}\gamma_n^i < \infty$ *for all* $i = 1, 2, ..., N$. *Then*

(a) There exists positive sequence $\{k_n\}$ *with* $\lim_{n\to\infty} k_n = 0$ *such that*

$$\|x_{n+1} - p\| \leq (1+k_n)\|x_n - p\| + d_n^{(N)},$$

for all $n \in \mathbf{N},\ p \in F$. *where* $\Sigma_{n=1}^{\infty} d_n^{(N)} < \infty$.

(b) There exists a constant $M > 0$, *such that* $\|x_{n+m} - p\| \leq M\|x_n - p\| + \Sigma_{k=n}^{n+m-1} d_k^N,\ \forall m, n \in \mathbf{N},\ \forall p \in F$.

Proof. Proof of (a). Let $p \in F$. For each $n \geq 1$, let $u_n = \max\{u_n^{(1)}, ..., u_n^{(N)}\}$. Then $u_n \geq 0$ and $\lim_{n\to\infty} u_n = 0$. It follows from (2.1) that

$$\begin{aligned}\|x_n^{(1)} - p\| &\leq \alpha_n^{(1)}\|x_n - p\| + \beta_n^{(1)}\|T_1^n x_n - p\| + \gamma_n^{(1)}\|v_n^{(1)} - p\| \\ &\leq \alpha_n^{(1)}\|x_n - p\| + \beta_n^{(1)}(1+u_n)\|x_n - p\| + \gamma_n^{(1)}\|v_n^{(1)} - p\| \\ &= (\alpha_n^{(1)} + \beta_n^{(1)}(1+u_n))\|x_n - p\| + d_n^{(1)},\end{aligned}$$

where $d_n^{(1)} = \gamma_n^{(1)}\|v_n^{(1)} - p\|$. Since $\Sigma_{n=1}^{\infty}\gamma_n^{(1)} < \infty$, we infer that $\Sigma_{n=1}^{\infty} d_n^{(1)} < \infty$. Next, we note that

$$\begin{aligned}\|x_n^{(2)} - p\| &\leq \alpha_n^{(2)}\|x_n - p\| + \beta_n^{(2)}\|T_2^n x_n^{(1)} - p\| + \gamma_n^{(2)}\|v_n^{(2)} - p\| \\ &\leq \alpha_n^{(2)}\|x_n - p\| + \beta_n^{(2)}(1+u_n)\|x_n^{(1)} - p\| + \gamma_n^{(2)}\|v_n^{(2)} - p\| \\ &\leq \alpha_n^{(2)}\|x_n - p\| + \beta_n^{(2)}(1+u_n)((\alpha_n^{(1)} + \beta_n^{(1)}(1+u_n))\|x_n - p\| + d_n^{(1)}) \\ &\quad +\gamma_n^{(2)}\|v_n^{(2)} - p\| \\ &\leq (\alpha_n^{(2)} + \alpha_n^{(1)}\beta_n^{(2)}(1+u_n) + \beta_n^{(1)}\beta_n^{(2)}(1+u_n)^2)\|x_n - p\| + d_n^{(2)} \\ &\leq (\alpha_n^{(2)} + \alpha_n^{(1)}\beta_n^{(2)} + \beta_n^{(1)}\beta_n^{(2)})(1+u_n)^2\|x_n - p\| + d_n^{(2)}\end{aligned}$$

where $d_n^{(2)} = \beta_n^{(2)}(1+u_n)d_n^{(1)} + \gamma_n^2\|v_n^{(2)} - p\|$. Since $\Sigma_{n=1}^{\infty} d_n^{(1)} < \infty$ and $\Sigma_{n=1}^{\infty}\gamma_n^{(2)} < \infty$, it follows that $\Sigma_{n=1}^{\infty} d_n^{(2)} < \infty$. Moreover, we have

$$\begin{aligned}\|x_n^{(3)} - p\| &\leq \alpha_n^{(3)}\|x_n - p\| + \beta_n^{(3)}\|T_3^n x_n^{(2)} - p\| + \gamma_n^{(3)}\|v_n^{(3)} - p\| \\ &\leq \alpha_n^{(3)}\|x_n - p\| + \beta_n^{(3)}(1+u_n)\|x_n^{(2)} - p\| + \gamma_n^{(3)}\|v_n^{(3)} - p\| \\ &\leq \alpha_n^{(3)}\|x_n - p\| + \beta_n^{(3)}(1+u_n)((\alpha_n^{(2)} + \alpha_n^{(1)}\beta_n^{(2)} \\ &\quad +\beta_n^{(1)}\beta_n^{(2)})(1+u_n)^2\|x_n - p\| + d_n^{(2)}) + \gamma_n^{(3)}\|v_n^{(3)} - p\|\end{aligned}$$

$$\leq (\alpha_n^{(3)} + \alpha_n^{(2)}\beta_n^{(3)}(1+u_n) + \alpha_n^{(1)}\beta_n^{(2)}\beta_n^{(3)}(1+u_n)^2 + \beta_n^{(1)}\beta_n^{(2)}\beta_n^{(3)}(1+u_n)^3)\|x_n - p\| + d_n^{(3)},$$
$$\leq (\alpha_n^{(3)} + \alpha_n^{(2)}\beta_n^{(3)} + \alpha_n^{(1)}\beta_n^{(2)}\beta_n^{(3)} + \beta_n^{(1)}\beta_n^{(2)}\beta_n^{(3)})(1+u_n)^3\|x_n - p\| + d_n^{(3)},$$

where $d_n^{(3)} = \beta_n^{(3)}(1+u_n)d_n^{(2)} + \gamma_n^{(3)}\|u_n^{(3)} - p\|$. So that $\Sigma_{n=1}^{\infty} d_n^{(3)} < \infty$. By continuing the above method, there exists a nonnegative real sequence $\{d_n^{(N)}\}$ such that $\Sigma_{n=1}^{\infty} d_n^{(N)} < \infty$ and

$$\|x_{n+1} - p\| \leq (\alpha_n^{(N)} + \alpha_n^{(N-1)}\beta_n^{(N)} + \alpha_n^{(N-2)}\beta_n^{(N-1)}\beta_n^{(N)} + ... + \beta_n^{(1)}\beta_n^{(2)}\cdots\beta_n^{(N)})(1+u_n)^N\|x_n - p\| + d_n^{(N)}$$
$$\leq (1+u_n)^N\|x_n - p\| + d_n^{(N)}.$$

This implies that

$$\|x_{n+1} - p\| = \|x_n^{(N)} - p\| \leq (1+k_n)\|x_n - p\| + d_n^{(N)},$$

where $k_n = C_1^N u_n + C_2^N u_n^2 + ... + C_N^N u_n^N$. Since $\lim_{n\to\infty} u_n = 0$, it follows that

$$\lim_{n\to\infty} k_n = 0.$$

This completes (a).

Proof of (b). From (a), we obtain

$$\|x_{n+m} - p\| \leq (1+u_{n+m-1})^N\|x_{n+m-1} - p\| + d_{n+m-1}^N$$
$$\leq e^{Nu_{n+m-1}}\|x_{m+n-1} - p\| + d_{n+m-1}^N$$
$$\leq e^{N(u_{n+m-1}+u_{m+n-2})}\|x_{n+m-2} - p\| + e^{Nu_{n+m-1}}d_{n+m-2}^N + d_{n+m-1}^N$$
$$\leq e^{N(u_{n+m-1}+u_{m+n-2})}\|x_{n+m-2} - p\| + e^{Nu_{n+m-1}}(d_{n+m-2}^N + d_{n+m-1}^N)$$
$$\leq e^{N\Sigma_{k=n}^{n+m-1}u_k}\|x_n - p\| + e^{N\Sigma_{k=n}^{n+m-1}u_k}\Sigma_{k=n}^{n+m-1}d_k^N$$
$$\leq M\|x_n - p\| + M\Sigma_{k=n}^{n+m-1}d_k^N,$$

where $M = e^{N\Sigma_{k=n}^{\infty}u_k}$. This completes the proof of (b). ◇

Theorem 3.2 *Let X be a nonempty convex subset of uniformly convex Banach space. Let $T_1, T_2, ..., T_N$ be asymptotically quasi-nonexpansive mappings of X, with sequences $\{u_n^{(1)}\}$, ..., $\{u_n^{(N)}\}$ and $F = \cap_{i=1}^N F(T_i) \neq \phi$. Let $\{x_n\}$ be the sequence as defined by (2.1) with $\Sigma_{n=1}^{\infty}\gamma_n^i < \infty$ for all $i = 1, 2, ..., N$. Then $\{x_n\}$ converges to a common fixed point of the mappings $\{T_1, T_2, ...T_N\}$ if and only if* $\liminf_{n\to\infty} d(x_n, F) = 0$.

Proof. The neccessity of the condition is obvious. Thus, we will only prove the sufficiency. Let $\{u_n\}$, $\{k_n\}$ and $\{d_n^{(N)}\}$ be sequences as in Lemma 3.1. Suppose that $\liminf_{n\to\infty} d(x_n, F) = 0$. From Lemma 3.1 (a), we have

$$\|x_{n+1} - p\| \leq (1+k_n)\|x_n - p\| + d_n^{(N)}$$

for all $p \in F$ and for all $n \in \mathbf{N}$. Hence

$$d(x_{n+1}, F) \leq (1+k_n)d(x_n, F) + d_n^{(N)}.$$

Since $\liminf_{n\to\infty} d(x_n, F) = 0$, and $\lim_{n\to\infty} d_n^{(N)} = 0$, we have

$$\lim_{n\to\infty} d(x_n, F) = 0. \tag{3.1}$$

We next show that $\{x_n\}$ is a cauchy sequence. Let $\epsilon > 0$. By Lemma 3.1 (b), there exists a constant $M > 0$ such that

$$\|x_{n+m} - p\| \le M\|x_n - p\| + M\Sigma_{k=n}^{n+m-1} d_k^{(N)}, \qquad \forall p \in F, \quad m, n \in \mathbf{N}.$$

By (3.1) and $\Sigma_{n=1}^{\infty} d_n^{(N)} < \infty$, there exists $N_1 \in \mathbf{N}$ such that

$$d(x_n, F) < \frac{\epsilon}{3M}, \text{ and } \Sigma_{k=n}^{n+m-1} d_k^{(N)} < \frac{\epsilon}{3M},$$

for all $n \ge N_1$. So $d(x_{N_1}, F) < \frac{\epsilon}{3M}$, there exists $p_1 \in F$ such that

$$\|x_{N_1} - p_1\| \le \frac{\epsilon}{3M}.$$

Hence for any $n \ge N_1$, and for all $m \in \mathbf{N}$, we have

$$\begin{aligned}\|x_{n+m} - x_n\| &\le \|x_{n+m} - p_1\| + \|x_n - p_1\| \\ &\le M\|x_{N_1} - p_1\| + M\|x_{N_1} - p_1\| + M\Sigma_{k=n}^{n+m-1} d_k^{(N)} \\ &< M\tfrac{\epsilon}{3M} + M\tfrac{\epsilon}{3M} + M\tfrac{\epsilon}{3M} = \epsilon.\end{aligned}$$

Thus $\{x_n\}$ is a cauchy sequence. Since X is complete, it follows that $\{x_n\}$ is converges. That $x_n \longrightarrow p \quad as\ n \to \infty$ for some $p \in X$. It will be proven that p is a common fixed point. Let $\rho > 0$. Since $x_n \longrightarrow p$, thus there exists $N_2 \in \mathbf{N}$ such that

$$\|x_n - p\| \le \frac{\rho}{2(2 + u_1)} \qquad \forall n \ge N_2.$$

By (3.1) there exists a natural number $N_3 \ge N_2$ such that

$$d(x_n, F) < \frac{\rho}{2(2 + u_1)} \qquad \forall n \ge N_3.$$

This implies that $d(x_{N_3}, F) < \frac{\rho}{2(2+u_1)}$, and hence there exists $p_2 \in F$ such that

$$\|x_{N_3} - p_2\| < \frac{\rho}{2(2 + u_1)}.$$

Hence for each $i = 1, 2, 3, ..., N$, we have

$$\begin{aligned}\|T_i p - p\| &\le \|T_i p - p_2\| + \|p_2 - p\| \\ &\le (1 + u_1^i)\|p - p_2\| + \|p_2 - p\| \\ &\le (2 + u_1)\|p - p_2\| \\ &< (2 + u_1)\|x_{N_3} - p\| + (2 + u_1)\|x_{N_3} - p_2\| \\ &< (2 + u_1)\tfrac{\rho}{2(2+u_1)} + (2 + u_1)\tfrac{\rho}{2(2+u_1)} = \rho,\end{aligned}$$

since ρ is an arbitrary positive number. Thus $T_i p = p$ is a common fixed point. This complete the proof of Theorem. ◇

Corollary 3.3 *Let X be a nonempty convex subset of uniformly convex Banach space. Let $T_1 = T_2 = ... = T_N \equiv T$ be asymptotically quasi-nonexpansive mappings of X, with $F = \cap_{i=1}^N F(T_i) \neq \phi$. Let $\{x_n\}$ be the sequence as defined by (2.2) with $\Sigma_{n=1}^{\infty}\gamma_n^i < \infty$ for all $i = 1, 2, ..., N$. Then $\{x_n\}$ converges to a fixed point of T if and only if* $\liminf_{n\to\infty} d(x_n, F) = 0$.

Corollary 3.4 *[6] Let X be a nonempty convex subset of uniformly convex Banach space. Let T be asymptotically quasi-nonexpansive mappings of X, with $F = \cap_{i=1}^N F(T_i) \neq \phi$. Let $\{x_n\}$ be the sequence as defined by (2.3) with $\Sigma_{n=1}^{\infty}\gamma_n^{'} < \infty, \Sigma_{n=1}^{\infty}\gamma_n < \infty$. Then $\{x_n\}$ converges to a fixed point of T if and only if* $\liminf_{n\to\infty} d(x_n, F) = 0$.

In the next result, we prove two sufficiency conditions of strong convergence theorems for the multi-step iterative scheme with errors for a finite family of asymptotically quasi-nonexpansive mappings. To do this, we need a lemma.

Lemma 3.5 *Let X be a uniformly convex Banach space. Let $T_1, T_2, ..., T_N$ be $(L-\alpha)$ uniform lipschitz asymptotically quasi-nonexpansive self mappings of subset of X. Then* $\lim_{n\to\infty}\|x_n - T_i x_n\| = 0$ *for all $i = 1, 2, ..., N$.*

Proof. It follows from Lemma 3.1 that $\|x_{n+1} - p\| \leq (1 + k_n)\|x_n - p\| + d_n^{(N)}$, For each $n \geq 1$, let $u_n = \max\{u_n^{(1)}, ..., u_n^{(N)}\}$ where $\Sigma_{n=1}^{\infty} d_n^{(N)} < \infty$. Hence, by Lemma 2.2, $\lim_{n\to\infty}\|x_n - p\| = c$ exists.
We note that

$$\limsup_{n\to\infty}\|x_n - p + \gamma_n^{(i)}(v_n^{(i)} - x_n)\| \leq \limsup_{n\to\infty}\|x_n - p\| + \limsup_{n\to\infty}\|\gamma_n^{(i)}(v_n^{(i)} - x_n)\| \leq c.$$

This implies that

$$\limsup_{n\to\infty}\|T_N^n x_n^{(N-1)} - p + \gamma_n^{(N)}(v_n^{(N)} - x_n)\| \leq \limsup_{n\to\infty}(1 + u_n)\|x_n^{(N-1)} - p\| + \limsup\nolimits_{n\to\infty}\|\gamma_n^{(N)}(v_n^{(N)} - x_n)\| \leq c.$$

Hence

$$\begin{aligned}
\lim\nolimits_{n\to\infty}\|x_n - p\| &= \lim\nolimits_{n\to\infty}\|x_{n+1} - p\| \\
&= \lim\nolimits_{n\to\infty}\|\alpha_n^{(N)}x_n + \beta_n^{(N)}T_N^n x_n^{(N-1)} + \gamma_n^{(N)}v_n^{(N)} - p\| \\
&= \lim\nolimits_{n\to\infty}\|\beta_n^{(N)}T_N^n x_n^{(N-1)} + (1 - \beta_n^{(N)})x_n - \gamma_n^{(N)}x_n + \gamma_n^{(N)}v_n^{(N)} \\
&\quad -(1 - \beta_n^{(N)})p - \beta_n^{(N)}p\| \\
&= \lim\nolimits_{n\to\infty}\|\beta_n^{(N)}(T_N^n x_n^{(N-1)} - p) + (1 - \beta_n^{(N)})(x_n - p) + \gamma_n^{(N)}(v_n^{(N)} - x_n)\| \\
&= \lim\nolimits_{n\to\infty}\|\beta_n^{(N)}(T_N^n x_n^{(N-1)} - p) + (1 - \beta_n^{(N)})(x_n - p) \\
&\quad +\beta_n^{(N)}(\gamma_n^{(N)}(v_n^{(N)} - x_n)) - \beta_n^{(N)}(\gamma_n^{(N)}(v_n^{(N)} - x_n)) + \gamma_n^{(N)}(v_n^{(N)} - x_n)\| \\
&= \lim\nolimits_{n\to\infty}\|\beta_n^{(N)}(T_N^n x_n^{(N-1)} - p + \gamma_n^{(N)}(v_n^{(N)} - x_n)) \\
&\quad +(1 - \beta_n^{(N)})(x_n - p + \gamma_n^{(N)}(v_n^{(N)} - x_n))\|
\end{aligned}$$

By Lemma 2.1, we have

$$\lim_{n\to\infty} \|T_N^n x_n^{(N-1)} - x_n\| = 0. \tag{3.2}$$

We observe that

$$\begin{aligned}\limsup_{n\to\infty} \|T_{N-1}^n x_n^{(N-2)} - p + \gamma_n^{(N-1)}(v_n^{(N-1)} - x_n)\| &\le \limsup_{n\to\infty}(1+u_n)\|x_n^{(N-2)} - p\| \\ &+ \limsup_{n\to\infty} \|\gamma_n^{(N-1)}(v_n^{(N-1)} - x_n)\| \le c,\end{aligned}$$

and
$$\begin{aligned}\|x_n - p\| &\le \|x_n - T_N^n x_n^{(N-1)}\| + \|T_N^n x_n^{(N-1)} - p\| \\ &\le \|x_n - T_N^n x_n^{(N-1)}\| + (1+u_n)\|x_n^{(N-1)} - p\|.\end{aligned}$$

Thus, we have

$$c = \lim_{n\to\infty} \|x_n - p\| \le \liminf_{n\to\infty} \|x_n^{(N-1)} - p\| \le \limsup_{n\to\infty} \|x_n^{(N-1)} - p\| \le c$$

$$\lim_{n\to\infty} \|x_n^{(N-1)} - p\| = c.$$

It implies that

$$\begin{aligned}c = \lim_{n\to\infty} \|x_n^{(N-1)} - p\| &= \lim_{n\to\infty} \|\alpha_n^{(N-1)} x_n + \beta_n^{(N-1)} T_{N-1}^n x_n^{(N-2)} \\ &+ \gamma_n^{(N-1)} v_n^{(N-1)} - p\| \\ &= \lim_{n\to\infty} \|\beta_n^{(N-1)} T_{N-1}^n x_n^{(N-2)} + (1-\beta_n^{(N-1)})x_n - \gamma_n^{(N-1)} x_n \\ &+ \gamma_n^{(N-1)} v_n^{(N-1)} - (1-\beta_n^{(N-1)})p - \beta_n^{(N-1)} p\| \\ &= \lim_{n\to\infty} \|\beta_n^{(N-1)}(T_{N-1}^n x_n^{(N-2)} - p) + (1-\beta_n^{(N-1)})(x_n - p) \\ &+ \gamma_n^{(N-1)}(v_n^{(N-1)} - x_n)\| \\ &= \lim_{n\to\infty} \|\beta_n^{(N-1)}(T_{N-1}^n x_n^{(N-2)} - p) + (1-\beta_n^{(N-1)})(x_n - p) \\ &+ \beta_n^{(N-1)}(\gamma_n^{(N-1)}(v_n^{(N-1)} - x_n)) - \beta_n^{(N-1)}(\gamma_n^{(N-1)}(v_n^{(N-1)} - x_n)) \\ &+ \gamma_n^{(N-1)}(v_n^{(N-1)} - x_n)\| \\ &= \lim_{n\to\infty} \|\beta_n^{(N-1)}(T_{N-1}^n x_n^{(N-2)} - p + \gamma_n^{(N-1)}(v_n^{(N-1)} - x_n)) \\ &+ (1-\beta_n^{(N-1)})(x_n - p + \gamma_n^{(N-1)}(v_n^{(N-1)} - x_n))\|.\end{aligned}$$

By Lemma 2.1, we have

$$\lim_{n\to\infty} \|T_{N-1}^n x_n^{(N-2)} - x_n\| = 0.$$

By continuing the above method, we have

$$\lim_{n\to\infty} \|T_i^n x_n^{(i-1)} - x_n\| = 0 \tag{3.3}$$

for $i = 2, 3, ..., N$. We note that

$$\begin{aligned}\limsup_{n\to\infty} \|T_1^n x_n - p + \gamma_n^{(1)}(v_n^{(1)} - x_n)\| &\le \limsup_{n\to\infty}(1+u_n)\|x_n - p\| \\ &+ \limsup_{n\to\infty} \|\gamma_n^{(1)}(v_n^{(1)} - x_n)\| \le c\end{aligned}$$

and
$$\begin{aligned}\|x_n - p\| &\le \|x_n - T_2^n x_n^{(1)}\| + \|T_2^n x_n^{(1)} - p\| \\ &\le \|x_n - T_2^n x_n^{(1)}\| + (1+u_n)\|x_n^{(1)} - p\|.\end{aligned}$$

Thus, we have

$$c = \lim_{n\to\infty} \|x_n - p\| \le \liminf_{n\to\infty} \|x_n^{(1)} - p\| \le \limsup_{n\to\infty} \|x_n^{(1)} - p\| \le c,$$

$$\lim_{n\to\infty} \|x_n^{(1)} - p\| = c.$$

It implies that

$$\begin{aligned}
c = \lim_{n\to\infty} \|x_n^{(1)} - p\| &= \lim_{n\to\infty} \|\alpha_n^{(1)} x_n + \beta_n^{(1)} T_1^n x_n + \gamma_n^{(1)} v_n^{(1)} - p\| \\
&= \lim_{n\to\infty} \|\beta_n^{(1)} T_1^n x_n + (1-\beta_n^{(1)}) x_n - \gamma_n^{(1)} x_n\| + \gamma_n^{(1)} v_n^{(1)} - (1-\beta_n^{(1)}) p - \beta_n^{(1)} p\| \\
&= \lim_{n\to\infty} \|\beta_n^{(1)} (T_1^n x_n - p) + (1-\beta_n^{(1)})(x_n - p) + \gamma_n^{(1)} (v_n^{(1)} - x_n)\| \\
&= \lim_{n\to\infty} \|\beta_n^{(1)} (T_1^n x_n - p) + (1-\beta_n^{(1)})(x_n - p) + \beta_n^{(1)} (\gamma_n^{(1)} (v_n^{(1)} - x_n)) \\
&\quad - \beta_n^{(1)} (\gamma_n^{(1)} (v_n^{(1)} - x_n)) + \gamma_n^{(1)} (v_n^{(1)} - x_n)\| \\
&= \lim_{n\to\infty} \|\beta_n^{(1)} (T_1^n x_n - p + \gamma_n^{(1)} (v_n^{(1)} - x_n)) \\
&\quad + (1-\beta_n^{(1)})(x_n - p + \gamma_n^{(1)} (v_n^1 - x_n))\|
\end{aligned}$$

By Lemma 2.1, we have

$$\lim_{n\to\infty} \|T_1^n x_n - x_n\| = 0. \tag{3.4}$$

Since $\|x_n^{(1)} - x_n\| \le \alpha_n^{(1)} \|x_n - x_n\| + \beta_n^{(1)} \|T_1^n x_n - x_n\| + \gamma_n^{(1)} \|v_n^{(1)} - x_n\|$ for all $n \in N$, it follows by (3.4) that $\lim_{n\to\infty} \|x_n^{(1)} - x_n\| = 0$. Similarly, since

$$\|x_n^{(2)} - x_n\| \le \alpha_n^{(2)} \|x_n - x_n\| + \beta_n^{(2)} \|T_2^n x_n^1 - x_n\| + \gamma_n^{(2)} \|v_n^{(2)} - x_n\|$$

it follows by (3.4) that $\lim_{n\to\infty} \|x_n^{(2)} - x_n\| = 0$. By continuing the above method, we have

$$\lim_{n\to\infty} \|x_n^{(i)} - x_n\| = 0, \tag{3.5}$$

for all $i = 1, 2, ..., N$.

By (3.4), we have $\lim_{n\to\infty} \|T_1^n x_n - x_n\|$. Consider, for $i = 2, 3, ..., N$ it follows that

$$\begin{aligned}
\|T_i^n x_n - x_n\| &\le \|T_i^n x_n - T_i^n x_n^{(i-1)}\| + \|T_i^n x_n^{(i-1)} - x_n\| \\
&\le L_i \|x_n - x_x^{i-1}\|^\alpha + \|T_i^n x_n^{(i-1)} - x_n\|,
\end{aligned}$$

by (3.3), (3.4) and (3.5) we have

$$\lim_{n\to\infty} \|T_i^n x_n - x_n\|. \tag{3.6}$$

Consider,

$$\begin{aligned}
\|T_1 x_n - x_n\| &\le \|T_1 x_n - T_1^{n+1} x_n\| + \|T_1^{n+1} x_n - T_1^{n+1} x_{n+1}\| \\
&\quad + \|T_1^{n+1} x_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| \\
&\le L_{1,1} \|x_n - T_1^n x_n\|^\alpha + L_{1,2} \|x_n - x_{n+1}\|^\alpha \\
&\quad + \|T_1^{n+1} x_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\|,
\end{aligned}$$

by (3.3), (3.4) and (3.5) we have $\lim_{n\to\infty} \|T_1 x_n - x_n\| = 0$. Next, we note that

$$\begin{aligned}
\|T_2 x_n - x_n\| &\le \|T_2 x_n - T_2^{n+1} x_n^{(1)}\| + \|T_2^{n+1} x_n^{(1)} - T_2^{n+1} x_n\| \\
&\quad + \|T_2^{n+1} x_n - T_2^{n+1} x_{n+1}\| + \|T_2^{n+1} x_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| \\
&\le L_{2,1} \|x_n - T_2^n x_n^{(1)}\|^\alpha + L_{2,2} \|x_n^{(1)} - x_n\|^\alpha + L_{2,3} \|x_n - x_{n+1}\|^\alpha \\
&\quad + \|T_2^{n+1} x_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\|,
\end{aligned}$$

by (3.3), (3.4) and (3.5) we have $\lim_{n\to\infty} \|T_2 x_n - x_n\| = 0$. Next, we note that

$$\begin{aligned}
\|T_3 x_n - x_n\| &\le \|T_3 x_n - T_3^{n+1} x_n^{(2)}\| + \|T_3^{n+1} x_n^{(2)} - T_3^{n+1} x_n\| \\
&\quad + \|T_3^{n+1} x_n - T_3^{n+1} x_{n+1}\| + \|T_3^{n+1} x_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| \\
&\le L_{2,1} \|x_n - T_3^n x_n^{(2)}\|^\alpha + L_{2,2} \|x_n^{(2)} - x_n\|^\alpha + L_{2,3} \|x_n - x_{n+1}\|^\alpha \\
&\quad + \|T_3^{n+1} x_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\|,
\end{aligned}$$

by (3.3), (3.4) and (3.5) we have $\lim_{n\to\infty}\|T_3x_n - x_n\| = 0$. By continuing the above method we have

$$\lim_{n\to\infty}\|T_ix_n - x_n\| = 0, \tag{3.7}$$

for all $i = 1, 2, ..., N$. This complete lemma. ◇

Theorem 3.6 *Let C be a nonempty compact convex subset of uniformly convex Banach space. Let $T_1, T_2, ..., T_N$ be $(L-\alpha)$ uniform lipschitz asymptotically quasi-nonexpansive mappings of X, with sequences $\{u_n^{(1)}\}$, ..., $\{u_n^{(N)}\}$ are $\Sigma_{n=1}^{\infty}u_n^i < \infty$ and $F = \cap_{i=1}^N F(T_i) \neq \phi$. Let $x_0 \in X$ and F, then the sequence $\{x_n\}$ define by (2.1) converge to common fixed point of $T_1, T_2, ..., T_N$.*

Proof. Since C is compact, thus $\{x_n\}$ has a convergence subsequence $\{x_{n_k}\}$ of $\{x_n\}$, say

$$\lim_{k\to\infty} x_{n_k} = q \tag{3.8}$$

and hence

$$\|x_{n_k+1} - x_{n_k}\| \leq \alpha_{n_k}^{(N)}\|x_{n_k} - x_{n_k}\| + \beta_{n_k}^{(N)}\|T_N^{n_k}x_{n_k} - x_{n_k}\| + \gamma_{n_k}^{(N)}\|v_{n_k}^{(N)} - x_{n_k}\| \longrightarrow 0 \text{ as } n \to \infty. \tag{3.9}$$

By Lemma 3.5, (3.4), (3.8)

$$\|T_N^{n_k}x_{n_k}^{(N-1)} - q\| \leq |T_N^{n_k}x_{n_k}^{(N-1)} - x_{n_k}\| + \|x_{n_k} - q\| \longrightarrow 0 \text{ as } n \to \infty. \tag{3.10}$$

By Lemma 3.5 and (3.8), it follows that

$$\|T_i^{n_k}x_{n_k}^{(i-1)} - q\| \leq \|T_i^{n_k}x_{n_k}^{(i-1)} - x_{n_k}\| + \|x_{n_k} - q\| \to 0 \text{ as } n \to \infty.$$

By Lemma 3.5 and (3.8) it follows that

$$\|T_1^{n_k}x_{n_k} - q\| \leq \|T_1^{n_k}x_{n_k} - x_{n_k}\| + \|x_{n_k} - q\| \to 0 \text{ as } n \to \infty.$$

From Lemma 3.5, (3.9) and (3.10) it follows that

$$\begin{aligned} 0 \leq \|q - T_iq\| &\leq \|q - T_i^{n_k+1}x_{n_k+1}^{(i-1)}\| + \|T_i^{n_k+1}x_{n_k+1}^{(i-1)} - T_i^{n_k+1}x_{n_k+1}\| \\ &\quad + \|T_i^{n_k+1}x_{n_k+1} - T_i^{n_k+1}x_{n_k}\| + \|T_i^{n_k+1}x_{n_k} - T_i^{n_k+1}x_{n_k}^{(i-1)}\| \\ &\quad + \|T_i^{n_k+1}x_{n_k}^{(i-1)} - T_iq\| \\ &\leq \|q - T_i^{n_k+1}x_{n_k+1}^{(i-1)}\| + L\|x_{n_k+1}^{(i-1)} - x_{n_k+1}\|^\alpha \\ &\quad + L\|x_{n_k+1} - x_{n_k}\|^\alpha + L\|x_{n_k} - x_{n_k}^{(i-1)}\|^\alpha + L\|T_i^{n_k}x_{n_k}^{(i-1)} - q\|^\alpha \\ &\qquad \longrightarrow 0 \text{ as } k \to \infty. \end{aligned}$$

Thus q is a common fixed point of $T_1, T_2, ..., T_N$. Since the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to q from (3.8), we have $\lim_{n\to\infty} x_n = q$. ◇

Let C be a nonempty subset of Banach space. A family $\{T_i : i = 1, 2, ..., N\}$ of self-mappings of C (i.e. $T_1, T_2, ..., T_N : C \to C$) with $F = \cap_{i=1}^N F(T_i) \neq \phi$ is said to satisfying condition (B) on C if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that for all $x \in C$

$$\max_{1\leq i\leq N}\{\|x - T_ix\|\} \geq f(d(x, F)).$$

Note that condition (B) reduces to condition (A) when $T_1 = T_2 = ... = T_N = T$.

Theorem 3.7 *Let C be a nonempty convex subset of uniformly convex Banach space. Let $T_1, T_2, ..., T_N$ be $(L-\alpha)$ uniform lipschitz asymptotically quasi-nonexpansive mappings of C and satisfying condition (B), with sequences $\{u_n\}$ as define by Lemma 3.1. Let $x_0 \in C$ and $F \neq \phi$, then the sequence $\{x_n\}$ define by (2.1) converge to common fixed point of $T_1, T_2, ..., T_N$.*

Proof. By Lemma 3.5, we have $\lim_{n\to\infty} \|T_i x_n - x_n\| = 0$ for all $i = 1, 2, ..., N$. Since $T_1 = T_2 = ... = T_N = T$ are satisfying condition (B) there exists $f : [0,\infty) \to [0,\infty)$ is nondecreasing function such that

$$\Sigma_{i=1}^{N} \|T_i x_n - x_n\| \geq f(d(x_n, F)).$$

Then $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since f is nondecreasing function and $f(0) = 0$,

$$\lim_{n\to\infty} d(x_n, F) = 0,$$

it follows, as in the proof of Theorem 3.2, that $\{x_n\}$ converges strongly to some common fixed point in F. This completes the proof. ◇

Theorem 3.8 *Let C be a nonempty convex subset of uniformly convex Banach space. Let $T_1 = T_2 = ... = T_N = T$ be $(L-\alpha)$ uniform lipschitz asymptotically quasi-nonexpansive mappings of C and satisfying condition (A), with sequences $\{u_n\}$ as define by Lemma 3.1. Let $x_0 \in C$ and $F \neq \phi$, then the sequence $\{x_n\}$ define by (2.2) converge to fixed point of T.*

Proof. By Lemma 3.5, we have $\lim_{n\to\infty} \|T_i x_n - x_n\| = 0$ for all $i = 1, 2, ..., N$. Since T is satisfying condition (A) there exists $f : [0,\infty) \to [0,\infty)$ is nondecreasing function such that

$$\|T x_n - x_n\| \geq f(d(x_n, F)).$$

Then $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since f is nondecreasing function and $f(0) = 0$,

$$\lim_{n\to\infty} d(x_n, F) = 0,$$

it follows, as in the proof of Theorem 3.2, that $\{x_n\}$ converges strongly to some fixed point in F. This completes the proof. ◇

For $N = 2$ and $T_1 = T_2 \equiv T$, then (2.1) reduces to the modified Ishikawa iterative scheme with errors, the sequence define by (2.3) converges to common fixed point of T_1, T_2.

Corollary 3.9 *[5] Let C be a nonempty compact subset of a uniformly convex Banach space, and T are $(L-\alpha)$ uniform lipschitz asymptotically quasi-nonexpansive mappings on C with $\Sigma_{n=1}^{\infty} u_n < \infty$. Let $x_0 \in C$ and F is nonempty. Then $\{x_n\}$ define by (2.3) converges to fixed point of T.*

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