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Viscosity Approximation to Common Fixed Points of Nonexpansive Semigroups in Hilbert Spaces[1]

Huimin He and Rudong Chen

Department of Mathematics
Tianjin Polytechnic University
Tianjin, China 300160
hehuimin20012000@yahoo.com.cn, chenrd@tjpu.edu.cn

Abstract. Let C be a closed convex subset of a Hilbert space H, let $\{T(t) : t \geq 0\}$be a strongly continuous semigroup of non-expansive mapping on C such that $\bigcap_{t\geq 0} F(T(t)) \neq \emptyset$, and $f : C \to C$ be a fixed contractive mapping. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers satisfying $0 < \alpha_n < 1,\ t_n > 0$ and $\lim_n t_n = \lim_n \frac{\alpha_n}{t_n} = 0$. Define a sequence $\{x_n\}$ in C by

$$x_n = \alpha_n f(x_n) + (1-\alpha_n)T(t_n)x_n, for\ n \in N.$$

Then $\{x_n\}$ converges strongly to the element of $\bigcap_{t\geq 0} F(T(t))$. Our results extent and improve corresponding ones of Tomonari Suzuki [Proc. Amer. Math. Soc., 131 (2002), 2133-2136] and Rudong Chen and Yunyan Song, Computers and Mathematics with Applications
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1. Introduction and preliminaries

In this paper, we denote by $\mathbb{N}$ and $\mathbb{R}_+$ the sets of positive integers and nonnegative real numbers, respectively. let C be a closed convex subset of a Hilbert space H, and Let $T : C \to C$ be a nonexpansive mapping (i.e., $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$). We use $F(T)$ to denote the set of fixed points of T; i.e. $F(T) = \{x \in C : x = Tx\}$. We know that $F(T)$ is nonempty if C is bounded, for more details see [1]. Recall that a self-mapping $f : C \to C$ is a fixed contraction on C if there exists a constant $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|,\ x, y \in\ C.$$

For a contraction on C and $\alpha \in (0, 1)$, there exists a unique point x_α of C satisfying $x_\alpha = (1 - \alpha)T(t)x_\alpha + \alpha f(x_\alpha)$, because the mapping $x \mapsto (1 -$

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$\alpha)T(t)x+\alpha f(x)$ is contractive. Let $\{T(t):t\in\mathbb{R}_+\}$ be a strongly continuous semigroup of non-expansive mappings on a closed convex subset C of a Hilbert space H, i.e.,

(1)for each $t\in\mathbb{R}_+$, $T(t)$ is a nonexpansive mapping on C;

(2)$T(0)x=x$ for all $x\in C$;

(3)$T(s+t)=T(s)\circ T(t)$ for all $s,t\in\mathbb{R}_+$;

(4)for each $x\in X$, the mapping $T(\cdot)x$ from $\mathbb{R}_+$ into C is continuous.

We put $F(T)=\bigcap_{t\geq 0}F(T(t))$. We know that $F(T)$ is nonempty if C is bounded, see reference [2].

In [6],Shioji-Takahashi introduce in a Hilbert space the implicit iteration

$$x_n=\alpha_n u+(1-\alpha_n)\sigma_{t_n}(x_n),\ n\geq 1, \tag{1.1}$$

Where $\{\alpha_n\}$ is a sequence in (0,1), $\{t_n\}$ a sequence of positive real numbers divergent to ∞, and for each $t\geq 0$ and $x\in C$, $\sigma_t(x)$ is the average given by

$$\sigma_t(x)=\frac{1}{t}\int_0^t T(s)xds.$$

under certain restrictions to the sequence $\{\alpha_n\}$, Shioji-Takahashi [6] prove strong convergence of $\{x_n\}$ to a member of $F(T)$.(see also [7]).

RuDong Chen [8] extended the results of Shioji-Takahashi [6],he studied the strong convergence of the following sequence (1.2) for a strongly continuous semigroup of nonexpansive mappings $T(t):t\geq 0$ with $F(T)\neq\phi$ in a Banach space.

$$x_n=\alpha_n f(x_n)+(1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)xds. \tag{1.2}$$

Note however that their iterate x_n at step n is constructed through the average of the semigroup over the interval $(0,t)$. In 2002, Tomonari Suzuki [3] is the first to introduce again in a Hilbert space the following implicit iteration process:

$$x_n=\alpha_n u+(1-\alpha_n)T(t_n)(x_n),\ n\geq 1, \tag{1.3}$$

for the strongly continuous semigroup of nonexpansive mappings case.

In this paper, motivated by the above result, we study viscosity approximation process for strongly continuous semigroup of nonexpansive mappings and prove another strong convergence theorem for a strongly continuous semigroup of nonexpansive mappings in Hilbert space, which is defined by

$$x_n=\alpha_n f(x_n)+(1-\alpha_n)T(t_n)x_n,$$

for $n\in\mathbb{N}$.

Lemma 1.1. *[4] Let C be a nonempty closed convex subset of a Hilbert space H. Given $x \in H$ and $y \in C$. Then $y = P_C x$ if and only if there satisfies $\langle x - y, y - z\rangle \geq 0\ \forall z \in C$.*

2. Main Results

It is well know that all Hilbert spaces satisfy Opial's condition.

Proposition 2.1. *[Opial [5]] Let H be a Hilbert space. If $\{x_n\}$ is a sequence in H and converges weakly to $z_0 \in H$,then $\liminf_n \|x_n - z_0\| < \liminf_n \|x_n - z\|$ for all $z \in H$ with $z \neq z_0$.*

Now we prove our main result.

Theorem 2.2. *Let C be a closed convex subset of a Hilbert space H. Let$\{T(t) : t \in \mathbb{R}_+\}$ be a strongly continuous semigroup of non-expansive mappings on C such that $F(T) \neq \emptyset$. Then $F(T)$ is closed convex subset of C.*

Proof. Since $T(t) : C \to C\ t > 0$ is expansive, we claim that $F(T)$ is closed. In fact, if $p_n \subset F(T) = \bigcap_{t\geq 0} F(T(t)),\ n \geq 1$, such that $\lim_{n\to\infty} p_n = p$, then we have

$$T(t)p = \lim_{n\to\infty} T(t)p_n = \lim_{n\to\infty} p_n = p\ \forall t \in \mathbb{R}_+$$

Thus $p \in F(T)$.

Secondly, we show that $F(T)$ is convex, we shall use the following identity in Hilbert space.

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2, \tag{2.1}$$

which holds $\forall x, y \in H$ and $\forall t \in [0, 1]$ indeed,

$$\begin{aligned} \|tx + (1-t)y\|^2 &= t^2\|x\|^2 + (1-t)^2\|y\|^2 + 2t(1-t)\langle x, y\rangle \\ &= t\|x\|^2 + (1-t)\|y\|^2 + 2t(1-t)\langle x, y\rangle \\ &\quad -t(1-t)\|x\|^2 - t(1-t)\|y\|^2 \\ &= t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)(\|x\|^2 + \|y\|^2 - 2\langle x, y\rangle) \\ &= t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2. \end{aligned}$$

Let $p_1, p_2 \in F(T)$ and $\forall t \in [0, 1]$, $p = tp_1 + (1-t)p_2$, then

$$p - p_1 = (1-t)(p_2 - p_1),\ p - p_2 = (1-t)(p_1 - p_2). \tag{2.2}$$

Form (2.1) and (2.2), we have

$$\begin{aligned} \|p - T(t)p\|^2 &= \|t(p_1 - T(t)p) + (1-t)(p_2 - T(t)p)\|^2 \\ &= t\|p_1 - T(t)p\|^2 + (1-t)\|p_2 - T(t)p\|^2 - t(1-t)\|p_1 - p_2\|^2 \\ &\leq t\|p_1 - p\|^2 + (1-t)\|p_2 - p\|^2 - t(1-t)\|p_1 - p_2\|^2 \\ &= t(1-t)^2\|p_1 - p_2\|^2 + t^2(1-t)\|p_1 - p_2\|^2 - t(1-t)\|p_1 - p_2\|^2 \\ &= t(1-t)(1-t+t-1)\|p_1 - p_2\|^2 \\ &= 0 \end{aligned}$$

Thus $p = T(t)p,\ \forall t > 0$, i.e. $p \in F(T)$.

The proof is complete. □

Theorem 2.3. *Let C be a closed convex subset of a Hilbert space H. Let $\{T(t) : t \in \mathbb{R}_+\}$ be a strongly continuous semigroup of non-expansive mappings on C such that $F(T) \neq \emptyset$, and $f : C \to C$ be a fixed contraction on C. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers satisfying $0 < \alpha_n < 1,\ t_n > 0$ and $\lim_n t_n = \lim_n \frac{\alpha_n}{t_n} = 0$. Define a sequence $\{x_n\}$ in C by*

$$x_n = \alpha_n f(x_n) + (1-\alpha_n)T(t_n)x_n, for\ n \in \mathbb{N}.$$

Then $\{x_n\}$ converges strongly to x, as $n \to \infty . x$ is the element of $F(T)$ and $x = P_{F(T)}f(x)$, i.e x satisfying the following variational inequality:

$$\langle x - f(x), x - z\rangle \leq 0\ \forall z \in F(T).$$

Proof. Let p be the element of $F(T)$, from

$$\begin{aligned} \|x_n - p\| &\leq \alpha_n \|f(x_n) - p\| + (1-\alpha_n)\|T(t_n)x_n - p\| \\ &\leq \alpha_n \|f(x_n) - f(p)\| + (1-\alpha_n)\|x_n - p\| + \alpha_n \|f(p) - p\| \\ &\leq \alpha\alpha_n \|x_n - p\| + (1-\alpha_n)\|x_n - p\| + \alpha_n \|f(p) - p\|. \end{aligned}$$

We have

$$\|T(t_n)x_n - p\| \leq \|x_n - p\| \leq \frac{1}{1-\alpha}\|f(p) - p\|, \forall n \in N.$$

Therefore $\{x_n\}$ is bounded, so are$\{f(x_n)\}$and$\{T(t_n)x_n\}$.

We claim that $\{x_n\}$ is sequentially compact. Indeed, there exists a weakly convergence subsequence $\{x_{n_j}\} \subseteq \{x_n\}$ by reflexivity of H and boundedness of the sequence $\{x_n\}$, now we suppose $x_{n_j} \rightharpoonup x \in C(j \to \infty)$. Now we show that $x \in F(T)$. Put $x_j = x_{n_j}$, $\beta_j = \alpha_{n_j}$ and $s_j = t_{n_j}$ for $j \in \mathbb{N}$, fix $t > 0$. From

$$\begin{aligned} \|x_j - T(t)x\| &\leq \textstyle\sum_{k=0}^{[t/s_j]-1} \|T((k+1)s_j)x_j - T(ks_j)x_j\| \\ &+ \|T([t/s_j]s_j)x_j - T([t/s_j]s_j)x\| + \|T([t/s_j]s_j)x - T(t)x\| \\ &\leq [t/s_j]\|T(s_j)x_j - x_j\| + \|x_j - x\| + \|T(t - [t/s_j]s_j)x - x\| \\ &= [t/s_j]\beta_j\|T(s_j)x_j - f(x_j)\| + \|x_j - x\| + \|T(t - [t/s_j]s_j)x - x\| \\ &\leq t\beta_j/s_j\|T(s_j)x_j - f(x_j)\| + \|x_j - x\| \\ &+ \max\{\|T(s)x - x\| : 0 \leq s \leq s_j\}. \end{aligned}$$

For all $j \in \mathbb{N}$, we have

$$\liminf_{j\to\infty} \|x_j - T(t)x\| \leq \liminf_{j\to\infty} \|x_j - x\|.$$

By the Proposition 2.1, this implies $T(t)x = x$. Therefore $x \in F(T)$.

We next prove $\{x_j\}$ converges strongly to x. From

$$\begin{aligned} \|x_j - x\|^2 &= \beta_j\langle f(x_j) - x, x_j - x\rangle + (1-\beta_j)\langle T(s_j)x_j - x, x_j - x\rangle \\ &= \beta_j\langle f(x_j) - x, x_j - x\rangle + (1-\beta_j)\langle T(s_j)x_j - T(s_j)x, x_j - x\rangle \\ &\quad +(1-\beta_j)\langle T(s_j)x - x, x_j - x\rangle \\ &\leq \beta_j\langle f(x_j) - x, x_j - x\rangle + (1-\beta_j)\|x_j - x\|^2. \end{aligned}$$

we have

$$\begin{aligned}\|x_j - x\|^2 &\le \langle f(x_j) - x, x_j - x\rangle \\ &= \langle f(x_j) - f(x), x_j - x\rangle + \langle f(x) - x, x_j - x\rangle \\ &\le \alpha \|x_j - x\|^2 + \langle f(x) - x, x_j - x\rangle,\end{aligned}$$

i.e.

$$\|x_j - x\|^2 \le \frac{1}{1-\alpha}\langle f(x) - x, x_j - x\rangle$$

For all $j \in \mathbb{N}$ and $x_j \rightharpoonup x$, so $\{x_j\}$ converges strongly to x. This shows that $\{x_n\}$ is sequentially compact.

To prove that $\{x_n\}$ converges strongly to x, we only need prove that any subsequence of $\{x_n\}$ converges strongly to x. Suppose not, then there exists another subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges strongly to $y \neq x$ as $k \to \infty$. Similarly, we also have $y \in F(T)$.

By $x_n = \alpha_n f(x_n) + (1 - \alpha_n)T(t_n)x_n, for\ n \in \mathbb{N}$, we have

$$f(x_n) = \frac{1}{\alpha_n}x_n + \frac{\alpha_n - 1}{\alpha_n}T(t_n)x_n$$

$$x_n - f(x_n) = \frac{\alpha_n - 1}{\alpha_n}(x_n - T(t_n)x_n)$$

Thus for all $z \in F(T)$, we get

$$\begin{aligned}\langle x_n - f(x_n), x_n - z\rangle &= \tfrac{\alpha_n - 1}{\alpha_n}\langle x_n - T(t_n)x_n, x_n - z\rangle \\ &= \tfrac{\alpha_n - 1}{\alpha_n}\langle (x_n - T(t_n)x_n) - (z - T(t_n)z), x_n - z\rangle \\ &\le 0\end{aligned}$$

Since

$$\begin{aligned}&\langle (x_n - T(t_n)x_n) - (z - T(t_n)z), x_n - z\rangle \\ &= \|x_n - z\|^2 - \|T(t_n)x_n - T(t_n)z\| \cdot \|x_n - z\| \\ &\ge 0\end{aligned}$$

Applying limit at the above inequality, we have

$$\lim_{j\to\infty}\langle x_{n_j} - f(x_{n_j}), x_{n_j} - z\rangle \le 0$$

i.e.

$$\langle x - f(x), x - z\rangle \le 0.$$

We also have

$$\langle y - f(y), y - z\rangle \le 0.$$

Let z replaced by x or y by $x, y \in F(T)$, then

$$\langle x - f(x), x - y\rangle \le 0$$

and

$$\langle y - f(y), y - x\rangle \le 0.$$

Adding up two inequations above, we have that

$$(1-\alpha)\|x - y\|^2 \le \langle (I - f)x - (I - f)y, x - y\rangle \le 0.$$

Thus $x = y$. Which is a contradiction, so $\{x_n\}$ strongly converges to x.

From theorem 2.1, we know $F(T)$ is closed convex subset of C. So from lemma 1.2, we have $x = P_{F(T)}f(x)$. The proof is completed. □

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