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# Generalized Integral Operator and Univalent Functions

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**Abstract**

Let $A$ be the class of functions $f : f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ analytic in the open unit disc $U$. Let

$$ {}_q f_s \left(\alpha_1, \alpha_2, \dots, \alpha_q; \beta_1, \dots, \beta_s; z\right) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_q)_n}{(\beta_1)_n \cdots (\beta_s)_n} \frac{z^{n+1}}{n!}, \quad (q \leq s+1; q, s \in N_0). $$

We define ${}_q f_s^{(-1)} \left(\alpha_1, \alpha_2, \dots, \alpha_q; \beta_1, \dots, \beta_s; z\right)$ by using convolution $*$ as ${}_q f_s \left(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z\right) * {}_q f_s^{(-1)}(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) = z/(1-z)^2$. We use this definition to introduce a generalized integral operator $L_{q,s} \left(\alpha_1, \dots, \alpha_q; \beta_1, \dots \beta_s\right) f(z) = {}_q f_s^{(-1)} \left(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z\right) * f(z)$. In terms of this integral operator we introduce new classes of functions and derive some interesting properties of these classes.

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## 1 Introduction

Let $A$ denote the class of functions $f(z)$ of the form

$$ f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1} $$

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which are analytic in the unit disc $U = \{z : |z| < 1\}$.

If $f(z) \in A$ satisfies

$$\left|\arg\left(\frac{zf'(z)}{f(z)} - \gamma\right)\right| < \frac{\pi}{2}\beta \quad (z \in U; 0 < \beta \leq 1, \quad 0 \leq \gamma < 1), \tag{2}$$

then $f(z)$ is said to be strongly starlike of order $\beta$ and type $\gamma$ in $U$, and we denote this by $S^*(\gamma, \beta)$.

If $f(z) \in A$ satisfies

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)} - \gamma\right)\right| < \frac{\pi}{2}\beta \quad (z \in U; \quad 0 < \beta \leq 1, \quad 0 \leq \gamma < 1), \tag{3}$$

then $f(z)$ is said to be strongly convex of order $\beta$ and type $\gamma$ in $U$, and denoted by $C(\gamma, \beta)$. It is obvious that $f(z) \in A$ belongs to $C(\gamma, \beta)$ if and only if $zf'(z) \in S^*(\gamma, \beta)$. Further, we note that $S^*(\gamma, 1) = S^*(\gamma)$ and $C(\gamma, 1) = C(\gamma)$.

For $\alpha_j \in C$ $(j = 1, 2, 3, \ldots, q)$ and $\beta_j \in C - \{0, -1, -2, \ldots\}$ $(j = 1, 2, \ldots, s)$, the generalized hypergeometric function is defined by

$${}_qF_s\ (\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_q)_n}{(\beta_1)_n \cdots (\beta_s)_n} \frac{z^n}{n!},$$
$$(q \leq s + 1; q, s \in N_0 = \{0, 1, 2, \ldots\}),$$

where $(a)_n$ is the Pochhammer symbol defined by $(a)_n = \dfrac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\cdots(a+n-1)$ for $n \in N = \{1, 2, \ldots\}$ and 1 when $n = 0$. In particular, the incomplete beta function, related to the Gauss hypegeometric function, $\phi(a, c; z)$, is defined by

$$\phi(a, c; z) = zF(a, 1; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1}, \quad z \in U, \quad c \neq 0, -1, -2, \ldots .$$

Note that $\phi(a, 1; z) = \dfrac{z}{(1-z)^a}$. Moreover, $\phi(2, 1; z) = \dfrac{z}{(1-z)^2}$ is the Koebe function.

Corresponding to the function $h(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots \beta_s; z) = z\,{}_qF_s\ (\alpha_1, \ldots \alpha_q; \beta_1, \ldots, \beta_s; z)$ the Dziok-Srivastava operator [3], $H_{q,s}(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots \beta_s)$ is defined by

$$\begin{aligned} H_{q,s}(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s)f(z) &= h(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_{n-1} \cdots (\alpha_q)_{n-1}}{(\beta_1)_{n-1} \cdots (\beta_s)_{n-1}} \frac{a_n z^n}{(n-1)!} \end{aligned}$$

where "$*$" stands for convolution.

It is well known [3] that

$$\alpha_1 H_{q,s}(\alpha_1+1,\dots,\alpha_q;\beta_1,\dots,\beta_s)f(z)=z\,[H_{q,s}(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s)f(z)]' \\ +(\alpha_1-1)H_{q,s}(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s)f(z). \tag{4}$$

To make the notation simple, we write $H_{q,s}[\alpha_1]f(z)=H_{q,s}(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s)f(z)$. For $q=s+1$ and $\alpha_2=\beta_1,\dots,\alpha_q=\beta_s$, it is easy to see that $H_{s+1,s}[1]f(z)=f(z)$ and $H_{s+1,s}[2]f(z)=zf'(z)$.

We note that many subclasses of analytic functions, associated with the Dziok-Srivastava operator $H_{q,s}[\alpha_1]$ and many special cases, were investigated recently by Aghalary and Azadi [1], Dziok-Srivastava [4], Liu [6], Liu and Srivastava [8], [9] and others.

In this paper we define a generalized integral operator $L_{q,s}(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s): A\to A$ analogous to $H_{q,s}[\alpha_1];(q\le s+1, q,s\in N_0)$ where $\alpha_j\in C-\{0,-1,-2,\dots\}$, $(j=1,2,\dots,q)$ as follows.

Let

$${}_qf_s(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s;z)=h(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s;z) \\ =z\,{}_qF_s(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s;z)$$

and let ${}_qf_s^{(-1)}(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s;z)$ be defined such that

$${}_qf_s(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s;z)*{}_qf_s^{(-1)}(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s;z)=\phi(2,1;z). \tag{5}$$

Then

$$L_{q,s}(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s)f(z)={}_q f_s^{(-1)}(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s;z)*f(z). \tag{6}$$

For $q=s+1$ and $\alpha_2=\beta_1,\dots,\alpha_q=\beta_s$, we note that $L_{q,s}(1,\alpha_2,\dots,\alpha_q;\beta_1,\dots,\beta_s)$ $f(z)=zf'(z)$ and $L_{q,s}(2,\alpha_2,\dots,\alpha_q;\beta_1,\dots,\beta_s)f(z)=f(z)$.

From (5) and (6), it follows that

$$\alpha_1 L_{q,s}(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s)f=z[L_{q,s}(\alpha_1+1,\alpha_2,\dots,\alpha_q;\beta_1,\dots,\beta_s)f]' \\ +(\alpha_1-1)L_{q,s}(\alpha_1+1,\dots,\alpha_q;\beta_1,\dots,\beta_s)f. \tag{7}$$

The relation (7) plays an important and significant role in obtaining our results.

To make the notation simple, we write

$$L_{q,s}[\alpha_1]f(z)=L_{q,s}(\alpha_1,\alpha_2,\dots,\alpha_q;\beta_1,\dots,\beta_s)f(z).$$

Also, we note that a special case of this operator is the Noor integral operator [11].

Let $q, s \in N$, $0 < \beta \leq 1$ and $0 \leq \gamma < 1$. We now introduce the following classes in terms of the new operator $L_{q,s}[\alpha_1]$ :

Let $\overline{S}^*_{\alpha_1}(\gamma, \beta)$ be the class of functions $f \in A$ satisfying

$$\left|\arg\left(\frac{z(L_{q,s}[\alpha_1]f(z))'}{L_{q,s}[\alpha_1]f(z)} - \gamma\right)\right| < \frac{\pi}{2}\beta \qquad (z \in U; \quad 0 < \beta \leq 1, \quad 0 \leq \gamma < 1).$$

Observe that $L_{q,s}[\alpha_1]f \in S^*(\gamma, \beta)$ and $\dfrac{z(L_{q,s}[\alpha_1]f)'}{L_{q,s}[\alpha_1]f} \neq \gamma$. Let $\overline{C}_{\alpha_1}(\gamma, \beta)$ be the class of functions $f \in A$ satisfying

$$\left|\arg\left(1 + \frac{z(L_{q,s}[\alpha_1]f(z))''}{(L_{q,s}[\alpha_1]f(z))'} - \gamma\right)\right| < \frac{\pi}{2}\beta.$$

Observe that $L_{q,s}[\alpha_1]f \in C(\gamma, \beta)$ and $1 + \dfrac{z(L_{q,s}[\alpha_1]f)''}{L_{q,s}[\alpha_1]f} \neq \gamma$.

For $q = s + 1$ and $\alpha_2 = \beta_1, \ldots, \alpha_q = \beta_s$, it is easy to see that $\overline{S}^*_{n+1}(\gamma, \beta)$ is the subclass of strongly starlike functions defined by the Noor integral operator which was introduced by Liu [7]. Further, we note that $\overline{S}^*_2(\gamma, \beta) = S^*(\gamma, \beta)$ and $\overline{C}_2(\gamma, \beta) = C(\gamma, \beta)$.

Clearly, $f \in \overline{C}_{\alpha_1}(\gamma, \beta)$ if and only if $zf' \in \overline{S}^*_{\alpha_1}(\gamma, \beta)$.

Also, let $R_{\alpha_1}(\gamma, \beta, \eta, A, B)$ be the class of functions $f \in A$ satisfying the condition

$$\left|\arg\left(\frac{z(L_{q,s}[\alpha_1]f(z))'}{L_{q,s}[\alpha_1]g(z)} - \gamma\right)\right| < \frac{\pi}{2}\beta, \qquad (z \in U; \quad 0 < \beta \leq 1, 0 \leq \gamma < 1)$$

for some $g \in Q_{\alpha_1}(\eta, A, B)$ where

$$Q_{\alpha_1}(\eta, A, B) = \left\{g \in A : \frac{1}{1-\eta}\left(\frac{z(L_{q,s}[\alpha_1]g(z))'}{L_{q,s}[\alpha_1]g(z)} - \eta\right) \prec \frac{1+Az}{1+Bz}\right\}$$
$$(z \in U; 0 \leq \eta < 1, -1 \leq B < A \leq 1).$$

For $q = s+1$ and $\alpha_2 = \beta_1, \ldots, \alpha_q = \beta_s$, we note that $R_{n+1}(\gamma, \beta, \eta, A, B)$ is the subclass of strongly close-to-convex functions defined by the Noor integral operator which was introduced by Cho [2]. Also, note that $R_1(\gamma, 1, \eta, 1, -1)$ and $R_2(\gamma, 1, \eta, 1, -1)$ are the classes of quasi-convex and close-to-convex functions of order $\gamma$ and type $\eta$, respectively, introduced and studied by Noor and Alkharsani [10] and Silverman [13]. Further, $R_2(0, \beta, 0, 1, -1)$ is the class of strongly close-to-convex functions of order $\beta$ in the sense of Pommeranke [12].

# 2 Main Results

In proving our main results, we need the following lemmas.

**Lemma 2.1.** [5] *Let $h$ be convex univalnet in $U$ with $h(0) = 1$ and $Re(\lambda h(z)+\mu) > 0$ $(\lambda, \mu \in \mathbb{C})$. If $p$ is analytic in$U$ with $p(0) = 1$, then*

$$p(z) + \frac{zp'(z)}{\lambda p(z) + \mu} \prec h(z) \qquad (z \in U)$$

*implies*

$$p(z) \prec h(z) \qquad (z \in U).$$

**Lemma 2.2.** [9] *Let $h$ be convex in $U$ and let $E \geq 0$. Suppose $B(z)$ is analytic in $U$ with Re $B(z) \geq E$. If $g$ is analytic in $U$ and $g(0) = h(0)$, then*

$$Ez^2 g''(z) + B(z)zg'(z) + g(z) \prec h(z)$$

*implies*

$$g(z) \prec h(z).$$

**Theorem 2.3.** *Let $h$ be convex univalent in $U$ with $h(0) = 1$ and Re $h(z) > 0$. If a function $f \in A$ satisfies the condition*

$$\frac{1}{1-\eta}\left(\frac{z(L_{q,s}[\alpha_1]f(z))'}{L_{q,s}[\alpha_1]f(z)} - \eta\right) \prec h(z) \qquad (0 \leq \eta < 1; z \in U),$$

*then*

$$\frac{1}{1-\eta}\left(\frac{z(L_{q,s}[\alpha_1+1]f(z))'}{L_{q,s}[\alpha_1+1]f(z)} - \eta\right) \prec h(z) \qquad (0 \leq \eta < 1; z \in U).$$

*Proof.* Let

$$p(z) = \frac{1}{1-\eta}\left(\frac{z(L_{q,s}[\alpha_1+1]f(z))'}{L_{q,s}[\alpha_1+1]f(z)} - \eta\right),$$

where $p$ is an analytic function with $p(0) = 1$. By using the equation (7), we get

$$\alpha_1 - 1 + \eta + (1-\eta)p(z) = \alpha_1 \frac{L_{q,s}[\alpha_1]f(z)}{L_{q,s}[\alpha_1+1]f(z)}. \tag{8}$$

Differentiating both sides of (8) logarithmically, it follows that

$$p(z) + \frac{zp'(z)}{\alpha_1 - 1 + \eta + (1-\eta)p(z)} = \frac{1}{1-\eta}\left(\frac{z(L_{q,s}[\alpha_1]f(z))'}{L_{q,s}[\alpha_1]f(z)} - \eta\right) \qquad (z \in U).$$

Applying Lemma 2.1, it follows that $p \prec h$, that is,

$$\frac{1}{1-\eta}\left(\frac{z(L_{q,s}[\alpha_1+1]f(z))'}{L_{q,s}[\alpha_1+1]f(z)} - \eta\right) \prec h(z) \qquad (z \in U).$$

□

Taking $h(z) = (1 + Az)/(1 + Bz) \qquad (-1 \leq B < A \leq 1)$, in Theorem 2.3, we have

**Corollary 2.4.** *$Q_{\alpha_1}(\eta, A, B) \subset Q_{\alpha_1+1}(\eta, A, B)$ for Re $\alpha_1 > 1 - \eta$ and $0 \leq \eta < 1$.*

Taking $h(z) = ((1 + z)/(1 - z))^{\beta}, \quad 0 < \beta \leq 1$ in Theorem 2.1, we have

**Corollary 2.5.** *$\overline{S}^*_{\alpha_1}(\eta, \beta) \subset \overline{S}^*_{\alpha_1+1}(\eta, \beta)$ for Re $\alpha_1 > 1 - \beta, \quad 0 \leq \eta < 1$ and $0 < \beta \leq 1$.*

**Corollary 2.6.** *$\overline{C}_{\alpha_1}(\eta, \beta) \subset \overline{C}_{\alpha_1+1}(\eta, \beta)$ for Re $\alpha_1 > 1 - \beta, \quad 0 \leq \eta < 1$ and $0 < \beta \leq 1$.*

*Proof.*

$$\begin{aligned} f(z) \in \overline{C}_{\alpha_1}(\eta, \beta) &\Leftrightarrow zf'(z) \in \overline{S}^*_{\alpha_1}(\eta, \beta) \\ &\Leftrightarrow zf'(z) \in \overline{S}^*_{\alpha_1+1}(\eta, \beta) \\ &\Leftrightarrow L_{q,s}[\alpha_1 + 1](zf'(z)) \in S^*(\eta, \beta) \\ &\Leftrightarrow z(L_{q,s}[\alpha_1 + 1]f(z))' \in S^*(\eta, \beta) \\ &\Leftrightarrow L_{q,s}[\alpha_1 + 1]f(z) \in C(\eta, \beta) \\ &\Leftrightarrow f(z) \in \overline{C}_{\alpha_1+1}(\eta, \beta). \end{aligned}$$

□

**Theorem 2.7.** *Let $h$ be convex univalent in $U$ with $h(0) = 1$ and Re $h(z) > 0$. If a function $f \in A$ satisies the condition*

$$\frac{1}{1-\eta}\left(\frac{z(L_{q,s}[\alpha_1]f(z))'}{L_{q,s}[\alpha_1]f(z)} - \eta\right) \prec h(z) \qquad (0 \leq \eta < 1; z \in U),$$

*then*

$$\frac{1}{1-\eta}\left(\frac{z(L_{q,s}[\alpha_1]J_c(f)(z))'}{L_{q,s}[\alpha_1]J_c(f)(z)} - \eta\right) \prec h(z) \qquad (0 \leq \eta < 1; z \in U),$$

*where $J_c$ be the integral operator defined by*

$$J_c(f) := J_c(f)(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt \qquad (c \geq 0). \tag{9}$$

*Proof.* From (9), we have

$$z(L_{q,s}[\alpha_1]J_c(f)(z))' = (c+1)L_{q,s}[\alpha_1]f(z) - cL_{q,s}[\alpha_1]J_c(f)(z). \tag{10}$$

Let

$$p(z) = \frac{1}{1-\eta}\left(\frac{z(L_{q,s}[\alpha_1]J_c(f)(z))'}{L_{q,s}[\alpha_1]J_c(f)(z)} - \eta\right),$$

where $p$ is an analytic function with $p(0) = 1$. Then by using (10), we get

$$c + \eta + (1 - \eta)p(z) = (c + 1)\frac{L_{q,s}[\alpha_1]f}{L_{q,s}[\alpha_1]J_c(f)}. \tag{11}$$

Taking logarithmic derivatives in both sides of (11) and multiplying by $z$, we have

$$p(z) + \frac{zp'(z)}{c + \eta + (1 - \eta)p(z)} = \frac{1}{1 - \eta}\left(\frac{z(L_{q,s}[\alpha_1]f(z))'}{L_{q,s}[\alpha_1]f(z)} - \eta\right) \qquad (z \in U).$$

Therefore by Lemma 2.1, we obtain that

$$\frac{1}{1 - \eta}\left(\frac{z(L_{q,s}[\alpha_1]J_c(f)(z))'}{L_{q,s}[\alpha_1]J_c(f)(z)} - \eta\right) \prec h(z) \qquad (z \in U).$$

□

Letting $h(z) = (1 + Az)/(1 + Bz) \quad (-1 \leq B < A \leq 1)$ in Theorem 2.7, we have

**Corollary 2.8.** *For $c > -\eta$ and $0 \leq \eta < 1$. If $f \in Q_{\alpha_1}(\eta, A, B)$, then $J_c(f) \in Q_{\alpha_1}(\eta, A, B)$, where $J_c$ is the integral operator defined by (9).*

Letting $h(z) = ((1 + z)/(1 - z))^{\beta} \quad (0 < \beta \leq 1)$ in Theorem 2.7, we have

**Corollary 2.9.** *For $c > -\beta$, $0 < \beta \leq 1$ and $0 \leq \eta < 1$. If $f \in \overline{S}^*_{\alpha_1}(\eta, \beta)$, then $J_c(f) \in \overline{S}^*_{\alpha_1}(\eta, \beta)$.*

**Corollary 2.10.** *For $c > -\beta$, $0 < \beta \leq 1$ and $0 \leq \eta < 1$. If $f \in \overline{C}_{\alpha_1}(\eta, \beta)$, then $J_c(f) \in \overline{C}_{\alpha_1}(\eta, \beta)$.*

*Proof.*

$$\begin{aligned} f \in \overline{C}_{\alpha_1}(\eta, \beta) &\Leftrightarrow zf' \in \overline{S}^*_{\alpha_1}(\eta, \beta) \\ &\Leftrightarrow J_c(zf'(z)) \in \overline{S}^*_{\alpha_1}(\eta, \beta) \\ &\Leftrightarrow z(J_c(f)(z))' \in \overline{S}^*_{\alpha_1}(\eta, \beta) \\ &\Leftrightarrow J_c(f)(z) \in \overline{C}_{\alpha_1}(\eta, \beta). \end{aligned}$$

□

**Theorem 2.11.** *Let $Re\,\alpha_1 > 1 - \beta$, $\quad 0 < \beta \leq 1$ and $f \in A$. Then $R_{\alpha_1}(\gamma, \beta, \eta, A, B) \subset R_{\alpha_1+1}(\gamma, \beta, \eta, A, B)$.*

*Proof.* Let $f \in R_{\alpha_1}(\gamma, \beta, \eta, A, B)$, then by definition, we can write

$$\frac{1}{1-\gamma}\left(\frac{z(L_{q,s}[\alpha_1]f(z))'}{L_{q,s}[\alpha_1]g(z)} - \gamma\right) \prec \left(\frac{1+z}{1-z}\right)^{\beta} \qquad (z \in U) \tag{12}$$

for some $g \in Q_{\alpha_1}(\eta, A, B)$.

Letting $h(z) = \dfrac{z(L_{q,s}[\alpha_1+1]f(z))'}{L_{q,s}[\alpha_1+1]g(z)}$ and $H(z) = \dfrac{z(L_{q,s}[\alpha_1+1]g(z))'}{L_{q,s}[\alpha_1+1]g(z)}$, we observe that $h$ and $H$ are analytic in $U$ and $H(0) = H(0) = 1$. Now by Corollary 2.4, $g \in Q_{\alpha_1+1}(\eta, A, B)$ and so Re $H(z) > \eta$. Also, note that

$$z(L_{q,s}[\alpha_1+1]f(z))' = (L_{q,s}[\alpha_1+1]g(z))h(z). \tag{13}$$

Differentiating both sides of (13) yields

$$\begin{aligned}\frac{z(L_{q,s}[\alpha_1+1]zf'(z))'}{L_{q,s}[\alpha_1+1]g(z)} &= \frac{z(L_{q,s}[\alpha_1+1]g(z))'}{L_{q,s}[\alpha_1+1]g(z)}h(z) + zh'(z)\\ &= H(z)h(z) + zh'(z).\end{aligned}$$

Now using the identity (7), we obtain

$$\begin{aligned}\frac{z(L_{q,s}[\alpha_1]f(z))'}{L_{q,s}[\alpha_1]g(z)} &= \frac{L_{q,s}[\alpha_1](zf'(z))}{L_{q,s}[\alpha_1]g(z)}\\ &= \frac{z(L_{q,s}[\alpha_1+1](zf'(z)))' + (\alpha_1-1)L_{q,s}[\alpha_1+1](zf'(z))}{z(L_{q,s}[\alpha_1+1]g(z))' + (\alpha_1-1)L_{q,s}[\alpha_1+1]g(z))}\\ &= \frac{\dfrac{z(L_{q,s}[\alpha_1+1](zf'(z)))'}{L_{q,s}[\alpha_1+1]g(z)} + (\alpha_1-1)\dfrac{L_{q,s}[\alpha_1+1](zf'(z))}{L_{q,s}[\alpha_1+1]g(z)}}{\dfrac{z(L_{q,s}[\alpha_1+1]g(z))'}{L_{q,s}[\alpha_1+1]g(z)} + (\alpha_1-1)}\\ &= \frac{H(z)h(z) + zh'(z) + (\alpha_1-1)h(z)}{H(z) + (\alpha_1-1)}\\ &= h(z) + \frac{1}{H(z) + (\alpha_1-1)}zh'(z).\end{aligned} \tag{14}$$

From (12), (13), and (14), we conclude that

$$\frac{1}{1-\gamma}\left(h(z) + \frac{1}{H(z)+(\alpha_1-1)}zh'(z) - \gamma\right) \prec \left(\frac{1+z}{1-z}\right)^{\beta}.$$

On letting $E = 0$ and $B(z) = \dfrac{1}{1-\gamma}\dfrac{1}{H(z)+(\alpha_1-1)}$, we obtain

$$\mathrm{Re}(B(z)) = \frac{1}{1-\gamma}\frac{1}{|H(z)+(\alpha_1-1)|^2}\ \mathrm{Re}(H(z)+(\alpha_1-1)) > 0.$$

The above inequality satisfies the conditions required by Lemma 2.2. Hence $h(z) \prec \left(\dfrac{1+z}{1-z}\right)^{\beta}$ and so the proof is complete. □

**Theorem 2.12.** *Let $c > -\beta, \quad 0 < \beta \le 1$. If $f \in R_{\alpha_1}(\gamma, \beta, \eta, A, B)$, so is $J_c(f) \quad (0 \le \eta < 1, \ -1 \le B < A \le 1)$, where $J_c$ is defined by (9).*

*Proof.* Let $f \in R_{\alpha_1}(\gamma, \beta, \eta, A, B)$, then

$$\frac{1}{1-\gamma}\left(\frac{z(L_{q,s}[\alpha_1]f(z))'}{L_{q,s}[\alpha_1]g(z)} - \gamma\right) \prec \left(\frac{1+z}{1-z}\right)^{\beta} \tag{15}$$

for some $g \in Q_{\alpha_1}(\eta, A, B)$. Now from (10), we have

$$\begin{aligned}\frac{z(L_{q,s}[\alpha_1]J_c(f)(z))'}{L_{q,s}[\alpha_1]g(z)} &= \frac{z(L_{q,s}[\alpha_1]J_c(zf')(z))' + cL_{q,s}[\alpha_1]J_c(zf')(z)}{z(L_{q,s}[\alpha_1]J_c(g)(z))' + cL_{q,s}[\alpha_1]J_c(g)(z)} \\ &= \frac{\dfrac{z(L_{q,s}[\alpha_1]J_c(zf')(z))'}{L_{q,s}[\alpha_1]J_c(g)(z)} + \dfrac{cL_{q,s}[\alpha_1](zf')(z)}{L_{q,s}[\alpha_1]J_c(g)(z)}}{\dfrac{z(L_{q,s}[\alpha_1]J_c(g)(z))'}{L_{q,s}[\alpha_1]J_c(g)(z)} + c}.\end{aligned} \tag{16}$$

Since $g \in Q_{\alpha_1}(\eta, A, B)$, by Corollary 2.8, we have $J_c(g)(z) \in Q_{\alpha_1}(\eta, A, B)$.

Letting $H(z) = \dfrac{z(L_{q,s}[\alpha_1]J_c(g)(z))'}{L_{q,s}[\alpha_1]J_c(g)(z)}$, we note that $\mathrm{Re}(H(z)) > \eta$. Now, let $h$ be defined by

$$z(L_{q,s}[\alpha_1]J_c(f)(z))' = h(z)L_{q,s}[\alpha_1]J_c(g)(z). \tag{17}$$

Differentiating both sides of (17) yields

$$\begin{aligned}\frac{z(L_{q,s}[\alpha_1](zJ_c(f)(z))')'}{L_{q,s}[\alpha_1]g(z)} &= zh'(z) + h(z)\frac{z(L_{q,s}[\alpha_1]J_c(g)(z))'}{L_{q,s}[\alpha_1]J_c(g)(z)} \\ &= zh'(z) + H(z)h(z).\end{aligned} \tag{18}$$

Therefore, from (16) and (18), we obtain

$$\frac{z(L_{q,s}[\alpha_1]f(z))'}{L_{q,s}[\alpha_1]g(z)} = \frac{zh'(z) + H(z)h(z) + ch(z)}{H(z) + c}.$$

This in conjunction with (15) leads to

$$\frac{1}{1-\gamma}\left(\frac{zh'(z)}{H(z)+c} + h(z) - \gamma\right) \prec \left(\frac{1+z}{1-z}\right)^{\beta}. \tag{19}$$

Letting $B(z) = \dfrac{1}{1-\gamma}\cdot\dfrac{1}{H(z)+c}$ in (19), we note that $\mathrm{Re}\, B(z) > 0$ if $c > -\beta$. Now, for $E = 0$ and $B$ as described, we conclude the proof since the required conditions of Lemma 2.2 are satisfied. □

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