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A Linear Operator and its Applications on p-Valent Functions

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Abstract

Let $A(p)$ denote the class of functions of the form $f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p}$, which are analytic in the open unit disk $D = \{z : z \in C; |z| < 1\}$. The authers define the function $\phi^{(*)}(a, c; z)$ by using convolusion $*$ as $\phi_p(a, c; z) * \phi_p^{(*)}(a, c; z) = \frac{z^p}{(1-z)^{p+1}}$ and then introduce new subclass of p-valent functions using an integral operator defined as $IL_p(a, c; z)f(z) = \phi_p^{(*)}(a, c; z) * f(z)$ and derive some interesting properties of this generalized integral operator $IL_p(a, c; z)$.

1 Introduction

Let $A(p)$ denote the class of functions of the form $f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p}$, which are analytic in the open unit disk $D = \{z : z \in C; |z| < 1\}$. If f and g are analytic in D, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwartz function w in D such that $f(z) = g(w(z))$.

The function $\phi_p(a, c; z)$ is defined by

$$\begin{aligned} \phi_p(a, c; z) &= z^p + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} z^{k+p} \\ (a &\in R, c \in C - \{0, -1, -2, ...\}, z \in D) \end{aligned}$$

Corresponding to the function $\phi_p(a, c; z)$, Saitoh[6] introduced a linear operator $L_p(a, c)$ which is defined by Hadamard product $(*)$ (or convolution):

$$\begin{aligned} L_p(a, c; z)f(z) &= \phi_p(a, c; z) * f(z) \\ &= z^p + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} a_{k+p} z^{k+p}, \end{aligned} \tag{1.1}$$

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where $(a)_k$ is the Pochhammer symbol defined by

$$\begin{aligned}(a)_k &= \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1)...(a+k-1), k \in N = \{1, 2, ...\}\\ (a)_0 &= 1.\end{aligned}$$

The definition 1 of the linear operator $L_p(a, c)$ is motivated essentially by the familiar Carlson-Shaffer operator $L(a, c) := L_1(a, c)$[1], which has been used widely on such spaces of analytic and univalent functions in D . A linear operator $l_p(a, c)$, analogous to $L_p(a, c)$ considered here, was investigated recently by Liu and Srivastava [4] on the space of meromorphically p-valent functions in U. We remark in passing that a much more general convolution operator than the operator $L_p(a, c)$ considered here, involving the generalized hypergeometric function in the defining Hadamard product (or convolution), was introduced earlier by Dziok and Srivastava[2].

Recently, analogous to Saitoh [6] defined an integral operator as follows.

Let $\phi_p^{(*)}(a, c; z)$ be defined such that

$$\phi_p(a, c; z) * \phi_p^{(*)}(a, c; z) = \frac{z^p}{(1-z)^{p+1}}$$

Then

$$IL_p(a, c; z)f(z) = \phi_p^{(*)}(a, c; z) * f(z) \tag{1.2}$$

The operator $IL_p(a, c; z)$ satisfies that

$$\begin{aligned}&IL_p(a, c+1; z)f(z)\\ =\ &z(IL_p(a, c; z)f(z))' + (c-p)IL_p(a, c; z)f(z)\end{aligned} \tag{1.3}$$

Putting $a = p$ and $c = 1$ in 1, we have $pIL_p(a, c; z)f(z) = zf'(z)$, and when we put $a = p + 1, c = 1$ in 1, we obtain $IL_p(a, c; z)f(z) = f(z)$.Setting $a = n + p, c = 1$in 1 we obtain $IL_p(a, c; z)f(z) = I_{n+p-1}$, and when $a = p + 1, c = n + p$, we have $IL_p(a, c; z)f(z) = D^{n+p-1}f(z)$.

2 Main Results

In this secion, we shall derive some properties of the linear operator $IL_p(a, c; z)$.

Theorem 1 *Let* $f(z) \in A(p)$, $\beta \geq 1$, $c > 0$ *and* $0 \leq \alpha < 1$. *If*

$$\Re e\{\frac{IL_p(a, c+1; z)f(z)}{IL_p(a, c; z)f(z)}\} < 1 + \frac{1-\alpha}{c}\ , \tag{2.1}$$

then

$$\Re e\{\left(\frac{IL_p(a, c; z)f(z)}{z^p}\right)^{\frac{1}{2\beta(1-\alpha)}}\} > 2^{-\frac{1}{\beta}}. \tag{2.2}$$

In order to prove the above theorem we shall need the following lemma which is due to Eenigenburg, Miller, Mocanu, and Reade [3].

Lemma 2 *Let* γ, β *be complex constants and* $h(z)$*be univalently convex in the unit disk* D *with* $h(0) = p$ *and* $\Re e(\beta h(z) + \gamma) > 0$. *Let* $g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p} z^{k+p}$ *be analytic in* D. *Then*

$$g(z) + \frac{zg'(z)}{\beta g(z) + \gamma} \prec h(z) \Longrightarrow g(z) \prec h(z).$$

Proof. From 1 and 1, we have

$$\frac{IL_p(a, c+1; z)f(z)}{IL_p(a, c; z)f(z)} = \frac{1}{c}\{\frac{z(IL_p(a, c; z)f(z))'}{IL_p(a, c; z)f(z)} + (c - p)\}, \tag{2.3}$$

then

$$\Re e\{\frac{z(IL_p(a, c; z)f(z))'}{IL_p(a, c; z)f(z)}\} < p + 1 - \alpha, z \in D.$$

That is,

$$\frac{1}{2(1-\alpha)}[\frac{z(IL_p(a, c; z)f(z))'}{IL_p(a, c; z)f(z)} + p] \prec \frac{z}{1-z}.$$

Now, let

$$s(z) = \left(\frac{IL_p(a, c; z)f(z)}{z^p}\right)^{\frac{1}{2(1-\alpha)}} \tag{2.4}$$

then 1 may be written as

$$z(\log s(z))' \prec z\left(\log \frac{1}{1-z}\right)'. \tag{2.5}$$

Using a well-known result[7] to 1, we find that

$$s(z) \prec \frac{1}{1-z}$$

that is, that

$$\left(\frac{IL_p(a, c; z)f(z)}{z^p}\right)^{\frac{1}{2\beta(1-\alpha)}} = \left(\frac{1}{1-w(z)}\right)^{\frac{1}{\beta}}, \tag{2.6}$$

where$w(z)$ is analytic in D, $w(0) = 0$ and $|w(z)| < 1$ for $z \in D$.According to $\Re e(t^{\frac{1}{\beta}}) \geq (\Re e t)^{\frac{1}{\beta}}$ for $\Re e t > 0$ and $\beta \geq 1$, 1 yields

$$\Re e\left(\frac{IL_p(a, c; z)f(z)}{z^p}\right)^{\frac{1}{2\beta(1-\alpha)}} \geq \left(\Re e\frac{1}{1-w(z)}\right)^{\frac{1}{\beta}} > 2^{-\frac{1}{\beta}}$$

This completes the proof. ■

Theorem 3 *Let* $f(z) \in A(p)$, $c > 0, 0 \le \lambda < c+1$ *and* $\alpha. > 1$. *Then*

$$\Re e\left((1-\lambda)\frac{IL_p(a,c+1;z)f(z)}{IL_p(a,c;z)f(z)} + \lambda\frac{IL_p(a,c+2;z)f(z)}{IL_p(a,c+1;z)f(z)}\right) < \alpha, z \in D, \quad (2.7)$$

implies

$$\Re e\left(\frac{IL_p(a,c+1;z)f(z)}{IL_p(a,c;z)f(z)}\right) < \beta, z \in D, \quad (2.8)$$

where $\beta \in [\alpha, +\infty)$*is the positive root of the equation*

$$2(c+1-\lambda)x^2 + (3\lambda - 2(c+1)\alpha)x - \lambda = 0.$$

We need the following lemma to prove our theorem

Lemma 4 *(see [5]). Let* Ω *be a set in the complex plane* C *and let* b *be a complex number satisfying* $\Re eb > 0$. *Suppose that the function* $\psi : C^2 \times D \to C$ *satisfies the condition* $\psi(ix; y) \notin \Omega$ *for all real* $x; y \le \frac{-|b+ix|^2}{2\Re eb}$ *and all* $z \in D$. *If the function* $p(z)$*defined by* $p(z) = b + a_1 z + a_2 z^2 + \ldots.$ *is analytic in* D *and if* $\psi(p(z), zp'(z)) \in \Omega$, *then* $\Re ep(z) > 0$ *in* D.

Proof. Let

$$p(z) = \frac{1}{\beta - 1}\left(\beta - \frac{IL_p(a,c+1;z)f(z)}{IL_p(a,c;z)f(z)}\right), \quad (2.9)$$

then $p(z)$ is analytic in$D$ and $p(0) = 1$. Differentiating 1 and using 1 we deduce that

$$\begin{aligned} & (1-\lambda)\frac{IL_p(a,c+1;z)f(z)}{IL_p(a,c;z)f(z)} + \lambda\frac{IL_p(a,c+2;z)f(z)}{IL_p(a,c+1;z)f(z)} \\ = \; & \beta - \frac{\lambda(\beta-1)}{c+1}\left(1 + \frac{zp'(z)}{\beta - (\beta-1)p(z)}\right) - \frac{(\beta-1)(c+1-\lambda)}{c+1}p(z) \\ = \; & \psi(p(z), zp'(z)) \end{aligned}$$

where

$$\psi(r,s) = \beta - \frac{\lambda(\beta-1)}{c+1}\left(1 + \frac{s}{\beta - (\beta-1)r}\right) - \frac{(\beta-1)(c+1-\lambda)}{c+1}r. \quad (2.10)$$

Using 1 and 1, we have

$$\{\psi(p(z), zp'(z)) : z \in D\} \subset \Omega = \{w \in C : \Re ew < \alpha\}.$$

Now for all real $x; y \leq \frac{1+x^2}{2}$, we obtain

$$\begin{aligned}\Re e\psi(ix,y) &= \beta-\frac{\lambda(\beta-1)}{c+1}(1+y\frac{\beta}{\beta^2+(\beta-1)^2x^2})\\ &\geq \beta-\frac{\lambda(\beta-1)}{c+1}(1-\frac{\beta\,(1+x^2)}{2\,(\beta^2+(\beta-1)^2x^2)})\\ &\geq \beta-\frac{\lambda(\beta-1)}{c+1}(1-\frac{1}{2\beta})=\alpha\end{aligned}$$

where β is the positive root of the equation

$$2(c+1-\lambda)x^2+(3\lambda-2(c+1)\alpha)x-\lambda=0.$$

Note that $0 \leq \lambda < \alpha+1$ and $f(\alpha) = -\lambda(2\alpha-1)(\alpha-1) \leq 0$, then we have $\beta \in [\alpha, +\infty)$. Hence for each $z \in D$, $\psi(ix; y) \notin \Omega$. By Lemma 1, we get $\Re ep(z) > 0$. This proves 1. ■

We next prove

Theorem 5 *Let* α, λ, c *be real number with* $0 \leq \lambda, \alpha. > 1$*and* $c > 0$.*Let* $g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p}z^{k+p}$ *satisfy*

$$\Re e\left(\frac{IL_p(a,c;z)g(z)}{IL_p(a,c+1;z)g(z)}\right)>\gamma, 0\leq\gamma<1. \qquad (2.11)$$

If $f\ (z) \in A(p)$ *satisfies*

$$\Re e\left((1-\lambda)\frac{IL_p(a,c;z)f(z)}{IL_p(a,c;z)g(z)}+\lambda\frac{IL_p(a,c+1;z)f(z)}{IL_p(a,c+1;z)g(z)}\right)<\alpha, z\in D, \qquad (2.12)$$

then we have

$$\Re e\left(\frac{IL_p(a,c;z)f(z)}{IL_p(a,c;z)g(z)}\right)<\beta, z\in D, \qquad (2.13)$$

where $\beta = \frac{2\alpha c+\lambda\gamma}{2c+\lambda\gamma}$.

Proof. Let $\beta = \frac{2\alpha c+\lambda\gamma}{2c+\lambda\gamma}$,and consider the function

$$s(z)=\frac{1}{\beta-1}\left(\beta-\frac{IL_p(a,c;z)f(z)}{IL_p(a,c;z)g(z)}\right). \qquad (2.14)$$

The function $s(z)$ is analytic in D and $s(0) = 1$. Set

$$B(z)=\frac{IL_p(a,c;z)g(z)}{IL_p(a,c+1;z)f(z)}$$

then$\Re e B(z) > \gamma$. Differentiating $s(z)$ and using 1, we

$$\begin{aligned}
&(1-\lambda)\frac{IL_p(a,c;z)f(z)}{IL_p(a,c;z)g(z)} + \lambda\frac{IL_p(a,c+1;z)f(z)}{IL_p(a,c+1;z)g(z)} \\
= \;& \beta - (\beta-1)s(z) - \frac{\lambda(\beta-1)}{c}B(z)zs'(z).
\end{aligned}$$

Let

$$\psi(r,t) = \beta - (\beta-1)r - \frac{\lambda(\beta-1)}{c}B(z)t.$$

then from 1, we deduce that

$$\{\psi(s(z), zs'(z)) : z \in D\} \subset \Omega = \{w \in C : \Re e w < \alpha\}.$$

Now for all real $x; y \leq \frac{1+x^2}{2}$, we obtain

$$\begin{aligned}
\Re e\psi(ix,y) &= \beta - \frac{\lambda(\beta-1)y}{c+1}\Re e(B(z)) \\
&\geq \beta + \frac{\lambda(\beta-1)\gamma}{2c}\left(1+x^2\right) \\
&\geq \beta + \frac{\lambda(\beta-1)\gamma}{2c} = \beta + \frac{\lambda(\beta-1)(1-2\beta)}{2\beta(c+1)} = \alpha
\end{aligned}$$

Hence for each $z \in D$, $\psi(ix;y) \notin \Omega$. Thus by Lemma 1, $\Re e s(z) > 0$ in D. The proof of the theorem is complete. ■

From the proof of Theorem 3, we can easily have the following corollaries.

Corollary 6 *Let Let* α, λ, c *be real number with* $0 \leq \lambda, \alpha. > 1$*and* $c > 0$.*Let* $g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p}z^{k+p}$ *satisfy the condition 1. If* $f(z) \in A(p)$ *satisfies 1 ,then we have*

$$\Re e\left(\frac{IL_p(a,c+1;z)f(z)}{IL_p(a,c+1;z)g(z)}\right) < \frac{\alpha(2c+\gamma)+\gamma(\lambda-1)}{2c+\lambda\gamma}, z \in D.$$

Taking $a = n+p$ *and* $c = 1$ *in Theorem 3, we have*

Corollary 7 *Let Let* α, λ *be real number with* $0 \leq \lambda$*and* $\alpha. > 1$.*Let* $g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p}z^{k+p}$ *satisfy*

$$\Re e\left(\frac{I_{n+p-1}g(z)}{I_{n+p}g(z)}\right) > \gamma, 0 \leq \gamma < 1.$$

If $f(z) \in A(p)$ *satisfies*

$$\Re e\left((1-\lambda)\frac{I_{n+p-1}f(z)}{I_{n+p-1}g(z)} + \lambda\frac{I_{n+p}f(z)}{I_{n+p}g(z)}\right) < \alpha, z \in D,$$

then we have

$$\Re e\left(\frac{I_{n+p-1}f(z)}{I_{n+p-1}g(z)}\right) < \frac{2\alpha+\lambda\gamma}{2+\lambda\gamma}.$$

Putting $a = p+1$ and $c = n+p$ in Theorem 3, we obtain

Corollary 8 *Let α, λ be real number with $0 \le \lambda$and $\alpha. > 1$ .Let $g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p} z^{k+p}$ satisfy*

$$\Re e\left(\frac{D^{n+p-1}g(z)}{D^{n+p}g(z)}\right) > \gamma, 0 \le \gamma < 1.$$

If $f(z) \in A(p)$ satisfies

$$\Re e\left((1-\lambda)\frac{D^{n+p-1}f(z)}{D^{n+p-1}g(z)} + \lambda\frac{D^{n+p}f(z)}{D^{n+p}g(z)}\right) < \alpha, z \in D.......$$

then we have

$$\Re e\left(\frac{D^{n+p-1}f(z)}{D^{n+p-1}g(z)}\right) < \frac{2\alpha(n+p)+\lambda\gamma}{2(n+p)+\lambda\gamma}.$$

For a function $f(z) \in A(p)$, we define the integral operator $F_v(f)$ by

$$F_\nu(f) = \frac{\nu+p}{z^\nu}\int_0^z t^{\nu-1}f(t)dt, (\nu+p \ge 0), f(z) \in A(p). \tag{2.15}$$

Corollary 9 *Let α, λ ,νbe real number with $0 \le \lambda, \alpha. > 1$and $\nu > 0$ .Let $g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p} z^{k+p}$ satisfy*

$$\Re e\left(\frac{F_\nu g(z)}{g(z)}\right) > \gamma, 0 \le \gamma < 1.$$

If $f(z) \in A(p)$ satisfies

$$\Re e\left((1-\lambda)\frac{F_\nu f(z)}{F_\nu g(z)} + \lambda\frac{f(z)}{g(z)}\right) < \alpha, z \in D,$$

then we have

$$\Re e\left(\frac{F_\nu f(z)}{F_\nu g(z)}\right) < \frac{2\alpha\nu+\lambda\gamma}{2\nu+\lambda\gamma}.$$

Theorem 10 *Let $h(z)$ be convex univalent in D with $\Re e(h(z)) > 0$ and $h(0) = p$. If $f(z) \in A(p)$ satisfies the condition $\frac{z(IL_p(a,c;z)f(z))'}{IL_p(a,c;z)f(z)} \prec h(z)$, then $\frac{z(IL_p(a,c;z)F_v(f\ (z)))'}{IL_p(a,c;z)F_v(f\ (z))} \prec h(z)$, where F_v is the integral operator defined by* 1

Proof. From 1, we have

$$z(IL_p(a,c;z)F_vf\ (z))' = (c+p)IL_p(a,c;z)f\ (z) - cIL_p(a,c;z)F_vf\ (z). \quad (2.16)$$

Letting $s(z) = \dfrac{z(IL_p(a,c;z)F_vf\ (z))'}{IL_p(a,c;z)F_vf\ (z)}$ in 1, we can write

$$\frac{z(IL_p(a,c;z)F_vf\ (z))'}{IL_p(a,c;z)F_vf\ (z)} + c = (c+p)\frac{IL_p(a,c;z)f\ (z)}{IL_p(a,c;z)F_vf\ (z)}. \quad (2.17)$$

Differentiating 1 yields

$$s(z) + \frac{zs'(z)}{s(z)+c} = \frac{z(IL_p(a,c;z)f\ (z))'}{IL_p(a,c;z)f\ (z)}.$$

It follows that $s(z) \prec h(z)$, that is, $\dfrac{z(IL_p(a,c;z)F_vf\ (z))'}{IL_p(a,c;z)F_vf\ (z)} \prec h(z)$. ■

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