

New Representations for Weighted Drazin Inverse of Matrices

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Abstract

In this paper, the result are established in the following four ways: First, we present a general representation for the weighted Drazin inverse $A_{d,W}$ of an arbitrary rectangular matrix $A \in M_{m,n}$ involving Moore-Penrose inverse, which reduces to the well-known result if the matrix A is a square and $W = I_n$. Second, we find representations for the weighted Drazin inverse of the Tracy-Singh product $A \odot B$ of the two matrices $A \in M_{m,n}$ and $B \in M_{p,q}$ by using our approach. Third, the results are extended to the case of Tracy-Singh product of any finite number of matrices. The result lead to equalities involving Kronecker product, Drazin inverse and group inverse, as a special case. Finally, We apply our result to present the solution of restricted singular matrix equations.

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1 Introduction and Preliminary Notes

One of the important types of generalized inverses of matrices is the weighted Drazin inverse, which has several important applications such as, applications

in singular differential, difference equations, Marckov chains, statistic problems, control system analysis, curve fitting, numerical analysis and Kronecker product systems [e.g., 4,7,10,12,14,15,16]. Here we use the following notations. Let $M_{m,n}$ be the set of all matrices over the complex number field \mathbb{C} and when $m = n$, we write M_n instead of $M_{n,n}$. For matrix $A \in M_{m,n}$, let A^* , $R(A)$ and $rank(A)$ be the conjugate transpose, range and rank of A . If $A \in M_m$ is a given matrix, then the smallest non-negative integer k such that

$$rank(A^{k+1}) = rrank(A^k) \quad (1.1)$$

is called the *index of A* and is denoted by $Ind(A) = k$.

It is well known that the *Drazin inverse* (DI) of $A \in M_m$ with $Ind(A) = k$ is defined to be the unique solution $X \in M_m$ satisfying the following three matrix equations:

$$A^k X A = A^k, \quad X A X = X, \quad A X = X A, \quad (1.2)$$

and is often denoted by $X = A_d$. Note that the first equation in (1.2) can be written as $A^{k+1} X = A^k$. In particular, when $Ind(A) = 1$, the Drazin inverse of $A \in M_m$ is called the *group inverse* of A , and is often denoted by A_g , but when $Ind(A) = 0$ and $A \in M_m$ is a non-singular matrix, then $A_d = A^{-1}$.

Wang [12] gave that for $A \in M_m$ with $Ind(A) = k$,

$$A^k (A_d)^k A^k = A^k, \quad (A_d)^k A^k (A_d)^k = (A_d)^k, \quad A^k (A_d)^k = (A_d)^k A^k. \quad (1.3)$$

By the uniqueness of the DI, we have

$$(A^k)_d = (A_d)^k. \quad (1.4)$$

For more properties concerning Drazin inverses, see [e.g., 3,4,11,13].

Cline and Greville [5] extended the Drazin inverse of square matrix to rectangular matrix and called it as the *weighted Drazin inverse* (WDI). The WDI of $A \in M_{m,n}$ with respect to the matrix $W \in M_{n,m}$ is defined to be the unique solution $X \in M_{m,n}$ of the following three matrix equations:

$$(AW)^{k+1} X W = (AW)^k, \quad X W A W X = X, \quad A W X = X W A, \quad (1.5)$$

where

$$k = \max \{Ind(AW), Ind(WA)\}, \quad (1.6)$$

and is often denoted by $X = A_{d,W}$. In particular, when $A \in M_m$ and $W = I_m$, then $A_{d,W}$ reduce to A_d , i.e., $A_d = A_{d,I_m}$. If $A \in M_m$ is non-singular square matrix and $W = I_m$, it is easily seen that $Ind(A) = 0$ and $A_{d,W} = A_d = A^{-1}$ satisfies the matrix equations (1.5).

The properties of WDI can be found in [e.g.,8,17,18]. Some notable properties are: If $A \in M_{m,n}$ with respect to the matrix $W \in M_{n,m}$ and $k = \max \{Ind(AW), Ind(WA)\}$, then

(i)

$$A_{d,W} = A \{(WA)_d\}^2 = \{(AW)_d\}^2 A \tag{1.7}$$

(ii)

$$A_{d,W}W = (AW)_d \quad , \quad WA_{d,W} = (WA)_d \tag{1.8}$$

(iii)

$$WAWA_{d,W} = WA(WA)_d \quad , \quad A_{d,W}WAW = (WA)_dAW \tag{1.9}$$

(iv) One closed-form solution of $A_{d,W}$ for a rectangular matrix $A \in M_{m,n}$,

$$A_{d,W} = \begin{cases} \lim_{\alpha \rightarrow 0} (AW)^{l+2} + \alpha^2 I)^{-1} (AW)^l A & \text{if } l \geq k \\ \lim_{\alpha \rightarrow 0} A(WA)^l ((WA)^{l+2} + \alpha^2 I)^{-1} & \text{if } l \geq k \end{cases} . \tag{1.10}$$

The *Moore-Penrose inverse* (MPI) is a generalization of the inverse of non-singular matrix to the inverse of a singular and rectangular matrix. The MPI of a matrix $A \in M_{m,n}$ is defined to be the unique solution $X \in M_{n,m}$ of the following four Penrose equations:

$$AXA = A \quad , \quad XAX = X \quad , \quad (AX)^* = AX \quad , \quad (XA)^* = XA, \tag{1.11}$$

and is often denoted by $X = A^+$. Note that if $A \in M_m$ is non-singular matrix, then $A^+ = A^{-1}$. As to various basic properties concerning MPI, see [e.g., 2,3,4,11].

The *general algebraic structures* (GAS) of the matrices $A \in M_{m,n}, W \in M_{n,m}, A^+, W^+$, and $A_{d,W} \in M_{n,m}$ with $k = \max \{Ind(AW), Ind(WA)\}$ are (see [e.g., 4,17,18,19]):

$$A = L \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} Q^{-1} \quad , \quad W = Q \begin{bmatrix} W_{11} & 0 \\ 0 & W_{22} \end{bmatrix} L^{-1} \quad , \quad A^+ = Q \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} L^{-1}, \tag{1.12}$$

$$W^+ = L \begin{bmatrix} W_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q^{-1}, \quad A_{d,W} = L \begin{bmatrix} (W_{11}A_{11}W_{11})^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q^{-1}. \quad (1.13)$$

where L, Q, A_{11}, W_{11} are non-singular matrices, and $A_{22}, W_{22}, A_{22}W_{22}, W_{22}A_{22}$ are nilpotent matrices (A matrix $A \in M_n$ is called nilpotent if $A^k = 0$ for some positive integer k). In particular, when $A \in M_m$ with $Ind(A) = k$, $W = I_m$ and $L = Q$, then we have

$$A = L \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} L^{-1}, \quad A_d = L \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} L^{-1}. \quad (1.14)$$

Finally, let $A \otimes B$ and $A \odot B$ be the Kronecker and Tracy-Singh products, respectively, of $A \in M_{m,n}$ and $B \in M_{p,q}$. The definitions of the mentioned two matrix products are mentioned by Liu [8] and Al Zhou and Kiliçman [1,2,7] as follows:

(i) *Kronecker product*

$$A \otimes B = [a_{ij}B]_{ij} \in M_{mp,nq}, \quad (1.15)$$

where $A = [a_{ij}] \in M_{m,n}$ and $B = [b_{kl}] \in M_{p,q}$ are scalar matrices and $a_{ij}B$ is the ij -th block of order $p \times q$.

(ii) *Tracy-Singh product*

$$A \odot B = [A_{ij} \odot B]_{ij} = [[A_{ij} \otimes B_{kl}]_{kl}]_{ij} \in M_{mp,nq}, \quad (1.16)$$

where $A = [A_{ij}] \in M_{m,n}$ is partitioned with A_{ij} of order $m_i \times n_j$ as the ij -th submatrix, $B = [B_{kl}] \in M_{p,q}$ is partitioned with B_{kl} of order $p_k \times q_l$ as the kl -th submatrix ($m = \sum_{i=1}^r m_i, n = \sum_{j=1}^s n_j, p = \sum_{k=1}^t p_k, q = \sum_{l=1}^h q_l$), $A_{ij} \otimes B_{kl}$ is the kl -th submatrix of order $m_i p_k \times n_j q_l$, and $A_{ij} \odot B$ is the ij -th submatrix of order $m_i p \times n_j q$.

In addition, Liu [8] stated that the Tracy-Singh product can be viewed as a generalized Kronecker product, as follows: For non-partitioned matrices A and B , their $A \odot B$ is $A \otimes B$, that is

$$A \odot B = [a_{ij} \odot B]_{ij} = [[a_{ij} \otimes B_{kl}]_{kl}]_{ij} = [[a_{ij}B_{kl}]_{kl}]_{ij} = [a_{ij}B]_{ij} = A \otimes B. \quad (1.17)$$

Observe that, unlike $A \odot B$ depends on the partitiones of A and B . Take, for example, the identity matrices I_2 and I_3 with I_3 partitioned as

$$\begin{bmatrix} 1 & 0 & \vdots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 1 & \vdots & 0 \\ 0 & 0 & \vdots & 1 \end{bmatrix}$$

Then $I_2 \odot I_3 \neq I_6$. Note that $I_2 \odot I_3$ is not even symmetric. On the other hand, for non-partitioned I_2 and I_3 , then $I_2 \odot I_3 = I_2 \otimes I_3 = I_6$. Hence the Tracy-Singh product greatly depends on the partitions of the matrices.

For any compatible partitioned matrices A, B, C and D ; and any real number r , we shall make frequent use of the following properties of the Tracy-Singh products (see [e.g.,1,2,7,8]):

(i) If AC and BD are well defined, then.

$$(A \odot B)(C \odot D) = AC \odot BD \tag{1.18}$$

(ii) If A and B are square positive (semi) definite matrices, then

$$(A \odot B)^r = A^r \odot B^r \tag{1.19}$$

(iii)

$$\text{rank}(A \odot B) = \text{rank}(A)r(B) \tag{1.20}$$

(iv)

$$(A \odot B)^+ = A^+ \odot B^+. \tag{1.21}$$

In the present paper, some new matrix representations for the weighted Drazin inverses involving Moore-Penrose inverse and Tracy-Singh products of matrices are established. Finally, we apply our result to present the solution of restricted singular matrix equations $(WAW)X(RBR)^T = C$.

2 Main Result

Theorem 1 *Let $A \in M_{m,n}$ and $W \in M_{n,m}$ such that $A_{22}W_{22}$ and $W_{22}A_{22}$ are nilpotent matrices of index k in GAS form. Then the WDI of A with respect to the matrix W can be written as matrix expression involving MPI by*

$$A_{d,W} = \left\{ (AW)^k \left[(AW)^{2k+1} \right]^+ (AW)^k \right\} W^+, \tag{2.1}$$

where $k = \max \{ \text{Ind}(AW), \text{Ind}(WA) \}$.

Proof. Due to the GAS of A, A^+W, W^+ and $A_{d,W}$, there exists non-singular matrices L, Q, A_{11} and W_{11} , and nilpotent matrices A_{22} and W_{22} such that

$$A = L \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} Q^{-1}, \quad W = Q \begin{bmatrix} W_{11} & 0 \\ 0 & W_{22} \end{bmatrix} L^{-1}, \quad W^+ = L \begin{bmatrix} W_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q^{-1}.$$

Since $A_{22}W_{22}$ and $W_{22}A_{22}$ are nilpotent matrices of index k , then $(A_{22}W_{22})^k = 0$, and it is easy to show that

$$(AW)^k = L \begin{bmatrix} (A_{11}W_{11})^k & 0 \\ 0 & 0 \end{bmatrix} L^{-1}, \quad [(AW)^{2k+1}]^+ = L \begin{bmatrix} (A_{11}W_{11})^{-2k-1} & 0 \\ 0 & 0 \end{bmatrix} L^{-1}.$$

Computation shows that $\{(AW)^k [(AW)^{2k+1}]^+ (AW)^k\} W^+$

$$\begin{aligned} &= L \begin{bmatrix} (A_{11}W_{11})^k & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (A_{11}W_{11})^{-2k-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (A_{11}W_{11})^k & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} W_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q^{-1} \\ &= L \begin{bmatrix} (A_{11}W_{11})^k (A_{11}W_{11})^{-2k-1} (A_{11}W_{11})^k W_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q^{-1} \\ &= L \begin{bmatrix} (A_{11}W_{11})^{-1} W_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q^{-1} \\ &= L \begin{bmatrix} (W_{11}A_{11}W_{11})^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q^{-1} \\ &= A_{d,W}. \end{aligned}$$

This completes the proof of Theorem 1. ■

If A is a square matrix with $Ind(A) = k$ and set $W = I_n$ in Theorem 1, we obtain the following corollary which is given by Wang [13]:

Corollary 2 *Let $A \in M_n$ with $Ind(A) = k$, then*

$$A_d = A^k (A^{2k+1})^+ A^k. \quad (2.2)$$

Theorem 3 *Let $A = [A_{ij}] \in M_{m,n}$ and $B = [B_{hl}] \in M_{p,q}$ be partitioned with A_{ij} and B_{hl} of order $m_i \times n_j$ and $p_h \times q_l$, respectively ($m = \sum_{i=1}^r m_i$, $n = \sum_{j=1}^s n_j$, $p = \sum_{h=1}^t p_h$, $q = \sum_{l=1}^c q_l$). Also, let $W \in M_{n,m}$, $R \in M_{q,p}$ and $Z = W \odot R$ be partitioned matrices with*

$$k_1 = \max \{Ind(AW), Ind(WA)\}, \quad k_2 = \max \{Ind(BR), Ind(RB)\}. \quad (2.3)$$

Then

$$Ind \{(A \odot B) Z\} = k = \max \{k_1, k_2\}, \quad (2.4)$$

and

$$(A \odot B)_{d,Z} = A_{d,W} \odot B_{d,R}. \quad (2.5)$$

Proof. (i) By assumptions, we have

$$\text{rank}(AW)^{k_1} = r\text{rank}(AW)^{k_1+1} , \quad \text{rank}(BR)^{k_2} = \text{rank}(BR)^{k_2+1} .$$

From properties of Tracy-Singh products, we have

$$\begin{aligned} \text{rank} \{(A \odot B) Z\}^r &= \text{rank} \{(A \odot B) (W \odot R)\}^r = \text{rank} \{AW \odot BR\}^r \\ &= \text{rank} \{AW\}^r \text{rank} \{BR\}^r . \end{aligned}$$

Similarly,

$$\text{rank} \{(A \odot B) Z\}^{r+1} = \text{rank} \{AW\}^{r+1} \text{rank} \{BR\}^{r+1} .$$

It is obvious that the smallest non-negative integer such that

$$\text{rank} \{(A \odot B) Z\}^{r+1} = \text{rank} \{(A \odot B) Z\}^r .$$

is $k = \max \{k_1, k_2\}$. Hence (2.4) is true.

From (2.1) of Theorem 1, the WDI of A and B with respect to the matrices W and R can be written as

$$\begin{aligned} A_{d,W} &= \left\{ (AW)^k [(AW)^{2k+1}]^+ (AW)^k \right\} W^+ , \\ B_{d,R} &= \left\{ (BR)^k [(BR)^{2k+1}]^+ (BR)^k \right\} R^+ , \end{aligned}$$

where

$$k = \max \{ \text{Ind}(AW), \text{Ind}(WA), \text{Ind}(BR), \text{Ind}(RB) \} .$$

Now using properties of Tracy-Singh product, we have

$$\begin{aligned} A_{d,W} \odot B_{d,R} &= \left(\left\{ (AW)^k [(AW)^{2k+1}]^+ (AW)^k \right\} W^+ \right) \odot \left(\left\{ (BR)^k [(BR)^{2k+1}]^+ (BR)^k \right\} R^+ \right) \\ &= \left\{ (AW \odot BR)^k [(AW \odot BR)^{2k+1}]^+ (AW \odot BR)^k \right\} (W^+ \odot R^+) \\ &= ((A \odot B)(W \odot R))^k [((A \odot B)(W \odot R))^{2k+1}]^+ ((A \odot B)(W \odot R))^k (W \odot R)^+ \\ &= \left\{ ((A \odot B)Z)^k [((A \odot B)Z)^{2k+1}]^+ ((A \odot B)Z)^k \right\} Z^+ \\ &= (A \odot B)_{d,Z} . \end{aligned}$$

This completes the proof of Theorem 3. ■

Corollary 4 Let $A_i \in M_{m(i),n(i)}$ and $W_i \in M_{n(i),m(i)}$ ($1 \leq i \leq r, r \geq 2$) be partitioned matrices ($m(i) = \sum_{j=1}^s m_j(i)$, $n(i) = \sum_{j=1}^s n_j(i)$) with

$$k_i = \max \{ \text{Ind}(A_i W_i), \text{Ind}(W_i A_i) \} : i = 1, 2, \dots, r . \tag{2.6}$$

Then

$$Ind \left\{ \left(\prod_{i=1}^r \odot A_i \right) Z \right\} = k, \tag{2.7}$$

and

$$\left(\prod_{i=1}^r \odot A_i \right)_{d,Z} = \prod_{i=1}^r \odot (A_i)_{d,W_i}, \tag{2.8}$$

where

$$k = \max \{k_1, k_2, \dots, k_r\}, \quad Z = \prod_{i=1}^r \odot W_i. \tag{2.9}$$

In particular,

(i) if $A_i \in M_{m(i)}$ and $W_i = I_{n_i}$ ($1 \leq i \leq k$, $k \geq 2$), we then have

$$Ind \left(\prod_{i=1}^r \odot A_i \right) = k, \quad \left(\prod_{i=1}^r \odot A_i \right)_d = \prod_{i=1}^r \odot (A_i)_d, \tag{2.10}$$

where

$$k = \max \{Ind(A_i) : i = 1, 2, \dots, r\}, \tag{2.11}$$

(ii) if $Ind(A_1) = Ind(A_2) = \dots = Ind(A_r) = 1$, we then have

$$\left(\prod_{i=1}^r \odot A_i \right)_g = \prod_{i=1}^r \odot (A_i)_g. \tag{2.12}$$

Proof. The proof (2.7) of Corollary 4 is by induction on r . The base case (when $r = 2$) has been established in (2.4) of Theorem 3. In the induction hypothesis, we assume that

$$\begin{aligned} Ind \left\{ \left(\prod_{i=1}^{r-1} \odot A_i \right) \left(\prod_{i=1}^{r-1} \odot W_i \right) \right\} &= Ind \left\{ \prod_{i=1}^{r-1} \odot A_i W_i \right\} \\ &= \gamma = \max \{k_1, k_2, \dots, k_{r-1}\}. \end{aligned}$$

Now

$$\begin{aligned} Ind \left\{ \left(\prod_{i=1}^r \odot A_i \right) Z \right\} &= Ind \left\{ \left(\prod_{i=1}^r \odot A_i \right) \left(\prod_{i=1}^r \odot W_i \right) \right\} \\ &= Ind \left\{ \prod_{i=1}^r \odot A_i W_i \right\} = Ind \left\{ \left(\prod_{i=1}^{r-1} \odot A_i W_i \right) \odot (A_r W_r) \right\} \\ &= \max \{\gamma, k_r\} = \max \{k_1, k_2, \dots, k_r\} = k. \end{aligned}$$

The proof (2.8) of Corollary 4 is also by induction on r . The base case (when $r = 2$) has been established in (2.5) of Theorem 3. In the induction hypothesis, we assume that

$$\left(\prod_{i=1}^{r-1} \odot A_i \right)_{d, \prod_{i=1}^{r-1} \odot W_i} = \prod_{i=1}^{r-1} \odot (A_i)_{d, W_i} .$$

Now

$$\begin{aligned} \left(\prod_{i=1}^r \odot A_i \right)_{d, Z} &= \left[\left(\prod_{i=1}^{r-1} \odot A_i \right)_{d, \prod_{i=1}^{r-1} \odot W_i} \right] \odot (A_r)_{d, W_r} \\ &= \left(\prod_{i=1}^{r-1} \odot (A_i)_{d, W_i} \right) \odot (A_r)_{d, W_r} \\ &= \prod_{i=1}^r \odot (A_i)_{d, W_i} . \end{aligned}$$

The proof of two special cases (2.10)-(2.12) are straightforward. ■

If A_i and $W_i \in M_{n(i), m(i)}$ ($1 \leq i \leq r, r \geq 2$) are non-partitioned (i.e. scalar) matrices in Corollary 4 we obtain the following corollary:

Corollary 5 Let $A_i \in M_{m(i), n(i)}$ and $W_i \in M_{n(i), m(i)}$ ($1 \leq i \leq r, r \geq 2$) be matrices with

$$k_i = \max \{ \text{Ind}(A_i W_i), \text{Ind}(W_i A_i) \} , \quad i = 1, 2, \dots, r . \tag{2.13}$$

Then

$$\text{Ind} \left\{ \left(\prod_{i=1}^r \otimes A_i \right) Z \right\} = k , \tag{2.14}$$

and

$$\left(\prod_{i=1}^r \otimes A_i \right)_{d, Z} = \prod_{i=1}^r \otimes (A_i)_{d, W_i} , \tag{2.15}$$

where $k = \max \{ k_1, k_2, \dots, k_r \}$ and $Z = \prod_{i=1}^r \otimes W_i$. In particular,
 (i) if $A_i \in M_{m(i)}$ and $W_i = I_{n_i}$ ($1 \leq i \leq k, k \geq 2$), we then have

$$\text{Ind} \left(\prod_{i=1}^r \otimes A_i \right) = k , \quad \left(\prod_{i=1}^r \otimes A_i \right)_d = \prod_{i=1}^r \otimes (A_i)_d , \tag{2.16}$$

where $k = \max \{ \text{Ind}(A_i), i = 1, 2, \dots, r \}$,

(ii) if $\text{Ind}(A_1) = \text{Ind}(A_2) = \dots = \text{Ind}(A_r) = 1$, we then have

$$\left(\prod_{i=1}^r \otimes A_i \right)_g = \prod_{i=1}^r \otimes (A_i)_g . \quad (2.17)$$

One of the important application of Corollary 5 is that the weighted Drazin inverse of Kronecker product arise naturally in solving the so-called *restricted singular matrix equations* (RSME) as follows.

Theorem 6 Let $A \in M_{m,n}$, $W \in M_{n,m}$, $B \in M_{p,q}$, $R \in M_{q,p}$ and $C \in M_{n,q}$ be given scalar matrices and $X \in M_{m,p}$ be an unknown matrix to be solved. Also, let

$$L = R \otimes W, \quad k_1 = \text{Ind}((B \otimes A) L), \quad k_2 = \text{Ind}(L(B \otimes A)) \quad (2.18)$$

such that

$$\text{rank}((B \otimes A) L)^{k_1} = \text{rank}(L(B \otimes A))^{k_2},$$

$$\text{Vec}C \in R(L(B \otimes A))^{k_2}, \quad \text{Vec}X \in R((B \otimes A)L)^{k_1}. \quad (2.19)$$

Then the unique solution of the following RSME

$$(WAW) X (RBR)^T = C \quad (2.20)$$

is given by

$$X = A_{d,W} C B_{d,R}^T . \quad (2.21)$$

Proof. From properties of the Kronecker product, it is well known that:

$$\text{Vec}(AXB^T) = (B \otimes A) \text{Vec}(X), \quad (2.22)$$

where

$$\text{Vec}(X) = [x_{11} \quad \dots \quad x_{m1} \quad x_{12} \quad \dots \quad x_{m2} \quad \dots \quad x_{1m} \quad \dots \quad x_{mn}]^T \quad (2.23)$$

denotes vectorization by columns of arbitrary matrix $X \in M_{m,n}$, it is not difficult to transform (41) into the vector form as:

$$(L(B \otimes A)L) \text{Vec}X = \text{Vec}C . \quad (2.24)$$

It is easy to verify under conditions (2.19) that the unique solution of (2.24) is

$$\begin{aligned} \text{Vec}X &= (B \otimes A)_{d,L} \text{Vec}C = (B_{d,R} \otimes A_{d,W}) \text{Vec}C \\ &= \text{Vec}(A_{d,W} C B_{d,R}^T), \end{aligned}$$

which is the required result. ■

An important particular case of Theorem 6 is that when $m = n$, $p = q$, $W = I_m$ and $R = I_p$, we obtain the following corollary:

Corollary 7 Let $A \in M_m$, $B \in M_p$ and $C \in M_{m,p}$ be given constant matrices and $X \in M_{m,p}$ be an unknown matrix to be solved. Then the unique solution of the following RSME

$$AXB^T = C \quad : \text{Vec}C \quad , \quad \text{Vec}X \in R(B \otimes A)^k \quad , \quad k = \text{Ind}(B \otimes A) \quad (2.25)$$

is given by

$$X = A_d C B_d^T. \quad (2.26)$$

3 Conclusion

In this paper, we have presented two general representations for weighted Drazin inverse related to Moore-penrose inverse and Tracy-Singh and Kronecker products of two and several matrices. These representations are viewed as a generalization of Wang's results in [13, Lemma 1.1, and 12, Theorem 2.2]. Although the results are applied to solve the restricted singular matrix equations, the idea adopted can be easily extended to solve the coupled restricted singular matrix equations. It is natural to ask if we can extend our results to the Minkowski inverse in Minkowski space, this will be the future research.

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