

Viscosity Approximation of a Zero of Accretive Operator in Banach Space¹

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Abstract. This paper introduces a iteration scheme for viscosity approximation of a zero of accretive operator in a reflexive Banach space with weakly continuous duality mapping. This new iteration scheme is defined as follows:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily} \\ y_n = \alpha_n x_n + (1 - \alpha_n) J_{r_n} x_n, & n \geq 0 \\ x_{n+1} = \beta_n f(x_n) + (1 - \beta_n) y_n, & n \geq 0 \end{cases}$$

where f is a contraction and J_r denotes resolvent $(I + rA)^{-1}$ for $r > 0$, sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in $(0, 1)$. Under certain appropriate assumptions on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$, strong convergence of the algorithm $\{x_n\}$ is proved. The results improve and extend the results of T.H.Kim, H.K.Xu and some others.

Keywords: Fixed point; nonexpansive mapping; m-accretive operator; reflexive Banach space; weakly continuous duality mapping

1. Introduction and Preliminaries

Let E be a real Banach space, C a nonempty closed convex subset of E , and $T : C \rightarrow C$ a mapping. Recall that T is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $Fix(T)$ the set of fixed points of T , that is, $Fix(T) = \{x \in C : Tx = x\}$. It is assumed throughout the paper that T is a nonexpansive mapping such that $Fix(T) \neq \emptyset$. Recall that a self mapping $f : C \rightarrow C$ is a contraction on C if there exists a constant $\alpha \in (0, 1)$ such that $\|f(x) - f(y)\| \leq \alpha \|x - y\|$,

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$x, y \in C$. The normalized duality mapping J from E into 2^{E^*} is given by $J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}$, $x \in E$ where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing.

We define a mapping $T_t x = tf(x) + (1-t)Tx$, $x \in C$. It is obviously that T_t is a contraction on C . In fact, for $x, y \in C$, we obtain

$$\begin{aligned} \|T_t x - T_t y\| &\leq \|t(f(x) - f(y)) + (1-t)(Tx - Ty)\| \\ &\leq \alpha t \|x - y\| + (1-t)\|Tx - Ty\| \\ &\leq \alpha t \|x - y\| + (1-t)\|x - y\| \\ &= (1-t(1-\alpha))\|x - y\|. \end{aligned}$$

Let x_t be the unique fixed point of T_t . That is, x_t is the unique solution of the fixed point equation

$$x_t = tf(x_t) + (1-t)Tx_t,$$

A special case has been considered by Browder [1] in a Hilbert space as follows. Fix $u \in C$ and define a contraction S_t on C by

$$S_t x = tu + (1-t)Tx, \quad x \in C. \quad (1.1)$$

If we use z_t to denote the unique fixed point of S_t , which yield that

$$z_t = tu + (1-t)Tz_t.$$

In 2004, Hong-Kun Xu[2] proves the following theorem in a uniformly smooth Banach space :

Theorem 1.1 Let E be a uniformly smooth Banach space space, C be a closed convex subset of X , $T : C \rightarrow C$ a nonexpansive mapping with $F(T) \neq \emptyset$, and $f \in \Pi_C$, where Π_C denotes the set of all contractions on C . Then $\{x_n\}$ defined by the following:

$$x_t = tf(x_t) + (1-t)Tx_t, \quad x \in C \quad (1.2)$$

converges strongly to a fixed point of T . If we define $Q : \Pi_C \rightarrow F(T)$ by

$$Q(f) := \lim_{t \rightarrow 0} x_t, f \in \Pi_C$$

then $Q(f)$ solves the variation inequality

$$\langle (I - f)Q(f), J(Q(f) - p) \rangle \leq 0, f \in \Pi_C, p \in F(T).$$

In 2006, H.K.XU[2] proved the strong convergence of $\{x_t\}$ defined by (1.1) in a reflexive Banach space with a weakly continuous duality mapping J_φ with gauge φ .

Recently Chen and Zhu[3] proved the following theorem in a reflexive Banach space and has a weakly continuous duality mapping J_φ with gauge φ .

Theorem 1.2 Let E be a real reflexive Banach space and has a weakly continuous duality mapping J_φ with gauge φ . Suppose C is a closed convex subset of E and $T : C \rightarrow C$ is a nonexpansive mapping, $f : C \rightarrow C$ be a fixed contractive mapping. For $t \in (0, 1)$, $\{x_t\}$ is defined by (1.2), Then T has a

fixed point if and only of $\{x_t\}$ remains bounded as $t \rightarrow 0^+$, and in this case, $\{x_t\}$ converges strongly to a fixed point of T as $t \rightarrow 0^+$.

Recall that an operator A with $D(A)$ and $R(A)$ in E is said to be *accretive*, if for each $x_i \in D(A)$ and $y_i \in A(x_i)(i = 1, 2)$, there is a $j \in J(x_2 - x_1)$ such that

$$\langle y_2 - y_1, J(x_2 - x_1) \rangle \geq 0,$$

where J is the duality mapping from E to the E^* given by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}, \quad x \in E.$$

An accretive operator A is m -accretive if $R(I + \lambda A) = E$ for all $\lambda > 0$

Denote by F the zero set of A : i.e.,

$$F := A^{-1}(0) = \{x \in D(A) : 0 \in Ax\}.$$

Throughout the rest of this paper it is always assumed that A is m -accretive and F is nonempty.

Denote by J_r the resolvent of A for $r > 0$:

$$J_r = (I + rA)^{-1}.$$

It is well known that J_r is nonexpansive mapping from E to $C := D(A)$ which will be assumed convex.

Chen and Zhu[3] studied the sequence generated by the algorithm

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)J_{r_n}x_n, \quad n \geq 0. \tag{1.3}$$

and proved strongly convergence of scheme(1.3) in a reflexive Banach space with a weakly continuous duality mapping J_φ with gauge φ .

Recently Q and Su[4] introduced the following iterative sequence

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily} \\ y_n = \alpha_n x_n + (1 - \alpha_n)J_{r_n}x_n, & n \geq 0 \\ x_{n+1} = \beta_n u + (1 - \beta_n)y_n, & n \geq 0 \end{cases} \tag{1.4}$$

and proved strongly convergence of scheme (1.4) in the framework of uniformly smooth Banach spaces.

Inspired and motivated by the iterative sequences (1.3) and (1.4), this paper given the following iterative sequences

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily} \\ y_n = \alpha_n x_n + (1 - \alpha_n)J_{r_n}x_n, & n \geq 0 \\ x_{n+1} = \beta_n f(x_n) + (1 - \beta_n)y_n, & n \geq 0 \end{cases} \tag{1.5}$$

where f is a contraction, T is a nonexpansive mapping from C into C and sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $(0, 1)$. Under certain appropriate assumptions on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$, we prove, $\{x_n\}$ defined by (1.5) converges to a zero of A .

If $\beta_n = 0$ in (1.5) then we have the iterative sequence defined by (1.3).

The purpose of this paper is to introduce this composite iteration scheme for approximating a fixed point of a accretive mapping by viscosity approximation methods in a reflexive Banach spaces. We establish the strong convergence of the composite iteration scheme $\{x_n\}$ defined by (1.5). The results improve and extend results of Kim and Chen[3] and Q [4] and some others.

In order to prove our main results, we need the following definitions and Lemmas for the proof of our main results.

Lemma 1.1 In a Banach space E , there holds the inequality

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle, \quad x, y \in E$$

where $j(x + y) \in J(x + y)$.

Lemma 1.2 (Xu,[5,6]) Let $\Sigma_{n=0}^{\infty}\{\alpha_n\}$ be a sequence of nonnegative real numbers satisfying the condition

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \gamma_n\sigma_n, \quad n \geq 0,$$

where $\{\gamma_n\}_{n=0}^{\infty} \in (0, 1)$ and $\{\sigma_n\}_{n=0}^{\infty}$ such that

- (i) $\lim_{n \rightarrow \infty} \gamma_n = 0$ and $\Sigma_{n=0}^{\infty}\{\gamma_n\} = \infty$,
- (ii) either $\lim_{n \rightarrow \infty} \sup \sigma_n \leq 0$ or $\Sigma_{n=0}^{\infty}|\gamma_n\sigma_n| < \infty$.

Then $\{\alpha_n\}_{n=0}^{\infty}$ converges to zero.

Lemma 1.3 (The Resolvent Identity[7]) For $\lambda > 0$ and $\mu > 0$ and $x \in E$,

$$J_{\lambda}x = J_{\mu}\left(\frac{\mu}{\lambda} + \left(1 - \frac{\mu}{\lambda}\right)J_{\lambda}\right)x$$

Recall that a gauge is a continuous strictly increasing function $\varphi : [0, \infty) \rightarrow [0, \infty)$ such that $\varphi(0) = 0$ and $\varphi(t) \rightarrow \infty$ as $t \rightarrow \infty$. Associated to a gauge φ is the duality mapping $J_{\varphi} : E \rightarrow E^*$ defined by

$$J_{\varphi}(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|\varphi(\|x\|), \|x\|^* = \varphi(\|x\|)(\|x\|)\}, \quad x \in E.$$

Following Browder [1], we say that a Banach space E has a weakly continuous duality map if there exists a gauge φ for which the duality map J_{φ} is single-valued and *weak - to - weak** sequence continuous (i.e., if x_n is a sequence in E weakly convergent to a point x , then the sequence $J_{\varphi}(x_n)$ convergent weakly to $J_{\varphi}(x)$). It is known that l^p has a weakly continuous duality map for all $1 < p < \infty$. Set

$$\Phi(t) = \int_0^t \varphi(\tau)d(\tau), t \geq 0.$$

Then

$$J_{\varphi}(x) = \partial\Phi(\|x\|), x \in E.$$

where ∂ denotes the subdifferential in the sense of convex analysis. The first part of the next lemma is an immediate consequence of the subdifferential

inequality and the proof of the can be found in [8].

Lemma 1.4 Assume that E has a weakly duality map J_φ with gauge φ

(i) For all $x, y \in E$, there holds the inequality

$$\Phi(\|x + y\|) \leq \Phi(\|x\|) + \langle y, J_\varphi(x + y) \rangle.$$

(ii) Assume a sequence x_n in E is weakly convergent to a point x , then there holds the inequality

$$\limsup_{n \rightarrow \infty} \Phi(\|x_n - y\|) = \limsup_{n \rightarrow \infty} \Phi(\|x_n - x\|) + \Phi(\|y - x\|), x, y \in E.$$

remark” \rightarrow ” stands for strong convergence and ” \rightharpoonup ” for weak convergence

2. Main Results

Theorem 2.1 Let E be a real reflexive and has a weakly duality map J_φ with gauge φ Banach space and A is a m -accretive map in E such that $C = \overline{D(A)}$ is convex with $F = A^{-1}(0) \neq \emptyset$, and $f : C \rightarrow C$ be a fixed contraction mapping, $\{\alpha_n\}_{n=0}^\infty$ in $(0,1)$ and $\{\beta_n\}_{n=0}^\infty$ in $[0,1]$, suppose $\{\alpha_n\}_{n=0}^\infty$, $\{\beta_n\}_{n=0}^\infty$ and $\{\gamma_n\}_{n=0}^\infty$ satisfy the following conditions :

- (i) $\sum_{n=0}^\infty \beta_n = \infty$, $\beta_n \rightarrow 0$, as $n \rightarrow \infty$;
 - (ii) $\beta_n \in [0, a)$, for some $a \in (0, 1)$ and $\gamma_n \geq \varepsilon$ for all n ;
 - (iii) $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^\infty |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^\infty |\gamma_{n+1} - \gamma_n| < \infty$
- Let $\{x_n\}_{n=1}^\infty$ be the composite process defined by

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily} \\ y_n = \alpha_n x_n + (1 - \alpha_n) J_{r_n} x_n, & n \geq 0 \\ x_{n+1} = \beta_n f(x_n) + (1 - \beta_n) y_n, & n \geq 0 \end{cases} \quad (2.1)$$

Then $\{x_n\}_{n=0}^\infty$ converges strongly to a zero of A .

Proof: First we observe that $\{x_n\}_{n=0}^\infty$ is bounded. Indeed, if we take a point $p \in F = A^{-1}(0)$, nothing that

$$\begin{aligned} \|y_n - p\| &= \|\alpha_n x_n + (1 - \alpha_n) J_{r_n} x_n - p\| \\ &= \|\alpha_n (x_n - p) + \alpha_n p + (1 - \alpha_n) J_{r_n} x_n - p\| \\ &\leq \|\alpha_n (x_n - p) + (1 - \alpha_n) (J_{r_n} x_n - p)\| \\ &\leq \alpha_n \|x_n - p\| + (1 - \alpha_n) \|x_n - p\| = \|x_n - p\| \end{aligned} \quad (2.2)$$

We have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\beta_n f(x_n) + (1 - \beta_n)y_n - p\| \\
&= \|(1 - \beta_n)(y_n - p) + \beta_n(f(x_n) - f(p) + \beta_n(f(p) - p))\| \\
&\leq (1 - \beta_n)\|y_n - p\| + \alpha\beta_n\|x_n - p\| + \beta_n\|f(p) - p\| \\
&\leq (1 - \beta_n)\|x_n - p\| + \alpha\beta_n\|x_n - p\| + \beta_n\|f(p) - p\| \\
&\leq (1 - \beta_n + \alpha\beta_n)\|x_n - p\| + \beta_n\|f(p) - p\| \\
&= (1 - (1 - \alpha)\beta_n)\|x_n - p\| + \beta_n\|f(p) - p\| \\
&\leq \max\left\{\frac{1}{1 - \alpha}\|f(p) - p\|, \|x_n - p\|\right\}
\end{aligned}$$

Now, an induction gives that

$$\|x_n - p\| \leq \max\left\{\frac{1}{1 - \alpha}\|f(p) - p\|, \|x_0 - p\|\right\}, \quad n \geq 0.$$

This implies that $\{x_n\}$, $\{f(x_n)\}$ and $\{y_n\}$ are bounded.

Furthermore, from the definition of x_{n+1} and condition (i), we obtain that

$$\begin{aligned}
\|x_{n+1} - y_n\| &= \|\beta_n f(x_n) + (1 - \beta_n)y_n - y_n\| \\
&= \beta_n\|f(x_n) - y_n\| \rightarrow 0
\end{aligned} \tag{2.3}$$

Next we show that

$$\|x_{n+1} - x_n\| \rightarrow 0$$

In order to prove (2.3), we calculate $y_{n+1} - y_n$ firstly.

From (2.1), we have

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n)J_{r_n} x_n, \\ y_{n-1} = \alpha_{n-1} x_{n-1} + (1 - \alpha_{n-1})J_{r_{n-1}} x_{n-1} \end{cases}$$

Simple calculations show that

$$\begin{aligned}
y_n - y_{n+1} &= (1 - \alpha_n)(J_{r_n} x_n - J_{r_{n-1}} x_{n-1}) \\
&\quad + (\alpha_n - \alpha_{n-1})(x_{n-1} - J_{r_{n-1}} x_{n-1}) + \alpha_n(x_n - x_{n-1})
\end{aligned} \tag{2.4}$$

It follows that

$$\begin{aligned}
\|y_n - y_{n-1}\| &\leq (1 - \alpha_n)\|J_{r_n} x_n - J_{r_{n-1}} x_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}|\|x_{n-1} - J_{r_{n-1}} x_{n-1}\| + \alpha_n\|x_n - x_{n-1}\|
\end{aligned} \tag{2.5}$$

From Lemma 1.3 (the resolvent identity) implies that

$$J_{r_n} x_n = J_{r_{n-1}} \left(\frac{r_{n-1}}{r_n} + \left(1 - \frac{r_{n-1}}{r_n}\right) J_{r_n} x_n \right)$$

If $r_{n-1} \leq r_n$, using the resolvent identity

$$\begin{aligned}
\|J_{r_n}x_n - J_{r_{n-1}}x_{n-1}\| &\leq \left\| \frac{r_{n-1}}{r_n}x_n + \left(1 - \frac{r_{n-1}}{r_n}\right)J_{r_n}x_n - x_{n-1} \right\| \\
&\leq \left\| \frac{r_{n-1}}{r_n}(x_n - x_{n-1}) + \left(1 - \frac{r_{n-1}}{r_n}\right)(J_{r_n}x_n - x_{n-1}) \right\| \\
&\leq \|x_n - x_{n-1}\| + \left(\frac{r_n - r_{n-1}}{r_n}\right)\|J_{r_n}x_n - x_{n-1}\| \\
&\leq \|x_n - x_{n-1}\| + \left(\frac{r_n - r_{n-1}}{\varepsilon}\right)\|J_{r_n}x_n - x_{n-1}\|
\end{aligned} \tag{2.6}$$

Substituting (2.6) to (2.5), we have

$$\begin{aligned}
\|y_n - y_{n-1}\| &\leq (1 - \alpha_n)\left(\|x_n - x_{n-1}\| + \left(\frac{r_n - r_{n-1}}{\varepsilon}\right)\|J_{r_n}x_n - x_{n-1}\|\right) \\
&\quad + |\alpha_n - \alpha_{n-1}|\|x_{n-1} - J_{r_{n-1}}x_{n-1}\| + \alpha_n\|x_n - x_{n-1}\| \\
&\leq \|x_n - x_{n-1}\| + (1 - \alpha_n)\left(\frac{r_n - r_{n-1}}{\varepsilon}\right)\|J_{r_n}x_n - x_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}|\|x_{n-1} - J_{r_{n-1}}x_{n-1}\| \\
&\leq \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|x_{n-1} - J_{r_{n-1}}x_{n-1}\| \\
&\quad + \left(\frac{r_n - r_{n-1}}{\varepsilon}\right)\|J_{r_n}x_n - x_{n-1}\| \\
&\leq \|x_n - x_{n-1}\| + M_1(|\alpha_n - \alpha_{n-1}| + |r_n - r_{n-1}|)
\end{aligned} \tag{2.7}$$

where M_1 is a constant such that

$$M_1 > \max\left\{\|x_{n-1} - J_{r_{n-1}}x_{n-1}\|, \frac{\|J_{r_n}x_n - x_{n-1}\|}{\varepsilon}\right\}$$

On the other hand, we have

$$\begin{cases} x_{n+1} = \beta_n f(x_n) + (1 - \beta_n)y_n \\ x_n = \beta_{n-1} f(x_{n-1}) + (1 - \beta_{n-1})y_{n-1} \end{cases}$$

Simple calculations show that

$$\begin{aligned}
x_{n+1} - x_n &= (1 - \beta_n)(y_n - y_{n-1}) + (\beta_n - \beta_{n-1})(f(x_{n-1}) - y_{n-1}) \\
&\quad + \beta_n(f(x_n) - f(x_{n-1}))
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq (1 - \beta_n)\|y_n - y_{n-1}\| + \alpha\beta_n\|x_n - x_{n-1}\| \\
&\quad + (\beta_n - \beta_{n-1})\|f(x_{n-1}) - y_{n-1}\|
\end{aligned} \tag{2.8}$$

Substituting (2.7) into (2.8) we obtain

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq (1 - \beta_n)(\|x_n - x_{n-1}\| + M_1(|\alpha_n - \alpha_{n-1}| + |r_n - r_{n-1}|)) \\
&\quad + \alpha\beta_n\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|f(x_{n-1}) - y_{n-1}\| \\
&= (1 - (1 - \alpha)\beta_n)\|x_n - x_{n-1}\| + (1 - \beta_n)M_1(|\alpha_n - \alpha_{n-1}| + |r_n - r_{n-1}|) \\
&\quad + \|f(x_{n-1}) - y_{n-1}\||\beta_n - \beta_{n-1}| \\
&\leq (1 - (1 - \alpha)\beta_n)\|x_n - x_{n-1}\| + M_1(|\alpha_n - \alpha_{n-1}| + |r_n - r_{n-1}|) \\
&\quad + \|f(x_{n-1}) - y_{n-1}\||\beta_n - \beta_{n-1}| \\
&\leq (1 - (1 - \alpha)\beta_n)\|x_n - x_{n-1}\| + M_2(|\alpha_n - \alpha_{n-1}| + (|\beta_n - \beta_{n-1}|) + |r_n - r_{n-1}|)
\end{aligned} \tag{2.9}$$

where M_2 is a constant such that

$$M_2 > \max\{M_1, \|f(x_{n-1}) - y_n\|\}$$

Similarly we can prove (2.9) if $\gamma_{n-1} \geq \gamma_n$. From the conditions (i)-(iii), we have that

$$\sum_{n=0}^{\infty} \beta_n = \infty, \quad \beta_n \rightarrow 0, \quad \text{as } n \rightarrow \infty$$

and

$$\sum_{n=1}^{\infty} (|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| + |r_n - r_{n-1}|) < \infty$$

Hence, applying Lemma 1.2 to (2.9), we obtain

$$\|x_{n+1} - x_n\| \rightarrow 0 \tag{2.10}$$

Next, we will prove $\|x_n - J_{r_n}x_n\| \rightarrow 0$

From (2.1), we have

$$\|J_{r_n}x_n - y_n\| = \|\alpha_n J_{r_n}x_n - \alpha_n x_n\| = \alpha_n \|J_{r_n}x_n - x_n\|$$

Therefore, we obtain

$$\begin{aligned}
\|J_{r_n}x_n - x_n\| &\leq \|J_{r_n}x_n - y_n\| + \|x_{n+1} - x_n\| + \|y_n - x_{n+1}\| \\
&= \alpha_n \|J_{r_n}x_n - x_n\| + \|x_{n+1} - x_n\| + \|y_n - x_{n+1}\|
\end{aligned}$$

That is

$$(1 - \alpha_n)\|J_{r_n}x_n - x_n\| \leq \|x_{n+1} - x_n\| + \|x_{n+1} - y_n\|$$

From (2.3) and (2.10), we imply

$$\begin{aligned}
&\|J_{r_n}x_n - x_n\| \rightarrow 0 \tag{2.11} \\
\|x_{n+1} - J_{r_n}x_n\| &\leq \|x_{n+1} - y_n\| + \|y_n - J_{r_n}x_n\| \\
&\leq \beta_n \|f(x_n) - y_n\| + \alpha_n \|x_n - J_{r_n}x_n\|
\end{aligned}$$

That is

$$\|x_{n+1} - J_{r_n}x_n\| \rightarrow 0.$$

We next prove

$$\limsup_{n \rightarrow \infty} \langle f(p) - p, J_{\varphi}(x_n - p) \rangle \leq 0, \quad p \in F \tag{2.12}.$$

By theorem 1.2, put $p = \lim_{n \rightarrow \infty} x_t$, we take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n \rightarrow \infty} \langle f(p) - p, J_\varphi(x_n - p) \rangle = \limsup_{n \rightarrow \infty} \langle f(p) - p, J_\varphi(x_{n_k} - p) \rangle \quad (2.13).$$

Since E is reflexive, we may assume that $x_{n_k} \rightharpoonup \bar{x}$. Moreover, since

$$\|J_{r_n} x_n - x_n\| \rightarrow 0$$

We obtain

$$J_{r_{n_k}-1} x_{r_{n_k}-1} \rightharpoonup \bar{x}.$$

Taking the limit as $k \rightarrow \infty$ in the relation

$$[J_{r_{n_k}-1} x_{r_{n_k}-1}, A_{r_{n_k}-1} x_{r_{n_k}-1}] \in A,$$

we get $[\bar{x}, 0] \in A$, i.e. $\bar{x} \in F$. Hence by (2.12) and (2.13) we have

$$\limsup_{n \rightarrow \infty} \langle f(p) - p, J_\varphi(x_n - p) \rangle = \langle f(p) - p, J_\varphi(\bar{x} - p) \rangle \leq 0$$

That is (2.12) holds. Finally we prove that $x_n \rightarrow p$.

$$\begin{aligned} \Phi(\|y_n - p\|) &= \Phi(\|\alpha_n(x_n - p) + (1 - \alpha_n)(J_{r_n} x_n - p)\|) \\ &\leq \Phi(\|\alpha_n\|x_n - p\| + (1 - \alpha_n)\|J_{r_n} x_n - p\|) \\ &\leq \Phi(\|x_n - p\|) \end{aligned}$$

That is

$$\Phi(\|y_n - p\|) \leq \Phi(\|x_n - p\|).$$

Therefore, we apply Lemma 1.4 to get

$$\begin{aligned} \Phi(\|x_{n+1} - p\|) &= \Phi(\|\beta_n(f(x_n) - p) + (1 - \beta_n)(y_n - p)\|) \\ &= \Phi(\|\beta_n(f(x_n) - f(p) + f(p) - p) + (1 - \beta_n)(y_n - p)\|) \\ &\leq \Phi(\|(1 - \beta_n)(y_n - p) + \beta_n(f(x_n) - f(p))\|) + \beta_n \langle f(p) - p, J_\varphi(x_{n+1} - p) \rangle \\ &\leq \Phi((1 - \beta_n)\|y_n - p\| + \beta_n\|f(x_n) - f(p)\|) + \beta_n \langle f(p) - p, J_\varphi(x_{n+1} - p) \rangle \\ &\leq \Phi((1 - \beta_n)\|x_n - p\| + \alpha\beta_n\|x_n - p\|) + \beta_n \langle f(p) - p, J_\varphi(x_{n+1} - p) \rangle \\ &\leq (1 - (1 - \alpha)\beta_n)\Phi(\|x_n - p\|) + \beta_n \langle f(p) - p, J_\varphi(x_{n+1} - p) \rangle \end{aligned}$$

Applying Lemma 1.2, we get

$$\Phi(\|x_n - p\|) \rightarrow 0$$

That is $\|x_n - p\| \rightarrow 0$, i.e. $x_n \rightarrow p$.

The proof is complete.

REFERENCES

- [1] F.E. Brower, Fixed point theorems for noncompact mappings in Hilbert space, Proc. Natl. Acad. Sci. USA 53 (1965) 1272-1276.
- [2] H.K.Xu, viscosity approximation methods for nonexpansive mappings, J. Math. Anal. Appl.298(2004)279-291.
- [3] Chen, Rudong and Zhu, Zhichuan, Viscosity approximation fixed points for nonexpansive and m -accretive operators, Fixed Point Theory and Applications.2006, Article ID 81325, 10 pages, 2006.
- [4] Xiaolong Qin, Yongfu Su, approximation of a zero point of accretive operator in Banach spaces. J.Math.Anal.Appl. 2006(in press)
- [5] H.K.Xu, An iterative approach to quadratic optimization, J.Optim.Theory Appl. 116(2003)659-678.
- [6] H.K.Xu, Iterative algorithms for nonlinear operators, J.London Math.Soc.66(2002)240-256.
- [7] V.Barbu, Nonlinear Semigroups and Differential Equations in Banach spaces. Noordhoff, 1976.
- [8] T.C.Lim and H.K.Xu, Fixed point theorems for asymptotically nonexpansive mappings, Nonlinear Anal.22(1994)1345-1355.

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