

Equilibrium Problems for KKM Set-Valued Maps

Weili Kong

Department of Primary Education
Qujing Normal University
Qujing City, Yunnan Province, 655011, P.R. China

Abstract

In this paper, we discuss the equilibrium problems of KKM set-valued maps by using the Fan KKM lemma. We obtain several existence theorems of solutions for KKM set-valued equilibrium problems. Our results extend the results of equilibrium problems for single-valued maps.

Mathematics Subject Classifications: 49K27, 49J52

Keywords: equilibrium problem; KKM map; Fan KKM lemma; set-valued map; transfer closed-valued map

1. Introduction

Let X be a real topological vector space, K be a nonempty convex subset of X and $f : K \times K \rightarrow R$ be a given function. An important problem in the nonlinear analysis is the so-called equilibrium problem, i.e., find an element $\bar{x} \in K$ such that

$$f(\bar{x}, y) \geq 0, \quad \forall y \in K.$$

As is well known, equilibrium problems theory provides us a natural, novel and unified framework to study a wide class of problems arising in economics, finance, transportation, network and structural analysis, elasticity and optimization. The ideas and techniques of this theory are being used in a variety of diverse areas and proved to be productive and innovative. Equilibrium problems are also among the most interesting and intensively studied classes of problems. They include fundamental mathematical problems like optimization problems, variational inequalities problems, minimax inequality problems, complementarity problems, fixed point problems and many other problems as special cases(see [2,4,5,6,7]).

We know that generalized KKM maps were introduced by Park in [8], and followed by some others. Recently, the famous Knaster-Kuratowski-Mazurkiewicz lemma and Ky Fan's lemma have been used widely as very versatile tools in modern nonlinear analysis. Some versions of the Fan KKM lemma are obtained and applied to variational inclusions problems and equilibrium problems (see [2,5,8]).

Inspired and motivated by [1,4,5,7], we discuss KKM set-valued equilibrium problems in this paper. we will introduce some notations and definitions in the section 2, which will be used in the paper. In the third section, we discuss the equilibrium problems of KKM set-valued maps by using the Fan KKM lemma. We obtain several existence theorems of solutions for KKM set-valued equilibrium problems. Our results extend some results of equilibrium problems for single-valued maps in [5].

2. Preliminaries

In this paper, let X, Y be two topological vector spaces, we shall denote by 2^X the family of all subsets of X and by $\Gamma(X)$ the family of all nonempty finite subsets of X .

Definition 2.1^[5] Let K be a subset of X , a set-valued map $F:K \rightarrow 2^X$ is called a KKM map if, for each $A \in \Gamma(K)$, $coA \subseteq \bigcup_{x \in A} F(x)$.

Definition 2.2^[5] A set-valued map $F : Y \rightarrow 2^X$ is said to be transfer closed-valued if, for any $(y, x) \in Y \times X$ with $x \notin F(y)$, there exists $y' \in Y$ such that

$$x \notin cl_X F(y').$$

It is easy to know that this definition is equivalent to saying that

$$\bigcap_{y \in Y} F(y) = \bigcap_{y \in Y} cl_X F(y).$$

If $Y = X$, then we will call F transfer closed-valued on X . If $B \subseteq Y$ and $A \subseteq X$, then we call $F : B \rightarrow 2^A$ transfer closed-valued if the set-valued map $y \mapsto F(y) \cap A$ is transfer closed-valued. In the case where $X = Y$ and $A = B$, we call F transfer closed-valued on A . Obviously, every closed-valued map is transfer closed-valued map.

Inspired by [5], we give the following definitions 2.3 and 2.4.

Definition 2.3 A set-valued map $F : X \times Y \rightarrow 2^R$ is called transfer lower semicontinuous (transfer l.s.c.) in the second variable if, for each $(x, y) \in X \times Y$ with $F(x, y) \subseteq R_+$, there exist $x' \in X$ and a neighborhood $U(y)$ of y in Y such that

$$F(x', z) \subseteq R_+, \quad \forall z \in U(y).$$

We call F transfer l.s.c. in the second variable on A , if $Y = X$ and $A \subseteq X$, and $F|_{A \times A}$ is transfer l.s.c. in the second variable.

When $f : X \times Y \rightarrow R$ is a single-valued function, then f is called transfer lower semicontinuous (transfer l.s.c.) in the second variable if, for each $(x, y) \in X \times Y$ with $f(x, y) > 0$, there exist $x' \in X$ and a neighborhood $U(y)$ of y in Y such that

$$f(x', z) > 0, \quad \forall z \in U(y)$$

It is easy to see that a l.s.c. function in the second variable is transfer l.s.c. in the second variable (see [5]).

Definition 2.4 Let $F, G : X \times X \rightarrow 2^R$ be two set-valued maps. Then F is called pseudomonotone with respect to G at y if, for each $x \in X$, the following implication holds

$$F(x, y) \subseteq R_+ \quad \Rightarrow \quad G(y, x) \cap R_- \neq \emptyset.$$

If the above is true for every $y \in X$, then we say that F is pseudomonotone with respect to G .

Let $f, g : X \times X \rightarrow R$ be two single-valued functions, then f is called pseudomonotone with respect to g at y if, for each $x \in X$, the following implication holds

$$f(x, y) \geq 0 \Rightarrow g(y, x) \leq 0$$

(see [5]). If the above is true for every $y \in X$, then we say that f is pseudomonotone with respect to g .

Definition 2.5^[3] A set-valued map $F : X \rightarrow 2^Y$ is called upper semicontinuous at $x \in \text{Dom}(F)$ (u.s.c. at x) if and only if for any open set U with $F(x) \subset U$, there exists a neighborhood V of x such that $F(V) \subset U$.

F is said to be upper semicontinuous if and only if it is upper semicontinuous at every point $x \in \text{Dom}(F)$.

Let M be a subset of Y , $F : X \rightarrow 2^Y$ be a set-valued map. We denote $F^{-1}(M) = \{x \in X : F(x) \cap M \neq \emptyset\}$, and $F^{+1}(M) = \{x \in X : F(x) \subseteq M\}$. The subset $F^{-1}(M)$ is called the inverse image of M by F and $F^{+1}(M)$ is called the core of M by F .

Proposition 2.1^[3] A set-valued map $F : X \rightarrow 2^Y$ with nonempty values is upper semicontinuous (u.s.c.) if and only if the inverse image of any closed subset is closed, and F is lower semicontinuous (l.s.c.) if and only if the core of any closed subset is closed.

We shall use the following Fan KKM lemma in this paper.

Lemma 2.1^[5] Let K be a nonempty convex subset of a topological vector space X and $F : K \rightarrow 2^K$ be a set-valued map such that

- (H1) F is a KKM map;
- (H2) for some $A_0 \in \Gamma(K)$, $cl_K(\bigcap_{x \in A_0} F(x))$ is compact;
- (H3) for each $A \in \Gamma(K)$ with $A_0 \subseteq A$, F is transfer closed-valued on coA ;

(H4) for each $A \in \Gamma(K)$ with $A_0 \subseteq A$,

$$(cl_K \bigcap_{x \in coA} F(x)) \cap coA = (\bigcap_{x \in coA} F(x)) \cap coA.$$

Then, $\bigcap_{x \in K} F(x) \neq \emptyset$.

The following lemma is the other version of the Fan KKM lemma.

Lemma 2.2^[5] Let K_0 be a nonempty subset of K and $F : K_0 \rightarrow 2^K$ satisfying the following conditions

- (i) F is a KKM map;
- (ii) for some $A_0 \in \Gamma(K_0)$, $cl_K(\bigcap_{x \in A_0} F(x))$ is compact;
- (iii) for each $A \in \Gamma(K_0)$ with $A_0 \subseteq A$, $F : A \rightarrow 2^{coA}$ is transfer closed-valued;
- (iv) for each $A \in \Gamma(K_0)$ with $A_0 \subseteq A$, $cl_K(\bigcap_{x \in A} F(x)) = (\bigcap_{x \in A} F(x))$.

Then, $\bigcap_{x \in K} F(x) \neq \emptyset$.

In the Lemma 2.2, if $X_0 = X$, then the result is weaker than Lemma 2.1.

3. Main results

Theorem 3.1 Let K be a nonempty convex subset of a Hausdorff topological vector space X , suppose that $F, G : K \times K \rightarrow 2^R$ are two set-valued maps such that

- (i) G is pseudomonotone with respect to F ;
- (ii) for each finite set $\{y_1, y_2, \dots, y_n\} \subset K$,

$$co\{y_1, y_2, \dots, y_n\} \subset \{x \in K : \exists i = 1, 2, \dots, n, G(x, y_i) \subseteq R_+\};$$

(iii) there exist a nonempty compact subset $K_0 \subset K$ and $A_0 \in \Gamma(K)$ such that for every $x \in K \setminus K_0$, there exists $y \in A_0$ such that $F(y, x) \subseteq R_+^*$;

(iv) for each $A \in \Gamma(K)$ with $A_0 \subseteq A$, F is transfer l.s.c. in the second variable on coA ;

(v) for each $A \in \Gamma(K)$ with $A_0 \subseteq A$, $x, y \in coA$ and (x_α) is a net on K converging to x , then the implication holds: if $F(tx + (1-t)y, x_\alpha) \cap R_- \neq \emptyset$, for all $t \in [0, 1]$, then $F(y, x) \cap R_- \neq \emptyset$.

Then, there exists $\bar{x} \in K$, such that $F(y, \bar{x}) \cap R_- \neq \emptyset$, for all $y \in K$.

Proof. Assume that $S, T : K \rightarrow 2^K$ are defined by

$$S(y) = \{x \in K : G(x, y) \subseteq R_+\},$$

$$T(y) = \{x \in K : F(y, x) \cap R_- \neq \emptyset\}.$$

From (i) we have that

$$S(y) \subseteq T(y), \quad \forall y \in K.$$

And by (ii), S is a KKM map, so T is a KKM map too. We know from (iii) that

$$\bigcap_{y \in A_0} T(y) \subseteq K_0.$$

Indeed, supposing the contrary, there exists $x \in \bigcap_{y \in A_0} T(y)$ but $x \notin K_0$, then there exists $y_0 \in A_0$, such that

$$F(y_0, x) \subseteq R_+^*,$$

but

$$x \in \bigcap_{y \in A_0} T(y) \subseteq T(y_0),$$

so

$$F(y_0, x) \cap R_- \neq \emptyset,$$

which is a contradiction with $F(y_0, x) \subseteq R_+^*$, so

$$\bigcap_{y \in A_0} T(y) \subseteq K_0,$$

thus

$$cl_K(\bigcap_{y \in A_0} T(y)) \subseteq K_0.$$

Since K_0 is a nonempty compact subset of K , then, $cl_K(\bigcap_{y \in A_0} T(y))$ is compact. Therefore conditions (H1) and (H2) of Lemma 2.1 hold for the set-valued map T .

From (iv), for each $A \in \Gamma(K)$ with $A_0 \subseteq A$, F is transfer l.s.c. in the second variable on coA , then for each $(y, x) \in coA \times coA$ with $F(y, x) \subseteq R_+^*$, there exist $y' \in coA$ and a neighborhood $U(x)$ of x in coA , such that

$$F(y', z) \subseteq R_+^*, \quad \forall z \in U(x),$$

It means that for any $(y, x) \in coA \times coA$ with $x \notin T(y)$, there exists $y' \in coA$ and a neighborhood $U(x)$ of x in coA , such that $z \notin T(y')$ for all $z \in U(x)$, that is

$$U(x) \cap T(y') = \emptyset.$$

Then

$$x \notin cl_{coA} T(y'),$$

so condition (H3) of Lemma 2.1 holds for T .

Now, for each $A \in \Gamma(K)$ with $A_0 \subseteq A$, if

$$x \in (cl_K \bigcap_{y \in coA} T(y)) \cap coA$$

then $x \in coA$, and there exists a net (x_α) converging to x such that

$$F(y, x_\alpha) \cap R_- \neq \emptyset, \quad \forall y \in coA.$$

Therefore, it follows that

$$F(tx + (1-t)y, x_\alpha) \cap R_- \neq \emptyset, \quad \forall y \in coA, t \in [0, 1].$$

Hence from (v) we get

$$F(y, x) \cap R_- \neq \emptyset, \quad \forall y \in coA,$$

that is

$$x \in \left(\bigcap_{y \in coA} T(y) \right) \cap coA.$$

Thus, according to Lemma 2.1, we get

$$\bigcap_{y \in K} T(y) \neq \emptyset.$$

Then, there exists $\bar{x} \in K$ such that $F(y, \bar{x}) \cap R_- \neq \emptyset$, for all $y \in K$.

By this theorem, we obtain the following corollary.

Corollary 3.1[5, Theorem 2.1] Let K be a nonempty convex subset of a Hausdorff topological vector space X . Suppose that f and g are two single-valued functions on $K \times K \rightarrow R$ such that

- (i) f is pseudomonotone with respect to g ;
- (ii) for each $A \in \Gamma(K)$, $\inf_{x \in coA} \max_{y \in A} f(x, y) \geq 0$;
- (iii) there exist a nonempty compact subset K_0 of K and $A_0 \in \Gamma(K)$ such that for every $y \in K \setminus K_0$, there exists $x \in A_0$ such that $g(x, y) > 0$;
- (iv) for each $A \in \Gamma(K)$ with $A_0 \subseteq A$, g is transfer lower semicontinuous in the second variable on coA ;
- (v) for each $A \in \Gamma(K)$ with $A_0 \subseteq A$, $x, y \in coA$ and (x_α) is a net on K converging to x , then the implication holds: if $g(tx + (1-t)y, x_\alpha) \leq 0$, for all $t \in [0, 1]$, then $g(y, x) \leq 0$.

Then, there exists $\bar{x} \in K$, such that $g(y, \bar{x}) \leq 0$, for all $y \in K$.

Proof. Let $F = g$, $G = f$. We can see easily that all of the conditions of Theorem 3.1 are satisfied. The conclusion follows.

Theorem 3.2 Let K be a nonempty convex subset of a Hausdorff topological vector space X and K_0 be a nonempty subset of K . Suppose that $F, G : K \times K \rightarrow 2^R$ are two set-valued maps satisfying the following conditions

- (i) G is pseudomonotone with respect to F at every $y \in K_0$;
- (ii) for each finite set $\{y_1, y_2, \dots, y_n\} \subset K_0$,

$$co\{y_1, y_2, \dots, y_n\} \subset \{x \in K : \exists i = 1, 2, \dots, n, G(x, y_i) \subseteq R_+\};$$

(iii) there exist a compact subset U of K and $A_0 \in \Gamma(K_0)$ such that, for every $x \in K \setminus U$, there exists $y \in A_0$ such that $F(y, x) \subseteq R_+^*$;

(iv) for each $A \in \Gamma(K_0)$ with $A_0 \subseteq A$, $F : A \times coA \rightarrow 2^R$ is transfer l.s.c. in the second variable;

(v) for each $A \in \Gamma(K_0)$ with $A_0 \subseteq A$ and (x_α) is a net on K converging to x , then the implication holds: if $F(y, x_\alpha) \cap R_- \neq \emptyset$, for all $y \in A$, then $F(y, x) \cap R_- \neq \emptyset$, for all $y \in A$.

Then, there exists $\bar{x} \in K$ such that $F(y, \bar{x}) \cap R_- \neq \emptyset$, for all $y \in K_0$.

Proof. Assume that $S, T : K_0 \rightarrow 2^K$ are defined by

$$S(y) = \{x \in K : G(x, y) \subseteq R_+\},$$

$$T(y) = \{x \in K : F(y, x) \cap R_- \neq \emptyset\}.$$

From condition (i) we have that

$$S(y) \subseteq T(y), \quad \forall y \in K_0.$$

From the condition (ii), S is a KKM map, so T is a KKM map too. By the same proof as that of Theorem 3.1, we can show that the conditions (iii)-(v) imply that all of the conditions (ii)-(iv) of Lemma 2.2 are satisfied. Thus, there exists $\bar{x} \in K$ such that $F(y, \bar{x}) \cap R_- \neq \emptyset$ for all $y \in K_0$.

Remark 3.1 In Theorem 3.2, if $X_0 = X$, then the result will be weaker than Theorem 3.1 according to Lemma 2.2.

From this theorem, we obtain the following corollary.

Corollary 3.2[5, Theorem 2.3] Let K be a nonempty convex subset of a Hausdorff topological vector space X and K_0 be a nonempty subset of K . Suppose that f and g are two single-valued functions on $K \times K \rightarrow R$ which satisfy the following conditions

- (i) f is pseudomonotone with respect to g at every $y \in K_0$;
- (ii) for each $A \in \Gamma(K_0)$, $\inf_{x \in coA} \max_{y \in A} f(x, y) \geq 0$;
- (iii) there exist a compact subset U of K and $A_0 \in \Gamma(K_0)$, such that for every $y \in K \setminus U$, there exists $x \in A_0$ such that $g(x, y) > 0$;

(iv) for each $A \in \Gamma(K_0)$ with $A_0 \subseteq A$, $g : A \times coA \rightarrow R$ is transfer l.s.c. in the second variable;

(v) for each $A \in \Gamma(K_0)$ with $A_0 \subseteq A$ and (y_α) is a net on K converging to y , then the implication holds: if $g(x, y_\alpha) \leq 0$, for all $x \in A$, then $g(x, y) \leq 0$, for all $x \in A$.

Then, there exists $\bar{x} \in K$, such that $g(y, \bar{x}) \leq 0$ for all $y \in K_0$.

Proof. Let $F = g$, $G = f$. We can see easily that all of the conditions of Theorem 3.2 are satisfied. The conclusion follows.

Theorem 3.3 Let X, Y be Hausdorff topological vector spaces, K a nonempty convex subset of X , M a nonempty subset of Y and $F : K \times K \rightarrow 2^Y$ a set-valued map with nonempty values such that

(i) for each finite set $\{y_1, y_2, \dots, y_n\} \subset K$,

$$co\{y_1, y_2, \dots, y_n\} \subset \{x \in K : \exists i = 1, 2, \dots, n, F(y_i, x) \cap (Y \setminus M) \neq \emptyset\};$$

(ii) there exist a nonempty compact subset K_0 of K and $A_0 \in \Gamma(K)$ such that for every $x \in K \setminus K_0$, there exists $y \in A_0$ such that $F(y, x) \subseteq M$;

(iii) for each $A \in \Gamma(K)$ with $A_0 \subseteq A$, and any $(y, x) \in coA \times coA$ with $F(y, x) \subseteq M$, there exists $y_0 \in coA$ such that $x \notin cl_{coA}\{x \in coA : F(y_0, x) \cap (Y \setminus M) \neq \emptyset\}$;

(iv) for each $A \in \Gamma(K)$ with $A_0 \subseteq A$, any $x, y \in coA$ and (x_α) is a net on K converging to x , then the implication holds:

if $F(tx + (1-t)y, x_\alpha) \cap (Y \setminus M) \neq \emptyset$, for all $t \in [0, 1]$, then

$$F(y, x) \cap (Y \setminus M) \neq \emptyset.$$

Then, there exists $\bar{x} \in K$ such that $F(y, \bar{x}) \cap (Y \setminus M) \neq \emptyset$, for all $y \in K$.

Proof. Assume that $G : K \rightarrow 2^K$ is defined by

$$G(y) = \{x \in K : F(y, x) \cap (Y \setminus M) \neq \emptyset\},$$

We shall show that all of the conditions of Lemma 2.1 are satisfied for G . From assumption (i), G is a KKM map. We can prove that $\bigcap_{y \in A_0} G(y) \subseteq K_0$ from (ii). Indeed, if $x \in \bigcap_{y \in A_0} G(y)$ but $x \notin K_0$, then there would exist $y_0 \in A_0$ such that

$$F(y_0, x) \subseteq M.$$

But

$$x \in \bigcap_{y \in A_0} G(y) \subseteq G(y_0),$$

that is

$$F(y_0, x) \cap (Y \setminus M) \neq \emptyset,$$

which is a contradiction, so we have that

$$\bigcap_{y \in A_0} G(y) \subseteq K_0.$$

Since K_0 is compact, so

$$cl_K\left(\bigcap_{y \in A_0} G(y)\right) \subseteq K_0,$$

and $cl_K(\bigcap_{y \in A_0} G(y))$ is compact. By assumption (iii), G is transfer closed-valued on coA . Finally we show that for each $A \in \Gamma(K)$ with $A_0 \subseteq A$, we have that

$$(cl_K \bigcap_{y \in coA} G(y)) \cap coA = (\bigcap_{y \in coA} G(y)) \cap coA.$$

For each $A \in \Gamma(K)$ with $A_0 \subseteq A$, if

$$x \in cl_K(\bigcap_{y \in coA} G(y)) \cap coA,$$

then $x \in coA$ and there exists a net (x_α) converging to x such that

$$F(y, x_\alpha) \cap (Y \setminus M) \neq \emptyset, \forall y \in coA.$$

Therefore, it follows that

$$F(tx + (1-t)y, x_\alpha) \cap (Y \setminus M) \neq \emptyset, \forall y \in coA, t \in [0, 1].$$

Hence, from assumption (iv), we know that

$$F(y, x) \cap (Y \setminus M) \neq \emptyset, \forall y \in coA;$$

that is

$$x \in (\bigcap_{y \in coA} G(y)) \cap coA.$$

Thus, according to Lemma 2.1, we have that $\bigcap_{y \in K} G(y) \neq \emptyset$, so there exists $\bar{x} \in K$ such that $F(y, \bar{x}) \cap (Y \setminus M) \neq \emptyset$ for all $y \in K$.

Theorem 3.4 Let X, Y, K and F as Theorem 3.3, M an open subset of Y . Suppose that

- (a) conditions (i) and (ii) of Theorem 3.3 hold for F ;
- (b) $\forall y \in K, x \mapsto F(y, x)$ is u.s.c. on K .

Then, there exists $\bar{x} \in K$ such that $F(y, \bar{x}) \cap (Y \setminus M) \neq \emptyset$, for all $y \in K$.

Proof. Assume that $G : K \rightarrow 2^K$ is defined by

$$G(y) = \{x \in K : F(y, x) \cap (Y \setminus M) \neq \emptyset\}.$$

Since M is open, from (b) and Proposition 2.1, we have that the set $G(y)$ are closed for all $y \in K$, so the conditions (H3) and (H4) of Lemma 2.1 are satisfied. The conclusion follows.

Remark 3.1 If we let $Y = R, M = R_+$ and replace (iii) of Theorem 3.3 by (iii)' for each $A \in \Gamma(K)$ with $A_0 \subseteq A$, F is transfer l.s.c. in the second variable on coA .

Then, we will obtain that there exists $\bar{x} \in K$ such that $F(y, \bar{x}) \cap R_* \neq \emptyset$, for all $y \in K$, certainly, $F(y, \bar{x}) \cap R_- \neq \emptyset$.

Proof. Assume that $G : K \rightarrow 2^K$ is defined by

$$G(y) = \{x \in K : F(y, x) \cap R_-^* \neq \emptyset\},$$

we will know that the condition (H3) of Lemma 2.1 holds by the same proof as that of Theorem 3.1, and the others conditions of Lemma 2.1 are certainly true. The conclusion follows.

Remark 3.2 If we let $Y = R$, $M = R_+^*$ in Theorem 3.3 and 3.4, then we get that there exists $\bar{x} \in K$ such that $F(y, \bar{x}) \cap R_- \neq \emptyset$.

Remark 3.3 Some researchers obtained different set-valued versions of Ky Fan's inequality using the classical Ky Fan's lemma (see [1]). In this paper, we obtain set-valued versions of Ky Fan's inequality by other versions of the Fan KKM lemma (for more details, see [1,5]).

References

- [1] A. Kristaly and C. Varga, Set-valued versions of Ky Fan's inequality with application to variational inclusion theory, *J. Math. Anal. Appl.*, 282 (2003), 8-20.
- [2] G. Tian, Generalized KKM theorems, minimax inequalities, and their applications, *J. Optim. Theory Appl.*, 83(1994), 375-389.
- [3] J.P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhauser, Boston, 1990.
- [4] M. Bianchi and S. Schaible, Generalized monotone bifunctions and equilibrium problems, *J. Optim. Theory Appl.*, 90(1996), 31-43.
- [5] M. Fakhar and J. Zafarani, Generalized equilibrium problems for quasi-monotone and pseudomonotone bifunctions, *J. Optim. Theory Appl.*, 123(2004), 349-364.
- [6] M.S.R. Chowdhury and K.K. Tan, Generalization of Ky Fan's minimax inequality with applications to generalized variational inequalities for pseudomonotone operators and fixed point theorem, *J. Math. Anal. Appl.*, 204(1996), 910-929.
- [7] O. Chadli, Z. Chbani and H. Riahi, Equilibrium problems with generalized monotone bifunctions and applications to variational inequalities, *J. Optim. Theory Appl.*, 105(2000), 299-323.
- [8] S. Park, Generalizations of Ky Fan's matching theorems and their applications, *J. Math. Anal. Appl.*, 141(1989), 164-176.

Received: July 6, 2006