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On a New Hardy-Hilbert's Type Inequality with a Parameter

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Abstract

By using the improved Euler-Maclaurin's summation formula and introducing a parameter α, a new Hardy-Hilbert's type inequality is built. As applications, the equivalent form and some particular results are considered. All the lemmas and the theorem provide some new estimates on this type of inequalities.

Mathematics Subject Classifications: 26D15

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1 Introduction

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, a_n, b_n \geq 0$, such that $0 < \sum_{n=1}^{\infty} a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then one has

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\ln(m/n)a_m b_n}{m-n} < \left[\frac{\pi}{\sin(\frac{\pi}{p})}\right]^2 \left\{\sum_{n=1}^{\infty} a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} b_n^q\right\}^{\frac{1}{q}}, \tag{1}$$

where the constant factor $[\pi/sin(\pi/p)]^2$ is the best possible (see [1]). Inequality (1) is one of the Hardy-Hilbert's type inequalities, and this type of inequalities are important in analysis and its applications (see[2]). In recent years, Pachpatte et. al [3,4,5,6,7,8,9] gave some new generalizations and improvements of them, and Kuang et. al [10] considered a strengthened version of (1) by using the improved Euler-Maclaurin's summation formula. More recently, Yang [11] gave an extension of (1) by introducing a parameter $\lambda \in (0, \min\{p, q\}]$ as

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\ln(\frac{m}{n})a_m b_n}{m^{\lambda}-n^{\lambda}} < \left[\frac{\pi}{\lambda\sin(\frac{\pi}{p})}\right]^2 \left\{\sum_{n=1}^{\infty} n^{(p-1)(1-\lambda)}a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} n^{(q-1)(1-\lambda)}b_n^q\right\}^{\frac{1}{q}}, \tag{2}$$

where the constant factor $[\frac{\pi}{\lambda\sin(\pi/p)}]^2$ is the best possible. And Yang [12,13] also built two different more accurate Mulholland's inequalities by introducing a parameter $\alpha \geq e^{7/6}$ as:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{mn\ln\alpha mn} < \frac{\pi}{\sin(\frac{\pi}{p})}\left\{\sum_{n=1}^{\infty}\frac{a_n^p}{n}\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\frac{b_n^q}{n}\right\}^{\frac{1}{q}}; \tag{3}$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{mn\ln\alpha mn} < \frac{\pi}{\sin(\frac{\pi}{p})}\left\{\sum_{n=1}^{\infty}(\ln\sqrt{\alpha}n)^{p-2}\frac{a_n^p}{n}\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}(\ln\sqrt{\alpha}n)^{q-2}\frac{b_n^q}{n}\right\}^{\frac{1}{q}}, \tag{4}$$

where the same constant factor $\frac{\pi}{\sin(\pi/p)}$ in the above inequalities is the best possible.

In this paper, by using the improved Euler-Maclaurin's summation formula and refinement of the way of weight coefficient as doing in [13], one still introduces a parameter α, and build a new Hardy-Hilbert's type inequality, which is a more accurate of (1) (for $p = q = 2$) related to the double series as

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\ln(\frac{m+\alpha}{n+\alpha})a_m b_n}{(m+\alpha)-(n+\alpha)} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\ln(\frac{m+\alpha}{n+\alpha})a_m b_n}{m-n} \quad (\alpha \geq \frac{1}{2}).$$

As applications, the equivalent form and some particular results are given. All the lemmas and the theorem provide some new estimates on this type of inequalities.

2 Some lemmas

First, we need the formula as (cf. [1,Ch.9]):

$$\int_0^{\infty}\frac{\ln u}{u-1}u^{-\frac{1}{p}}du = \left[\frac{\pi}{\sin(\pi/p)}\right]^2 \quad (p > 1). \tag{5}$$

LEMMA 2.1(the improved Euler-Maclaurin's summation formula, see [10,13]). If $f \in C^4[0,\infty)$, $(-1)^i f^{(i)}(x) > 0, f^{(i)}(\infty) = 0 (i = 0,1,2,3,4)$, then

$$\sum_{m=0}^{\infty} f(m) \leq \int_0^{\infty} f(x)dx + \frac{1}{2}f(0) - \frac{1}{12}f'(0). \tag{6}$$

LEMMA 2.2. For $\alpha > 0, r > 1$ and $n \in N_0$ (N_0 is the set of non-negative integers), setting $g(u) = \frac{\ln u}{u-1}$, $u \in (0,\infty)(g(1) := 1)$, and

$$f(x) = g\left(\frac{x+\alpha}{n+\alpha}\right)\left(\frac{x+\alpha}{n+\alpha}\right)^{-\frac{1}{r}}, x \in (-\alpha, \infty),$$

then $f(x)$ possesses the condition of (6).

Proof. One finds $g \in C^4(0,\infty)$, and

$$\begin{aligned}
g'(u) &= \frac{1}{(u-1)^2}(1 - \frac{1}{u} - \ln u), g'(1) := -1;\\
g''(u) &= \frac{1}{(u-1)^3}(2\ln u - 3 + \frac{4}{u} - \frac{1}{u^2}), g''(1) := \frac{2}{3};\\
g'''(u) &= \frac{1}{(u-1)^4}(-6\ln u + 11 - \frac{18}{u} + \frac{9}{u^2} - \frac{2}{u^3}), g'''(1) := -\frac{3}{2};\\
g^{(4)}(u) &= \frac{h(u)}{(u-1)^5}, h(u) = 24\ln u - 50 + \frac{96}{u} - \frac{72}{u^2} + \frac{32}{u^3} - \frac{6}{u^4}, g^{(4)}(1) := \frac{24}{5}.
\end{aligned}$$

It is obvious that $g^{(i)}(\infty) = 0 (i = 0,1,2,3,4)$. Since $h'(u) = \frac{24}{u}(1-\frac{1}{u})^4 > 0 (u \neq 1)$, then $h(u)$ is strict increasing in $(0,\infty)$. In view of $h(1) = 0$, one has $h(u) < 0$, $u \in (0,1); h(u) > 0, u \in (1,\infty)$, and then $g^{(4)}(u) > 0$ for $u \in (0,\infty)$. Hence $g'''(u)$ is strict increasing and $g'''(u) < 0$ since $g'''(\infty) = 0$. By the same way, it follows that $g''(u)$ is strict decreasing and $g''(u) > 0$ since $g''(\infty) = 0$, and $g'(u) < 0$ since $g'(\infty) = 0$ and $g'(u)$ is strict decreasing. Therefor one can concludes that $(-1)^i[g(u)u^{-\frac{1}{r}}]^{(i)} > 0$ $(i = 0,1,2,3,4)$, and then

$$(-1)^i f^{(i)}(x) = (-1)^i \left(g(u)u^{-\frac{1}{r}}\right)^{(i)}_{u=\frac{x+\alpha}{n+\alpha}} \frac{1}{(n+\alpha)^i} > 0 \ (x \in [0,\infty)),$$

and $f^{(i)}(\infty) = 0 (i = 0,1,2,3,4)$. The lemma is proved.

Note. By (6), one has

$$\sum_{m=0}^{\infty} \frac{\ln(\frac{m+\alpha}{n+\alpha})}{m-n}\left(\frac{n+\alpha}{m+\alpha}\right)^{\frac{1}{r}} = \frac{1}{n+\alpha}\sum_{m=0}^{\infty} f(m)$$

$$\begin{aligned}
&\leq \frac{1}{n+\alpha}\left[\int_0^\infty f(x)dx + \frac{1}{2}f(0) - \frac{1}{12}f'(0)\right]\\
&= \frac{1}{n+\alpha}\left[\int_{-\alpha}^\infty f(x)dx - R_\alpha(r,n)\right],
\end{aligned}$$

$$R_\alpha(r,n) := \int_{-\alpha}^0 f(x)dx - \frac{1}{2}f(0) + \frac{1}{12}f'(0). \tag{7}$$

Since $g''(u) > 0$, one obtains

$$\int_{-\alpha}^{0} f(x)dx = \frac{r(n+\alpha)}{r-1}\int_{-\alpha}^{0} g\left(\frac{x+\alpha}{n+\alpha}\right) d\left(\frac{x+\alpha}{n+\alpha}\right)^{1-\frac{1}{r}}$$

$$= \frac{r(n+\alpha)}{r-1}\left[g(\frac{\alpha}{n+\alpha})(\frac{\alpha}{n+\alpha})^{1-\frac{1}{r}} - \frac{1}{n+\alpha}\int_{-\alpha}^{0} g'(\frac{x+\alpha}{n+\alpha})(\frac{x+\alpha}{n+\alpha})^{1-\frac{1}{r}}dx\right]$$

$$= \frac{r(n+\alpha)}{r-1}g(\frac{\alpha}{n+\alpha})(\frac{\alpha}{n+\alpha})^{1-\frac{1}{r}} - \frac{r^2(n+\alpha)}{(2r-1)(r-1)}g'(\frac{\alpha}{n+\alpha})(\frac{\alpha}{n+\alpha})^{2-\frac{1}{r}}$$

$$+\frac{r^2}{(2r-1)(r-1)}\int_{-\alpha}^{0} g''\left(\frac{x+\alpha}{n+\alpha}\right)\left(\frac{x+\alpha}{n+\alpha}\right)^{2-\frac{1}{r}}dx$$

$$> \frac{r(n+\alpha)}{r-1}g\left(\frac{\alpha}{n+\alpha}\right)\left(\frac{\alpha}{n+\alpha}\right)^{1-\frac{1}{r}} - \frac{r^2(n+\alpha)}{(2r-1)(r-1)}g'\left(\frac{\alpha}{n+\alpha}\right)\left(\frac{\alpha}{n+\alpha}\right)^{2-\frac{1}{r}};$$

$$f'(0) = \frac{1}{(n+\alpha)}g'\left(\frac{\alpha}{n+\alpha}\right)\left(\frac{\alpha}{n+\alpha}\right)^{-\frac{1}{r}} - \frac{1}{r(n+\alpha)}g\left(\frac{\alpha}{n+\alpha}\right)\left(\frac{\alpha}{n+\alpha}\right)^{-\frac{1}{r}-1},$$

and $f(0) = g(\frac{\alpha}{n+\alpha})(\frac{\alpha}{n+\alpha})^{-\frac{1}{r}}$. Hence one obtains from (7) and the above results that

$$\begin{aligned} R_\alpha(r,n) &> g\left(\frac{\alpha}{n+\alpha}\right)\left(\frac{\alpha}{n+\alpha}\right)^{-\frac{1}{r}}\left(\frac{r\alpha}{r-1} - \frac{1}{2} - \frac{1}{12r\alpha}\right) \\ &\quad -g'\left(\frac{\alpha}{n+\alpha}\right)\left(\frac{\alpha}{n+\alpha}\right)^{1-\frac{1}{r}}\left[\frac{r^2\alpha}{(2r-1)(r-1)} - \frac{1}{12\alpha}\right]. \end{aligned} \tag{8}$$

LEMMA 2.3. For $r > 1, \alpha \geq \frac{1}{2}, n \in N_0$, define the weight coefficient $\omega_\alpha(r,n)$ as

$$\omega_\alpha(r,n) := \sum_{m=0}^{\infty} \frac{\ln(\frac{m+\alpha}{n+\alpha})}{m-n}\left(\frac{n+\alpha}{m+\alpha}\right)^{\frac{1}{r}}. \tag{9}$$

Then one has

$$\omega_\alpha(r,n) < \left[\frac{\pi}{\sin(\pi/r)}\right]^2 \ (n \in N_0). \tag{10}$$

Proof. For $r > 1$, $\alpha \geq \frac{1}{2}$, one has

$$\frac{r\alpha}{r-1} - \frac{1}{2} - \frac{1}{12r\alpha} = \frac{6r^2\alpha(2\alpha-1) + (6\alpha-1)r + 1}{12r(r-1)\alpha} > 0; \tag{11}$$

$$\frac{r^2\alpha}{(2r-1)(r-1)} - \frac{1}{12\alpha} = \frac{2r^2(6\alpha^2-1) + 3r - 1}{12(2r-1)(r-1)\alpha} > 0.$$

Since $g(u) > 0$ and $g'(u) < 0$, in view of (8), one has $R_\alpha(r,n) > 0$. Setting $u = (x+\alpha)/(n+\alpha)$, one finds from (5) that

$$\frac{1}{n+\alpha}\int_{-\alpha}^{\infty} f(x)dx = \int_0^\infty \frac{\ln u}{u-1} u^{-\frac{1}{r}} du = \left[\frac{\pi}{\sin(\pi/r)}\right]^2 .$$

In view of (9) and (7), one has (10). The lemma is proved.

Note. If $\alpha < \frac{1}{2}$, one can't conform that $R_\alpha(r,n) > 0$ by (11) for the more large enough number $r > 1$.

LEMMA 2.4. If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \alpha \geq \frac{1}{2}, 0 < \varepsilon < 1$, one has

$$\begin{aligned} I \quad : \quad &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\ln(\frac{m+\alpha}{n+\alpha})}{m-n} \left(\frac{1}{m+\alpha}\right)^{\frac{1}{q}+\frac{\varepsilon}{p}} \left(\frac{1}{n+\alpha}\right)^{\frac{1}{p}+\frac{\varepsilon}{q}} \\ &\geq \frac{1}{\varepsilon\alpha^\varepsilon}\{\left[\frac{\pi}{\sin(\pi/p)}\right]^2 + o(1)\}(\varepsilon \to 0^+) \qquad (12) \end{aligned}$$

Proof. For fixed y, setting $u = (x+\alpha)/(y+\alpha)$, since $g(u)$ is decreasing, one obtains

$$\begin{aligned} I \quad &\geq \int_0^\infty \left(\frac{1}{y+\alpha}\right)^{\frac{1}{p}+\frac{\varepsilon}{q}} [\int_0^\infty \frac{\ln(\frac{x+\alpha}{y+\alpha})}{x-y} \left(\frac{1}{x+\alpha}\right)^{\frac{1}{q}+\frac{\varepsilon}{p}} dx]dy \\ &= \int_0^\infty \left(\frac{1}{y+\alpha}\right)^{1+\varepsilon} [\int_{\frac{\alpha}{y+\alpha}}^\infty \frac{\ln u}{u-1} u^{-(\frac{1}{q}+\frac{\varepsilon}{p})} du]dy \\ &= \int_0^\infty \left(\frac{1}{y+\alpha}\right)^{1+\varepsilon} [\int_0^\infty \frac{\ln u}{u-1} u^{-(\frac{1}{q}+\frac{\varepsilon}{p})} du - \int_0^{\frac{\alpha}{y+\alpha}} \frac{\ln u}{u-1} u^{-(\frac{1}{q}+\frac{\varepsilon}{p})} du]dy \\ &= \frac{1}{\varepsilon\alpha^\varepsilon}\int_0^\infty \frac{\ln u}{u-1} u^{-(\frac{1}{q}+\frac{\varepsilon}{p})} du \\ &\quad - \int_0^\infty \frac{1}{(y+\alpha)^{1+\varepsilon}} [\int_0^{\frac{\alpha}{y+\alpha}} \frac{\ln u}{u-1} u^{-(\frac{1}{q}+\frac{\varepsilon}{p})} du]dy \\ &= \frac{1}{\varepsilon\alpha^\varepsilon}\int_0^\infty \frac{\ln u}{u-1} u^{-(\frac{1}{q}+\frac{\varepsilon}{p})} du \\ &\quad - \sum_{n=0}^{\infty} \int_0^\infty \frac{1}{(y+\alpha)^{1+\varepsilon}} [\int_0^{\frac{\alpha}{y+\alpha}} (-\ln u) u^{n-(\frac{1}{q}+\frac{\varepsilon}{p})} du]dy \\ &= \frac{1}{\varepsilon\alpha^\varepsilon}\left\{\left[\frac{\pi}{\sin(\pi/p)}\right]^2 + o(1)\right\} + \alpha^{n+\frac{1-\varepsilon}{p}} \sum_{n=0}^{\infty} \frac{1}{(n+\frac{1-\varepsilon}{p})(n+\frac{1}{p}+\frac{\varepsilon}{q})} \\ &\quad \times \int_0^\infty \left[-\ln(\frac{\alpha}{y+\alpha}) + \frac{1}{n+\frac{1-\varepsilon}{p}}\right] d(y+\alpha)^{-n-\frac{1-\varepsilon}{p}-\varepsilon} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\varepsilon\alpha^{\varepsilon}}\{[\frac{\pi}{\sin(\pi/p)}]^2 + o(1) \\
&\quad -\varepsilon\sum_{n=0}^{\infty}\frac{1}{(n+\frac{1-\varepsilon}{p})(n+\frac{1}{p}+\frac{\varepsilon}{q})}[\frac{1}{n+\frac{1-\varepsilon}{p}}+\frac{1}{n+\frac{1}{p}+\frac{\varepsilon}{q}}]\ \}.
\end{aligned}$$

Hence one has (12). The lemma is proved.

3 Main results and applications

THEOREM 3.1. If $p > 1, \frac{1}{p}+\frac{1}{q} = 1, \alpha \geq \frac{1}{2}, a_n, b_n \geq 0$, such that $0 < \sum_{n=0}^{\infty}(n+\alpha)^{p-2}a_n^p < \infty$ and $0 < \sum_{n=0}^{\infty}(n+\alpha)^{q-2}b_n^q < \infty$, then

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\ln(\frac{m+\alpha}{n+\alpha})a_m b_n}{m-n} < \left[\frac{\pi}{\sin(\frac{\pi}{p})}\right]^2 \left\{\sum_{n=0}^{\infty}(n+\alpha)^{p-2}a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=0}^{\infty}(n+\alpha)^{q-2}b_n^q\right\}^{\frac{1}{q}}, \tag{13}$$

where the constant factor $[\pi/\sin(\pi/p)]^2$ is the best possible. The equivalent form is

$$\sum_{n=0}^{\infty}(n+\alpha)^{p-2}\left[\sum_{m=0}^{\infty}\frac{\ln(\frac{m+\alpha}{n+\alpha})a_m}{m-n}\right]^p < \left[\frac{\pi}{\sin(\frac{\pi}{p})}\right]^{2p}\sum_{n=0}^{\infty}(n+\alpha)^{p-2}a_n^p, \tag{14}$$

where the constant factor $[\pi/\sin(\pi/p)]^{2p}$ is also the best possible. If particular, for $\alpha = 1$ in (13) and (14), replacing $a_{n-1}$by a_n, and $b_{n-1}$by b_n, one has the following equivalent inequalities:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\ln(m/n)a_m b_n}{m-n} < \left[\frac{\pi}{\sin(\frac{\pi}{p})}\right]^2 \left\{\sum_{n=1}^{\infty}n^{p-2}a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty}n^{q-2}b_n^q\right\}^{\frac{1}{q}}; \tag{15}$$

$$\sum_{n=1}^{\infty}n^{p-2}\left[\sum_{m=1}^{\infty}\frac{\ln(m/n)a_m}{m-n}\right]^p < \left[\frac{\pi}{\sin(\frac{\pi}{p})}\right]^{2p}\sum_{n=1}^{\infty}n^{p-2}a_n^p. \tag{16}$$

Proof. By *Hölder's* inequality with weight (see [14]) and using (9), one has

$$\begin{aligned}
H_\alpha(a_m, b_n) \quad : &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\ln(\frac{m+\alpha}{n+\alpha})a_m b_n}{m-n} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\ln(\frac{m+\alpha}{n+\alpha})}{m-n}\left[\frac{(m+\alpha)^{1/q^2}}{(n+\alpha)^{1/p^2}}a_m\right]\left[\frac{(n+\alpha)^{1/p^2}}{(m+\alpha)^{1/q^2}}b_n\right] \\
&\leq \left\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\ln(\frac{m+\alpha}{n+\alpha})}{m-n}\frac{(m+\alpha)^{\frac{p}{q^2}}}{(n+\alpha)^{\frac{1}{p}}}a_m^p\right\}^{\frac{1}{p}} \left\{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\ln(\frac{m+\alpha}{n+\alpha})}{m-n}\frac{(n+\alpha)^{\frac{q}{p^2}}}{(m+\alpha)^{\frac{1}{q}}}b_n^q\right\}^{\frac{1}{q}} \\
&= \left\{\sum_{m=0}^{\infty}\omega_\alpha(p,m)(m+\alpha)^{p-2}a_m^p\right\}^{\frac{1}{p}} \left\{\sum_{n=0}^{\infty}\omega_\alpha(q,n)(n+\alpha)^{q-2}b_n^q\right\}^{\frac{1}{q}}.
\end{aligned}$$

Hence by (10), since $\frac{\pi}{\sin(\pi/q)}=\frac{\pi}{\sin(\pi/p)}$, one has (13).

For $0<\varepsilon<1$, setting $\widetilde{a}_m,\widetilde{b}_n$ as

$$\widetilde{a}_m=\left(\frac{1}{m+\alpha}\right)^{\frac{1}{q}+\frac{\varepsilon}{p}},\widetilde{b}_n=\left(\frac{1}{n+\alpha}\right)^{\frac{1}{p}+\frac{\varepsilon}{q}},m,n\in N_0,$$

one has

$$\begin{aligned}&\left\{\sum_{n=0}^{\infty}(n+\alpha)^{p-2}\widetilde{a}_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=0}^{\infty}(n+\alpha)^{q-2}\widetilde{b}_n^q\right\}^{\frac{1}{q}}\\ =\;&1+\sum_{n=1}^{\infty}\left(\frac{1}{n+\alpha}\right)^{1+\varepsilon}<1+\int_0^{\infty}\left(\frac{1}{x+\alpha}\right)^{1+\varepsilon}dx=1+\frac{1}{\varepsilon\alpha^{\varepsilon}}.\end{aligned}\tag{17}$$

If the constant factor $[\pi/\sin(\pi/p)]^2$ in (13) is not the best possible, then there exists a positive number $k<[\pi/\sin(\pi/p)]^2$, such that (13) is still valid if one replaces $[\pi/\sin(\pi/p)]^2$ by k. In particular, by (12) and (17), one has

$$\begin{aligned}&\frac{1}{\alpha^{\varepsilon}}\{[\frac{\pi}{\sin(\pi/p)}]^2+o(1)\}\le\varepsilon I=\varepsilon H_{\alpha}(\widetilde{a}_m,\widetilde{b}_n)\\ <\;&\varepsilon k\left\{\sum_{n=0}^{\infty}(n+\alpha)^{p-2}\widetilde{a}_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=0}^{\infty}(n+\alpha)^{q-2}\widetilde{b}_n^q\right\}^{\frac{1}{q}}<k(\varepsilon+\frac{1}{\alpha^{\varepsilon}}),\end{aligned}$$

and then $[\pi/\sin(\pi/p)]^2\le k(\varepsilon\to 0^+)$. This contradicts the fact that $k<[\pi/\sin(\pi/p)]^2$. Hence the constant factor $[\pi/\sin(\pi/p)]^2$ in (13) is the best possible.

Setting b_n as

$$b_n:=(n+\alpha)^{p-2}\left[\sum_{m=0}^{\infty}\frac{\ln(\frac{m+\alpha}{n+\alpha})a_m}{m-n}\right]^{p-1},n\in N_0,$$

and use (13) to obtain

$$\begin{aligned}&\left\{\sum_{n=0}^{\infty}(n+\alpha)^{q-2}b_n^q\right\}^p\\ =\;&\left\{\sum_{n=0}^{\infty}(n+\alpha)^{p-2}\left[\sum_{m=0}^{\infty}\frac{\ln(\frac{m+\alpha}{n+\alpha})a_m}{m-n}\right]^p\right\}^p=\left\{\sum_{n=0}^{\infty}H_{\alpha}(a_m,b_n)\right\}^p\\ \le\;&\left[\frac{\pi}{\sin(\frac{\pi}{p})}\right]^{2p}\left\{\sum_{n=0}^{\infty}(n+\alpha)^{p-2}a_n^p\right\}\left\{\sum_{n=0}^{\infty}(n+\alpha)^{q-2}b_n^q\right\}^{p-1};\end{aligned}\tag{18}$$

$$\begin{aligned}0\;<\;&\sum_{n=0}^{\infty}(n+\alpha)^{q-2}b_n^q=\sum_{n=0}^{\infty}(n+\alpha)^{p-2}\left[\sum_{m=0}^{\infty}\frac{\ln(\frac{m+\alpha}{n+\alpha})a_m}{m-n}\right]^p\\ \le\;&\left[\frac{\pi}{\sin(\frac{\pi}{p})}\right]^{2p}\sum_{n=0}^{\infty}(n+\alpha)^{p-2}a_n^p<\infty.\end{aligned}\tag{19}$$

It follows that (17) takes the form of strict inequality by using (13); so does (19). Hence (14) holds.

On the other hand, if (14) holds, by *Hölder's* inequality, one has

$$\begin{aligned} H_\alpha(a_m,b_n) &= \sum_{n=0}^{\infty}\left[(n+\alpha)^{\frac{2-q}{q}}\sum_{m=0}^{\infty}\frac{\ln(\frac{m+\alpha}{n+\alpha})a_m}{m-n}\right][(n+\alpha)^{\frac{q-2}{q}}b_n] \\ &\le \left\{\sum_{n=0}^{\infty}(n+\alpha)^{p-2}\left[\sum_{m=0}^{\infty}\frac{\ln(\frac{m+\alpha}{n+\alpha})a_m}{m-n}\right]^p\right\}^{\frac{1}{p}}\left\{\sum_{n=0}^{\infty}(n+\alpha)^{q-2}b_n^q\right\}^{\frac{1}{q}} \end{aligned} \tag{20}$$

In view of (14), one has (13). It follows that (13) and (14) are equivalent.

If the constant factor $[\pi/\sin(\pi/p)]^{2p}$ in (14) is not the best possible, then by using (20), one can get a contradiction that the constant factor $[\pi/\sin(\pi/p)]^2$ in (13) is not the best possible. The theorem is proved.

REMARK 3.2. (i) For $\alpha = \frac{1}{2}$ in (13) and (14), one has the following new equivalent inequalities:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\ln(\frac{2m+1}{2n+1})a_mb_n}{m-n} < \left[\frac{\pi}{\sin(\frac{\pi}{p})}\right]^2\left\{\sum_{n=0}^{\infty}(2n+1)^{p-2}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=0}^{\infty}(2n+1)^{q-2}b_n^q\right\}^{\frac{1}{q}}; \tag{21}$$

$$\sum_{n=0}^{\infty}(2n+1)^{p-2}\left[\sum_{m=0}^{\infty}\frac{\ln(\frac{2m+1}{2n+1})a_m}{m-n}\right]^p < \left[\frac{\pi}{\sin(\frac{\pi}{p})}\right]^{2p}\sum_{n=0}^{\infty}(2n+1)^{p-2}a_n^p. \tag{22}$$

(ii) For $p = q = 2, \alpha \ge \frac{1}{2}$ in (13) and (14), one has the following new equivalent inequalities:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\ln(\frac{m+\alpha}{n+\alpha})a_mb_n}{m-n} < \pi^2\left\{\sum_{n=0}^{\infty}a_n^2\sum_{n=0}^{\infty}b_n^2\right\}^{\frac{1}{2}}; \tag{23}$$

$$\sum_{n=0}^{\infty}\left[\sum_{m=0}^{\infty}\frac{\ln(\frac{m+\alpha}{n+\alpha})a_m}{m-n}\right]^2 < \pi^4\sum_{n=0}^{\infty}a_n^2. \tag{24}$$

For $\alpha = 1$, (23) reduces to (1) (for $p = q = 2$). It follows that (23) is a best extension of (1) for $p = q = 2$. Since for $\frac{1}{2} \le \alpha < 1$ and $a_n, b_n > 0$ in Theorem 3.1, one has

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\ln(\frac{m+1}{n+1})a_mb_n}{m-n} < \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\ln(\frac{m+\alpha}{n+\alpha})a_mb_n}{m-n},$$

it follows that inequality (23) is more accurate than (1) for any $\frac{1}{2} \le \alpha < 1$ and $p = q = 2$.

(iii) Inequalities (15) and (1) are similar but different, although both of them are with the same best constant factor $[\frac{\pi}{\sin(\pi/p)}]^2$.

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