On a New Hardy-Hilbert’s Type Inequality with a Parameter

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Abstract

By using the improved Euler-Maclaurin’s summation formula and introducing a parameter $\alpha$, a new Hardy-Hilbert’s type inequality is built. As applications, the equivalent form and some particular results are considered. All the lemmas and the theorem provide some new estimates on this type of inequalities.

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1 Introduction

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, a_n, b_n \geq 0$, such that $0 < \sum_{n=1}^{\infty} a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then one has

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\ln(m/n)a_mb_n}{m-n} < \left[ \frac{\pi}{\sin(\pi/p)} \right]^2 \left\{ \sum_{n=1}^{\infty} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} b_n^q \right\}^{\frac{1}{q}},$$

where the constant factor $[\pi/\sin(\pi/p)]^2$ is the best possible (see [1]). Inequality (1) is one of the Hardy-Hilbert’s type inequalities, and this type of inequalities are important in analysis and its applications (see[2]). In recent years, Pachpatte et. al [3,4,5,6,7,8,9] gave some new generalizations and improvements of them, and Kuang et. al [10] considered a strengthened version of (1) by using the improved Euler-Maclaurin’s summation formula. More recently, Yang [11] gave an extension of (1) by introducing a parameter $\lambda \in (0, \min\{p, q\})$ as
If the same constant factor $\pi$ where the constant factor $\alpha$ is a parameter also built two different more accurate Mulholland’s inequalities by introducing a parameter $\alpha \geq e^{7/6}$ as:

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_mb_n}{mn \ln \alpha mn} < \frac{\pi}{\sin(\frac{\pi}{p})} \left\{ \sum_{n=1}^{\infty} \frac{a_n^p}{n} \right\}^\frac{1}{p} \left\{ \sum_{n=1}^{\infty} \frac{b_n^q}{n} \right\}^\frac{1}{q},$$

(3)

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_mb_n}{mn \ln \alpha mn} < \frac{\pi}{\sin(\frac{\pi}{p})} \left\{ \sum_{n=1}^{\infty} (\ln \sqrt{\alpha n})^{p-2} \frac{a_n^p}{n} \right\}^\frac{1}{p} \left\{ \sum_{n=1}^{\infty} (\ln \sqrt{\alpha n})^{q-2} \frac{b_n^q}{n} \right\}^\frac{1}{q},$$

(4)

where the same constant factor $\frac{\pi}{\sin(\pi/p)}$ in the above inequalities is the best possible.

In this paper, by using the improved Euler-Maclaurin’s summation formula and refinement of the way of weight coefficient as doing in [13], one still introduces a parameter $\alpha$, and build a new Hardy-Hilbert’s type inequality, which is a more accurate of (1) ( for $p = q = 2$ ) related to the double series as

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\ln(\frac{m+n}{n+\alpha})a_mb_n}{(m+\alpha) - (n+\alpha)} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\ln(\frac{m+n}{n+\alpha})a_mb_n}{m-n} \quad (\alpha \geq \frac{1}{2}).$$

As applications, the equivalent form and some particular results are given. All the lemmas and the theorem provide some new estimates on this type of inequalities.

2 Some lemmas

First, we need the formula as (cf. [1, Ch.9]):

$$\int_0^{\infty} \frac{\ln u}{u - 1} u^{-\frac{1}{p}} du = \left[ \frac{\pi}{\sin(\pi/p)} \right]^2 \quad (p > 1).$$

(5)

LEMMA 2.1 (the improved Euler-Maclaurin’s summation formula, see [10, 13]).

If $f \in C^4[0, \infty)$, $(-1)^i f^{(i)}(x) > 0$, $f^{(i)}(\infty) = 0 (i = 0, 1, 2, 3, 4)$, then

$$\sum_{m=0}^{\infty} f(m) \leq \int_0^{\infty} f(x) dx + \frac{1}{2} f(0) - \frac{1}{12} f'(0).$$

(6)
LEMMA 2.2. For $\alpha > 0, r > 1$ and $n \in N_0$ ($N_0$ is the set of non-negative integers), setting $g(u) = \frac{\ln u}{u-1}, u \in (0, \infty)$ ($g(1) := 1$), and

$$f(x) = g\left(\frac{x + \alpha}{n + \alpha}\right) \left(\frac{x + \alpha}{n + \alpha}\right)^{-\frac{1}{r}}, x \in (-\alpha, \infty),$$

then $f(x)$ possesses the condition of (6).

Proof. One finds $g \in C^4(0, \infty)$, and

\[
g'(u) = \frac{1}{(u-1)^2}(1 - \frac{1}{u} - \ln u), g'(1) := -1; \\
g''(u) = \frac{1}{(u-1)^3}(2 \ln u - 3 + \frac{4}{u} - \frac{1}{u^2}), g''(1) := \frac{2}{3}; \\
g'''(u) = \frac{1}{(u-1)^4}(-6 \ln u + 11 - \frac{18}{u} + \frac{9}{u^2} - \frac{2}{u^3}), g'''(1) := -\frac{3}{2}; \\
g^{(4)}(u) = \frac{h(u)}{(u-1)^5}, h(u) = 24 \ln u - 50 + \frac{96}{u} - \frac{72}{u^2} + \frac{32}{u^3} - \frac{6}{u^4}, g^{(4)}(1) := \frac{24}{5}. \]

It is obvious that $g^{(i)}(\infty) = 0$ $(i = 0, 1, 2, 3, 4)$. Since $h'(u) = \frac{24}{u}(1 - \frac{1}{u})^4 > 0$ $(u \neq 1)$, then $h(u)$ is strictly increasing in $(0, \infty)$. In view of $h(1) = 0$, one has $h(u) < 0, u \in (0, 1); h(u) > 0, u \in (1, \infty)$, and then $g^{(4)}(u) > 0$ for $u \in (0, \infty)$. Hence $g'''(u)$ is strictly increasing and $g'''(u) < 0$ since $g'''(\infty) = 0$. By the same way, it follows that $g''(u)$ is strictly decreasing and $g''(u) > 0$ since $g''(\infty) = 0$, and $g'(u) < 0$ since $g'(\infty) = 0$ and $g'(u)$ is strictly decreasing. Therefore one can concludes that $(-1)^{i}[g(u)u^{-\frac{1}{r}}]^{(i)} > 0$ $(i = 0, 1, 2, 3, 4)$, and then

\[
(-1)^{i}f^{(i)}(x) = (-1)^{i}\left[g(u)u^{-\frac{1}{r}}\right]^{(i)} \frac{1}{(n + \alpha)^{i}} > 0 \quad (x \in [0, \infty)),
\]

and $f^{(i)}(\infty) = 0$ $(i = 0, 1, 2, 3, 4)$. The lemma is proved.

Note. By (6), one has

\[
\sum_{m=0}^{\infty} \frac{\ln (\frac{m+\alpha}{n+\alpha})}{m-n} \left(\frac{n+\alpha}{m+\alpha}\right)^{\frac{1}{r}} = \frac{1}{n+\alpha} \sum_{m=0}^{\infty} f(m)
\leq \frac{1}{n+\alpha} \left[\int_{0}^{\infty} f(x) dx + \frac{1}{2} f(0) - \frac{1}{12} f'(0)\right]
= \frac{1}{n+\alpha} \left[\int_{-\alpha}^{\infty} f(x) dx - R_\alpha(r, n)\right],
\]

\[
R_\alpha(r, n) := \int_{-\alpha}^{0} f(x) dx - \frac{1}{2} f(0) + \frac{1}{12} f'(0). \quad (7)
\]
and

\[ f'(0) = \frac{1}{(n + \alpha)} g'\left(\frac{\alpha}{n + \alpha}\right) \left(\frac{\alpha}{n + \alpha}\right)^{-\frac{1}{r}} - \frac{1}{r(n + \alpha)} g\left(\frac{\alpha}{n + \alpha}\right) \left(\frac{\alpha}{n + \alpha}\right)^{-\frac{1}{r} - 1}, \]

and \( f(0) = g\left(\frac{\alpha}{n + \alpha}\right)\left(\frac{\alpha}{n + \alpha}\right)^{-\frac{1}{r}}. \) Hence one obtains from (7) and the above results that

\[
R_\alpha(r, n) > g\left(\frac{\alpha}{n + \alpha}\right) \left(\frac{\alpha}{n + \alpha}\right)^{-\frac{1}{r}} \left(\frac{r\alpha}{r - 1} - \frac{1}{2} - \frac{1}{12r\alpha}\right) - g'\left(\frac{\alpha}{n + \alpha}\right) \left(\frac{\alpha}{n + \alpha}\right)^{-\frac{1}{r} + \frac{1}{r}} \left[\frac{r^2\alpha}{(2r - 1)(r - 1)} - \frac{1}{12\alpha}\right]. \tag{8}
\]

**LEMMA 2.3.** For \( r > 1, \alpha \geq \frac{1}{2}, n \in N_0, \) define the weight coefficient \( \omega_\alpha(r, n) \) as

\[
\omega_\alpha(r, n) := \sum_{m=0}^{\infty} \frac{\ln\left(\frac{m+n}{n}\right)}{m-n} \left(\frac{n + \alpha}{m + \alpha}\right)^{\frac{1}{r}}. \tag{9}
\]

Then one has

\[
\omega_\alpha(r, n) < \left[\frac{\pi}{\sin(\pi/r)}\right]^2 (n \in N_0). \tag{10}
\]

Proof. For \( r > 1, \alpha \geq \frac{1}{2}, \) one has

\[
\frac{r\alpha}{r - 1} - \frac{1}{2} - \frac{1}{12r\alpha} = \frac{6r^2\alpha(2\alpha - 1) + (6\alpha - 1)r + 1}{12r(r - 1)\alpha} > 0; \tag{11}
\]

\[
\frac{r^2\alpha}{(2r - 1)(r - 1)} - \frac{1}{12\alpha} = \frac{2r^2(6\alpha^2 - 1) + 3r - 1}{12(2r - 1)(r - 1)\alpha} > 0.
\]
Since \( g(u) > 0 \) and \( g'(u) < 0 \), in view of (8), one has \( R_\alpha(r,n) > 0 \). Setting \( u = (x+\alpha)/(n+\alpha) \), one finds from (5) that

\[
\frac{1}{n+\alpha} \int_{-\alpha}^{\infty} f(x)dx = \int_{0}^{\infty} \frac{\ln u}{u-1} u^{-\frac{1}{p}} du = \left[ \frac{\pi}{\sin(\pi/r)} \right]^2.
\]

In view of (9) and (7), one has (10). The lemma is proved.

Note. If \( \alpha < \frac{1}{2} \), one can’t conform that \( R_\alpha(r,n) > 0 \) by (11) for the more large enough number \( r > 1 \).

**LEMMA 2.4.** If \( p > 1, \frac{1}{p} + \frac{1}{q} = 1, \alpha \geq \frac{1}{2}, 0 < \varepsilon < 1 \), one has

\[
\begin{align*}
I : & = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \ln(\frac{m+\alpha}{n+\alpha}) \left( \frac{1}{m+\alpha} \right)^{\frac{1}{q}+\varepsilon} \left( \frac{1}{n+\alpha} \right)^{\frac{1}{p}+\varepsilon} \\
& \geq \frac{1}{\varepsilon \alpha^2} \left\{ \left[ \frac{\pi}{\sin(\pi/p)} \right]^2 + o(1) \right\} (\varepsilon \to 0^+)
\end{align*}
\]

Proof. For fixed \( y \), setting \( u = (x+\alpha)/(y+\alpha) \), since \( g(u) \) is decreasing, one obtains

\[
I \geq \int_{0}^{\infty} \left( \frac{1}{y+\alpha} \right)^{\frac{1}{q}+\varepsilon} \left[ \int_{0}^{\infty} \frac{\ln(\frac{y+\alpha}{y+y})}{x-y} \left( \frac{1}{x+\alpha} \right)^{\frac{1}{q}+\varepsilon} dx \right] dy
\]

\[
= \int_{0}^{\infty} \left( \frac{1}{y+\alpha} \right)^{1+\varepsilon} \left[ \int_{0}^{\infty} \frac{\ln u}{u-1} u^{-\left(\frac{1}{q}+\varepsilon\right)} du \right] dy
\]

\[
= \int_{0}^{\infty} \left( \frac{1}{y+\alpha} \right)^{1+\varepsilon} \left[ \int_{0}^{\infty} \frac{\ln u}{u-1} u^{-\left(\frac{1}{q}+\varepsilon\right)} du - \int_{0}^{\frac{\alpha}{y+\alpha}} \frac{\ln u}{u-1} u^{-\left(\frac{1}{q}+\varepsilon\right)} du \right] dy
\]

\[
= \frac{1}{\varepsilon \alpha^2} \int_{0}^{\infty} \frac{\ln u}{u-1} u^{-\left(\frac{1}{q}+\varepsilon\right)} du
\]

\[
- \int_{0}^{\infty} \left( \frac{1}{y+\alpha} \right)^{1+\varepsilon} \left[ \int_{0}^{\frac{\alpha}{y+\alpha}} \frac{\ln u}{u-1} u^{-\left(\frac{1}{q}+\varepsilon\right)} du \right] dy
\]

\[
= \frac{1}{\varepsilon \alpha^2} \int_{0}^{\infty} \frac{\ln u}{u-1} u^{-\left(\frac{1}{q}+\varepsilon\right)} du
\]

\[
- \sum_{n=0}^{\infty} \int_{0}^{\infty} \left( \frac{1}{y+\alpha} \right)^{1+\varepsilon} \left[ \int_{0}^{\frac{\alpha}{y+\alpha}} (-\ln u) u^{-\left(\frac{1}{q}+\varepsilon\right)} du \right] dy
\]

\[
= \frac{1}{\varepsilon \alpha^2} \left\{ \left[ \frac{\pi}{\sin(\pi/p)} \right]^2 + o(1) \right\} + \alpha^{n+\frac{1}{p}} \sum_{n=0}^{\infty} \frac{1}{(n + \frac{1-\varepsilon}{p})(n + \frac{1}{p} + \frac{\varepsilon}{q})}
\]

\[
\times \int_{0}^{\infty} \left[ -\ln\left( \frac{\alpha}{y+\alpha} + \frac{1}{n + \frac{1-\varepsilon}{p}} \right) \right] d(y+\alpha)^{-n+\frac{1-\varepsilon}{p}+\varepsilon}
\]
\[
\frac{1}{\varepsilon \alpha} \left\{ \left[ \frac{\pi}{\sin(\pi/p)} \right]^2 + o(1) \right\} - \varepsilon \sum_{n=0}^{\infty} \frac{1}{(n + 1 - \varepsilon/p)(n + 1/p + \varepsilon/q)} \left\{ \frac{1}{n + 1 - \varepsilon/p} + \frac{1}{n + 1/p + \varepsilon/q} \right\}.
\]

Hence one has (12). The lemma is proved.

3 Main results and applications

THEOREM 3.1. If \( p > 1, \frac{1}{p} + \frac{1}{q} = 1, \alpha \geq \frac{1}{2}, a_n, b_n \geq 0, \) such that \( 0 < \sum_{n=0}^{\infty} (n + \alpha)^{p-2} a_n^p < \infty \) and \( 0 < \sum_{n=0}^{\infty} (n + \alpha)^{q-2} b_n^q < \infty, \) then

\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\ln((m+n)/n)}{m-n} < \left[ \frac{\pi}{\sin(\pi/p)} \right]^2 \left\{ \sum_{n=0}^{\infty} (n + \alpha)^{p-2} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=0}^{\infty} (n + \alpha)^{q-2} b_n^q \right\}^{\frac{1}{q}},
\]

(13)

where the constant factor \([\pi / \sin(\pi/p)]^2\) is the best possible. The equivalent form is

\[
\sum_{n=1}^{\infty} (n + \alpha)^{p-2} \left[ \sum_{m=0}^{\infty} \ln((m+n)/n) a_m \right]^p < \left[ \frac{\pi}{\sin(\pi/p)} \right]^{2p} \sum_{n=0}^{\infty} (n + \alpha)^{p-2} a_n^p,
\]

(14)

where the constant factor \([\pi / \sin(\pi/p)]^{2p}\) is also the best possible. If particular, for \( \alpha = 1 \) in (13) and (14), replacing \( a_{n-1} \) by \( a_n \), and \( b_{n-1} \) by \( b_n \), one has the following equivalent inequalities:

\[
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\ln((m+n)/n)}{m-n} < \left[ \frac{\pi}{\sin(\pi/p)} \right]^2 \left\{ \sum_{n=1}^{\infty} n^{p-2} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} n^{q-2} b_n^q \right\}^{\frac{1}{q}};
\]

(15)

\[
\sum_{n=1}^{\infty} n^{p-2} \left[ \sum_{m=1}^{\infty} \ln((m+n)/n) a_m \right]^p < \left[ \frac{\pi}{\sin(\pi/p)} \right]^{2p} \sum_{n=1}^{\infty} n^{p-2} a_n^p.
\]

(16)

Proof. By Hölder’s inequality with weight (see [14]) and using (9), one has

\[
H_{\alpha}(a_m, b_n) := \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\ln((m+n)/n) a_m b_n}{m-n}
\]

\[
= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\ln((m+n)/n)}{m-n} \left[ (m + \alpha)^{(1/q)^2} a_m \right] \left[ (n + \alpha)^{(1/p)^2} b_n \right]
\]

\[
\leq \left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \ln((m+n)/n) (m + \alpha)^{p/2} a_m \right\}^{\frac{1}{p}} \left\{ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\ln((m+n)/n)(n + \alpha)^{q/2} b_n}{m-n} \right\}^{\frac{1}{q}}
\]

\[
= \left\{ \sum_{m=0}^{\infty} \omega_{\alpha}(p, m)(m + \alpha)^{p-2} a_m \right\}^{\frac{1}{p}} \left\{ \sum_{n=0}^{\infty} \omega_{\alpha}(q, n)(n + \alpha)^{q-2} b_n \right\}^{\frac{1}{q}}.
\]
Hence by (10), since \( \frac{\pi}{\sin(\pi/q)} = \frac{\pi}{\sin(\pi/p)} \), one has (13).

For \( 0 < \varepsilon < 1 \), setting \( \tilde{a}_m, \tilde{b}_n \) as

\[
\tilde{a}_m = \left( \frac{1}{m + \alpha} \right)^{\frac{2 + \varepsilon}{p}}, \tilde{b}_n = \left( \frac{1}{n + \alpha} \right)^{\frac{2 + \varepsilon}{q}}, m, n \in N_0,
\]

one has

\[
\left\{ \sum_{n=0}^{\infty} (n + \alpha)^{p-2} \tilde{a}_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=0}^{\infty} (n + \alpha)^{q-2} \tilde{b}_n^q \right\}^{\frac{1}{q}}
= 1 + \sum_{n=1}^{\infty} \left( \frac{1}{n + \alpha} \right)^{1+\varepsilon} < 1 + \int_0^\infty \left( \frac{1}{x + \alpha} \right)^{1+\varepsilon} dx = 1 + \frac{1}{\varepsilon \alpha^{\varepsilon}}.
\]

If the constant factor \( \frac{\pi}{\sin(\pi/p)} \) in (13) is not the best possible, then there exists a positive number \( k < \frac{\pi}{\sin(\pi/p)} \), such that (13) is still valid if one replaces \( \frac{\pi}{\sin(\pi/p)} \) by \( k \). In particular, by (12) and (17), one has

\[
\frac{1}{\alpha^{\varepsilon}} \left( \frac{\pi}{\sin(\pi/p)} \right)^2 + o(1) \leq \varepsilon I = \varepsilon H_\alpha(\tilde{a}_m, \tilde{b}_n)
< \varepsilon k \left\{ \sum_{n=0}^{\infty} (n + \alpha)^{p-2} \tilde{a}_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=0}^{\infty} (n + \alpha)^{q-2} \tilde{b}_n^q \right\}^{\frac{1}{q}} < k(\varepsilon + \frac{1}{\alpha^{\varepsilon}}),
\]

and then \( \frac{\pi}{\sin(\pi/p)} \leq k(\varepsilon \to 0^+) \). This contradicts the fact that \( k \leq \frac{\pi}{\sin(\pi/p)} \). Hence the constant factor \( \frac{\pi}{\sin(\pi/p)} \) in (13) is the best possible.

Setting \( b_n \) as

\[
\tilde{b}_n := (n + \alpha)^{p-2} \left[ \sum_{m=0}^{\infty} \frac{\ln(m + \alpha)}{m - n} a_m \right]^{p-1}, n \in N_0,
\]

and use (13) to obtain

\[
\left\{ \sum_{n=0}^{\infty} (n + \alpha)^{q-2} \tilde{b}_n^q \right\}^{p} = \left\{ \sum_{n=0}^{\infty} \left( \frac{\ln(m + \alpha)}{m - n} a_m \right)^p \right\}^{p} = \left\{ \sum_{n=0}^{\infty} H_\alpha(a_m, b_n) \right\}^{p} \leq \left[ \frac{\pi}{\sin(\pi/p)} \right]^{2p} \left\{ \sum_{n=0}^{\infty} (n + \alpha)^{p-2} \tilde{a}_n^p \right\} \left\{ \sum_{n=0}^{\infty} (n + \alpha)^{q-2} \tilde{b}_n^q \right\}^{p-1};
\]

\[
0 < \sum_{n=0}^{\infty} (n + \alpha)^{q-2} \tilde{b}_n^q = \sum_{n=0}^{\infty} (n + \alpha)^{p-2} \left[ \sum_{m=0}^{\infty} \frac{\ln(m + \alpha)}{m - n} a_m \right]^{p} \leq \left[ \frac{\pi}{\sin(\pi/p)} \right]^{2p} \sum_{n=0}^{\infty} (n + \alpha)^{p-2} \tilde{a}_n^p < \infty.
\]
It follows that (17) takes the form of strict inequality by using (13); so does (19). Hence (14) holds.

On the other hand, if (14) holds, by Hölder’s inequality, one has

\[ H_\alpha(a_m, b_n) = \sum_{n=0}^{\infty} \left[ (n + \alpha)^{\frac{2-\alpha}{q}} \sum_{m=0}^{\infty} \frac{\ln(m + \alpha)}{m - n} a_m \right] \left[ (n + \alpha)^{\frac{2-\alpha}{q}} b_n \right] \]

\[ \leq \left\{ \sum_{n=0}^{\infty} (n + \alpha)^{p-2} \left[ \sum_{m=0}^{\infty} \frac{\ln(m + \alpha)}{m - n} a_m \right]^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=0}^{\infty} (n + \alpha)^{q-2} b_n^q \right\}^{\frac{1}{q}} \]

(20)

In view of (14), one has (13). It follows that (13) and (14) are equivalent.

If the constant factor \([\pi/\sin(\pi/p)]^2\) in (14) is not the best possible, then by using (20), one can get a contradiction that the constant factor \([\pi/\sin(\pi/p)]^2\) in (13) is not the best possible. The theorem is proved.

REMARK 3.2. (i) For \(\alpha = \frac{1}{2}\) in (13) and (14), one has the following new equivalent inequalities:

\[ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\ln(\frac{m+1}{2n+1}) a_m b_n}{m - n} < \left[ \frac{\pi}{\sin(\frac{\pi}{p})} \right]^2 \left\{ \sum_{n=0}^{\infty} (2n + 1)^{p-2} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=0}^{\infty} (2n + 1)^{q-2} b_n^q \right\}^{\frac{1}{q}} \]

(21)

\[ \sum_{n=0}^{\infty} (2n + 1)^{p-2} \left[ \sum_{m=0}^{\infty} \frac{\ln(\frac{m+1}{2n+1}) a_m}{m - n} \right]^p < \left[ \frac{\pi}{\sin(\frac{\pi}{p})} \right]^{2p} \sum_{n=0}^{\infty} (2n + 1)^{p-2} a_n^p. \]

(22)

(ii) For \(p = q = 2, \alpha \geq \frac{1}{2}\) in (13) and (14), one has the following new equivalent inequalities:

\[ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\ln(m + \alpha)}{m - n} a_m b_n < \pi^2 \left\{ \sum_{n=0}^{\infty} a_n^2 \sum_{n=0}^{\infty} b_n^2 \right\}^{\frac{1}{2}} \]

(23)

\[ \sum_{n=0}^{\infty} \left[ \sum_{m=0}^{\infty} \frac{\ln(m + \alpha)}{m - n} a_m \right]^2 < \pi^4 \sum_{n=0}^{\infty} a_n^2. \]

(24)

For \(\alpha = 1\), (23) reduces to (1) (for \(p = q = 2\)). It follows that (23) is a best extension of (1) for \(p = q = 2\). Since for \(\frac{1}{2} \leq \alpha < 1\) and \(a_n, b_n > 0\) in Theorem 3.1, one has

\[ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\ln(\frac{m+1}{n+\alpha}) a_m b_n}{m - n} < \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\ln(\frac{m+1}{n+\alpha}) a_m b_n}{m - n}, \]

it follows that inequality (23) is more accurate than (1) for any \(\frac{1}{2} \leq \alpha < 1\) and \(p = q = 2\).

(iii) Inequalities (15) and (1) are similar but different, although both of them are with the same best constant factor \([\pi/\sin(\pi/p)]^2\).
New Hardy-Hilbert’s type inequality

References


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