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Existence of Blow-up Solutions to a Class of $p(x)$-Laplacian Problems[1]

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Abstract

This paper investigates the problem

$$\begin{cases} -\triangle_{p(x)}u + f(x,u) = 0 \text{ in } \Omega, \\ u(x) \to +\infty \text{ as } d(x,\ \partial\Omega) \to 0, \end{cases}$$

where $-\triangle_{p(x)}u = -div(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian. The existence of blow-up solutions is discussed, and the singularity of blow-up solutions is given. In particular, we do not assume radial symmetric conditions.

Mathematics Subject Classification: 35D05, 35J70

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1 Introduction

The study of differential equations and variational problems with nonstandard $p(x)$-growth conditions is a new and interesting topic. It arises from nonlinear elasticity theory, electrorheological fluids, etc. (see [15,22]). Many results have been obtained on this kind of problems, for example [1-9,12,15-19]. On the regularity of weak solutions for differential equations with nonstandard $p(x)$-growth conditions, we refer to [1-3,9]. In this paper, we consider the problem

$$(P) \quad \begin{cases} -\triangle_{p(x)}u + f(x,u) = 0 \text{ in } \Omega, \\ u(x) \to +\infty \text{ as } d(x,\ \partial\Omega) \to 0, \end{cases}$$

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where $-\triangle_{p(x)}u = -div(|\nabla u|^{p(x)-2}\nabla u)$, $\Omega \subset \mathbb{R}^N$ is a C^2 domain. Our aim is to give the existence of blow-up solutions for problem (P), and the singularity of blow-up solutions is given.

Since $\Omega \subset \mathbb{R}^N$ is a C^2 domain, then there exists a positive constant ρ such that $d(x) = d(x, \partial\Omega) \in C^2(\overline{\Omega_\rho})$, and $|\nabla d(x)| = 1$ for any $x \in \overline{\Omega_\rho}$, where $\Omega_\rho = \{x \in \Omega \mid d(x) < \rho\}$. Throughout the paper, we assume that $p(x)$ and $f(x,u)$ satisfy

(H_1) $p(x) \in C^1(\overline{\Omega})$ and satisfies

$$1 < \inf_{x\in\Omega} p(x) \leq \sup_{x\in\Omega} p(x) < N,$$

and $p(x) = p(d(x))$ in Ω_ρ.

(H_2) $f(x,\cdot)$ is increasing and $f(x,0) = 0$ for any $x \in \Omega$.

(H_3) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function and satisfies

$$|f(x,t)| \leq C_1 + C_2 |t|^{\alpha(x)-1}, \forall (x,t) \in \Omega \times \mathbb{R},$$

where C_1, C_2 are positive constants, $\alpha \in C(\overline{\Omega})$ and $1 \leq \alpha(x) < p^*(x) := \frac{Np(x)}{N-p(x)}$.

The operator $-\triangle_{p(x)}u = -div(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian. Especially, if $p(x) \equiv p$ (a constant), (P) is the well known p-Laplacian problem. If $f(x,u)$ can be represented as $h(x)f(u)$, there are many papers on the blow-up solutions of p-Laplacian problems, for example [10,11,13,14]. In the investigation of existence of blow-up solutions for the following p-Laplacian problems (p is a constant)

$$-\triangle_p u + h(x)f(u) = 0 \text{ in } \Omega,$$

the following generalized Keller Osserman condition is crucial

$$\int_1^\infty \frac{1}{(F(t))^{\frac{1}{p}}} dt < +\infty, \text{ where } F(t) = \int_0^t f(s)ds.$$

In [21], the author had discussed the existence of positive solutions of (P), when the problem (P) is radial. In many cases, the radial symmetric conditions are effectively to deal with $p(x)$-Laplacian problems. There are many results about the radial $p(x)$-Laplacian problems (see[5,17-20]). Our aim is to give the existence of positive blow-up solutions for problem (P) without radial symmetric conditions. Through a new method to constructing sub-supersolution, this paper give the existence of positive weak solutions for problem (P). Because of the non-homogeneity of $p(x)$-Laplacian, $p(x)$-Laplacian problems are more complicated than those of p-Laplacian ones; and another difficulty of this paper is that $f(x,u)$ cannot be represented as $h(x)f(u)$. For the problem (P), since $p(x)$ is a function, the typical form is $-\triangle_{p(x)}u+|u|^{q(x)-2}u=0$, our results possess some difference from the generalized Keller Osserman condition. Our results partially generalized the results of [20,21].

2 Preliminary

In order to deal with $p(x)$-Laplacian problems, we need some theories on the spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and properties of $p(x)$-Laplacian which we will use later (see[4]). Let

$$L^{p(x)}(\Omega)=\left\{u \mid u \text{ is a measurable real-valued function, } \textstyle\int_\Omega |u(x)|^{p(x)}\,dx<\infty\right\}.$$

We can introduce the norm on $L^{p(x)}(\Omega)$ by

$$|u|_{p(x)}=\inf\left\{\lambda>0 \,\middle|\, \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx\le 1\right\}.$$

The space $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ becomes a Banach space. We call it generalized Lebesgue space. The space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega)=\left\{u\in L^{p(x)}(\Omega) \,\middle|\, |\nabla u|\in L^{p(x)}(\Omega)\right\},$$

and it can be equipped with the norm

$$\|u\|=|u|_{p(x)}+|\nabla u|_{p(x)}, \forall u\in W^{1,p(x)}(\Omega).$$

$W_0^{1,p(x)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable, reflexive and uniform convex Banach spaces (see [4, Theorem 1.10, 1.14, 2.1]). If $u\in W_{loc}^{1,p(x)}(\Omega)$, u is called a blow-up solution of (P) if it satisfies

$$\int_Q |\nabla u|^{p(x)-2}\nabla u\nabla q dx+\int_Q f(x,u)q dx=0, \forall q\in W_0^{1,p(x)}(Q),$$

for any domain $Q\Subset\Omega$, and $\max(k-u,0)\in W_0^{1,p(x)}(\Omega)$ for every positive integer k.

Let

$$W_{0,loc}^{1,p(x)}(\Omega)=\{u \mid \text{there is an open domain } Q\Subset\Omega \text{ s.t. } u\in W_0^{1,p(x)}(Q)\},$$

and define $A : W^{1,p(x)}_{loc}(\Omega) \to (W^{1,p(x)}_{0,loc}(\Omega))^*$, as

$$\langle Au, \varphi\rangle = \textstyle\int_\Omega (|\nabla u|^{p(x)-2} \nabla u \nabla\varphi + f(x, u)\varphi)dx,$$

$$\forall u \in W^{1,p(x)}_{loc}(\Omega),\ \forall\varphi \in W^{1,p(x)}_{0,loc}(\Omega);$$

then we have

Lemma 2.1 (see [7, Theorem 3.1]). Let $h \in W^{1,p(x)}(\Omega)$, $X = h + W^{1,p(x)}_0(\Omega)$. Then, $A : X \to (W^{1,p(x)}_0(\Omega))^*$ is strictly monotone.

Let $g \in (W^{1,p(x)}_{0,loc}(\Omega))^*$, if $\langle g, \varphi\rangle \geq 0$, $\forall\varphi \in W^{1,p(x)}_{0,loc}(\Omega)$, $\varphi \geq 0$ a.e. in Ω, then denote $g \geq 0$ in $(W^{1,p(x)}_{0,loc}(\Omega))^*$; correspondingly, if $-g \geq 0$ in $(W^{1,p(x)}_{0,loc}(\Omega))^*$, then denote $g \leq 0$ in $(W^{1,p(x)}_{0,loc}(\Omega))^*$.

Definition 2.2. Let $u \in W^{1,p(x)}_{loc}(\Omega)$. If $Au \geq 0$ ($Au \leq 0$) in $(W^{1,p(x)}_{0,loc}(\Omega))^*$, then u is called a weak super-solution (weak sub-solution) of equation (P).

Copying the proof of [16], we have

Lemma 2.3 (comparison principle). Let $u, v \in W^{1,p(x)}(\Omega)$ satisfy $Au - Av \geq 0$ in $(W^{1,p(x)}_0(\Omega))^*$. Let $\varphi(x) = \min\{u(x)-v(x), 0\}$. If $\varphi(x) \in W^{1,p(x)}_0(\Omega)$ (i.e. $u \geq v$ on $\partial\Omega$), then $u \geq v$ a.e. in Ω.

Lemma 2.4 (see [9]). Under the conditions (H_1) and (H_3), if $u \in W^{1,p(x)}(\Omega)$ is a weak solution of $-\triangle_{p(x)}u + f(x, u) = 0$ in Ω, $u = w_0$ on $\partial\Omega$, where $w_0 \in W^{1,p(x)}(\Omega)$, then $u \in C^{1,\alpha}_{loc}(\Omega)$, where $\alpha \in (0, 1)$ is a constant.

3 Main results and proofs

Theorem 3.1 If there exists a $\delta \in (0, \rho)$ such that

$$f(x, u) \geq au^{q-1}(\text{as } u \to +\infty) \text{ for } x \in \Omega_\delta \text{ uniformly},$$

where a and q are positive constants, and $q - p(x) \geq \frac{1}{n}$ (where n is an integer and $n > 3$) for $x \in \Omega_\delta$ uniformly, then there exists a function $\Phi_1(x)$ which satisfies $\Phi_1(x) \to +\infty$ (as $d(x) \to 0$), and such that, if u is a weak solution of problem (P) then $u(x) \leq \Phi_1(x)$.

Proof Since $p(x) = p(d(x))$, $\forall x \in \overline{\Omega_\rho}$, then $p(x) = p(d(x)) \equiv p(0)$, $\forall x \in \partial\Omega$. Define the function $g(r, \epsilon)$ on $[0, +\infty)$ as

$$g(r,\epsilon) = \begin{cases} Cr^{-s} + 2k, \ 0 \le r < R_0, \\ \\ CR_0^{-s} + 2k - \int_{R_0}^{r} [CsR_0^{-s-1}]^{\frac{p(R_0)-1}{p(t)-1}} [\sin\varepsilon(\delta - t)]^{\frac{1}{p(t)-1}} dt, \\ \qquad R_0 < r < \delta \ , \\ \\ CR_0^{-s} + 2k - \int_{R_0}^{\delta} [CsR_0^{-s-1}]^{\frac{p(R_0)-1}{p(t)-1}} [\sin\varepsilon(\delta - t)]^{\frac{1}{p(t)-1}} dt, \\ \qquad r \ge \delta, \end{cases}$$

where $s = \frac{p(0)}{q-p(0)}$, $R_0 \in (0, \frac{\delta}{2})$ and R_0 is small enough, $\varepsilon = \frac{\pi}{2(\delta - R_0\ )}$, $0 < \epsilon$ is a small enough constant and

$$C = C_\epsilon = (1+\epsilon)[\frac{1}{a}s^{p(0)-1}(s+1)(p(0)-1)]^{\frac{1}{q-p(0)}},$$

$$k = [C(\frac{R_0}{2})^{-s-1}]^{\frac{p^+-1}{p^--1}} + \int_\rho^{R_0} [Cs(\frac{R_0}{2})^{-s-1}]^{\frac{p(R_0)-1}{p(t)-1}} [\sin\varepsilon(\delta - t)]^{\frac{1}{p(t)-1}} dt,$$

where $p^- = \inf_{x \in \Omega_\delta} p(x)$, $p^+ = \sup_{x \in \Omega_\delta} p(x)$.

Obviously, for any positive constant ϵ, $g(r,\epsilon) \in C^1[0, +\infty)$. We consider $g(x,\epsilon) = g(d,\epsilon) = g(d(x),\epsilon)$, then $g(x,\epsilon) \in C^1(\overline{\Omega})$.

Since $|\nabla d(x)| = 1$, when $x \in \Omega_{R_0}$, we have

$$\begin{aligned} -\triangle_{p(x)} g &= -div(|g'|^{p(d)-2}\, g'\nabla d) \\ &= -|g'|^{p(d)-2}\, g''[(p(d)-1) - \frac{d}{s+1}p' \ln|g'| - \frac{d}{s+1}(div\nabla d)]. \end{aligned}$$

Since $d \in C^2(\overline{\Omega_\rho})$, then there exists a positive constant D such that $|div\nabla d(x)| \le D$, $\forall x \in \overline{\Omega_\rho}$, when R_0 is small enough, by computation, we have

$$-\triangle_{p(x)} g + f(x,g) \ge 0, \forall x \in \Omega_{R_0}. \tag{1}$$

Obviously, if R_0 is small enough, then k is big enough, so we have

$$\begin{aligned} \triangle_{p(x)} g &= \varepsilon[CsR_0^{-s-1}]^{(p(R_0)-1)} \cos\varepsilon(\delta - d) - [CsR_0^{-s-1}]^{(p(R_0)-1)}(div\nabla d)\sin\varepsilon(\delta - d) \\ &\le ak^{q-1} \le ag^{q-1} \le f(x,g), \ \forall x \in \Omega_\delta \backslash \Omega_{R_0}. \end{aligned}$$

Thus

$$-\triangle_{p(x)} g + f(x,g) \ge 0, \forall x \in \Omega_\delta \backslash \Omega_{R_0}. \tag{2}$$

Obviously

$$-\triangle_{p(x)}g + f(x,g) \geq 0, \forall x \in \Omega\backslash\Omega_\delta. \tag{3}$$

Since $g(x,\epsilon)$ is a C^1 function on Ω, if $0 < R_0$ is small enough (R_0 depends on R, p, q, n, s), from (1), (2) and (3), we can see that $g(x,\epsilon)$ is a super-solution of (P).

Define the function $g_m(r, \frac{\epsilon}{2})$ on $[\frac{1}{m}, +\infty)$ as

$$g_m(r,\epsilon) = \begin{cases} C(r-\frac{1}{m})^{-s} + 2k, \ \frac{1}{m} \leq r < R_0, \\ C(R_0-\frac{1}{m})^{-s} + 2k - \int_{R_0}^{r}[Cs(R_0-\frac{1}{m})^{-s-1}]^{\frac{p(R_0)-1}{p(t)-1}}[\sin\varepsilon(\delta-t)]^{\frac{1}{p(t)-1}}dt, \\ \qquad R_0 < r < \delta \ , \\ C(R_0-\frac{1}{m})^{-s} + 2k - \int_{R_0}^{\delta}[Cs(R_0-\frac{1}{m})^{-s-1}]^{\frac{p(R_0)-1}{p(t)-1}}[\sin\varepsilon(\delta-t)]^{\frac{1}{p(t)-1}}dt, \\ \qquad r \geq \delta, \end{cases}$$

where $s = \frac{p(0)}{q-p(0)}$, m is a big enough integer such that $0 < \frac{1}{m} \leq \frac{R_0}{2}$, $\varepsilon = \frac{\pi}{2(\delta-R_0\ )}$, $0 < \epsilon$ is a small enough constant, $C = C_{\frac{\epsilon}{2}}$, then $g_m(d(x), \frac{\epsilon}{2})$ is a super-solution of (P) on $\Omega\backslash\Omega_{\frac{1}{m}}$. If u is a solution of (P), according to the comparison principle, we get that $g_m(d(x), \frac{\epsilon}{2}) \geq u(x)$ for any $x \in \Omega\backslash\Omega_{\frac{1}{m}}$. For any $x \in \Omega_{R_0}\backslash\Omega_{\frac{1}{m}}$, we have $g_m(d(x), \frac{\epsilon}{2}) \geq g_{m+1}(d(x), \frac{\epsilon}{2})$. Then, for any $x \in \Omega_{R_0}$, we have

$$u(x) \leq \lim_{m\to+\infty} g_m(d(x), \frac{\epsilon}{2}).$$

When $d(x) > 0$ is small enough, we have

$$\lim_{m\to+\infty} g_m(d(x), \frac{\epsilon}{2}) \leq C_{\frac{2\epsilon}{3}}(d(x))^{-s} + k \leq g(d(x),\epsilon).$$

According to the comparison principle, we obtain that $g(d(x),\epsilon) \geq u(x), \forall x \in \Omega$; then $\Phi_1(x) = g(d(x),\epsilon)$ is an upper control function of all of the solutions of (P). The proof is completed.

Theorem 3.2 If there exists a $\delta \in (0,\rho)$ such that

$$f(x,u) \geq au^{q-1} (\text{as } u \to +\infty) \text{ for } x \in \Omega_\delta \text{ uniformly},$$

where a and q are positive constants, and $q - p(x) \geq \frac{1}{n}$ (where n is an integer and $n > 3$) for $x \in \Omega_\delta$ uniformly, then (P) possesses a blow-up solution.

Proof Let's consider the problem

$$\begin{cases} -\triangle_{p(x)}u + f(x,u) = 0 \text{ in } \Omega, \\ u(x) = j \text{ for } x \in \partial\Omega, \end{cases} \tag{4}$$

where $j = 1, 2, \cdots$. The relative functional is

$$\varphi = \int_\Omega \frac{1}{p(x)} \mid \nabla u(x) \mid^{p(x)} dx + \int_\Omega F(x,u)dx,$$

where $F(x,u) = \int_0^u f(x,t)dt$. Since φ is coercive in $X_j := j + W_0^{1,p(x)}(\Omega)$, then φ possesses a nontrivial minimum point u_j, then problem (4) possesses a weak solution u_j. According to the comparison principle, we get $u_j\ (x) \leq u_{j+1}\ (x)$ for any $x \in \Omega$ and $j = 1, 2, \cdots$. Since $\Phi_1(x)$ defined in theorem 3.1 is a super-solution, according to the comparison principle, we get $u_j(x) \leq \Phi_1(x)$ on Ω for all $j = 1, 2, \cdots$. Since every weak solution of (4) is a $C^{1,\alpha}_{loc}$ function, and $\Phi_1(x)$ is locally bounded, similarly to the proof of Lemma 2.1 of paper [13], we can prove that $\{u_j\}$ possesses a subsequence $\{u_{j_i}\}$, such that $u_{j_i} \to u$ is a blow-up solution of (P). The proof is completed.

Theorem 3.3 If there exists a $\delta \in (0, \rho)$ such that

$$f(x,u) \leq bu^{q-1} (\text{as } u \to +\infty) \text{ for } x \in \Omega_\delta \text{ uniformly},$$

where b and q are positive constants, and $q - p(x) \geq \frac{1}{n}$ (where n is an integer and $n > 3$) for $x \in \Omega_\delta$ uniformly, then there exists a function $\Phi_2(x)$ which satisfies $\Phi_2(x) \to +\infty$ (as $d(x) \to 0$), and such that, if $u(x)$ is a blow-up solution of problem (P) then $u(x) \geq \Phi_2(x)$.

Proof Define the function $v(r, \epsilon)$ on $B(0, R)$ as

$$v(r,\epsilon) = \begin{cases} C^* r^{-s} - 2k^*, \ 0 \leq r < R_0, \\ \\ C^* R_0^{-s} - 2k^* - \int_{R_0}^r [C^* s R_0^{-s-1}]^{\frac{p(R_0)-1}{p(t)-1}} [\sin \varepsilon(\delta - t)]^{\frac{1}{p(t)-1}} dt, \\ \qquad R_0 < r < \delta \ , \\ \\ C^* R_0^{-s} - 2k^* - \int_{R_0}^\delta [C^* s R_0^{-s-1}]^{\frac{p(R_0)-1}{p(t)-1}} [\sin \varepsilon(\delta - t)]^{\frac{1}{p(t)-1}} dt, \\ \qquad r \geq \delta, \end{cases}$$

where $s = \frac{p(0)}{q-p(0)}$, $R_0 \in (0, \delta)$ and R_0 is small enough, $\varepsilon = \frac{\pi}{2(\delta - R_0\ )}$, $0 < \epsilon$ is a small enough constant and

$$C^* = C^*_\epsilon = (1-\epsilon)[\frac{1}{b} s^{p(0)-1}(s+1)(p(0)-1)]^{\frac{1}{q-p(0)}},$$

$$k^* = [C^*(\frac{R_0}{2})^{-s-1}]^{\frac{p^+-1}{p^--1}} + \int_{R_0}^{\delta}[C^*s(\frac{R_0}{2})^{-s-1}]^{\frac{p(R_0)-1}{p(t)-1}}[\sin\varepsilon(\delta-t)]^{\frac{1}{p(t)-1}}dt.$$

Obviously, for any positive constant ϵ, $v(r,\epsilon)\in C^1[0,+\infty)$.

Consider $v(d(x),\epsilon)$. Similar to the proof of theorem 3.1, we have

$$div(|\nabla v|^{p(x)-2}\nabla v) = div(|v'|^{p(x)-2}v'\nabla d) \geq b[C^*(d(x))^{-s}]^{q-1} \geq f(x,v), \forall x\in\Omega_{R_0},$$

$$div(|\nabla v|^{p(x)-2}\nabla v) \geq -b(k^*)^{q-1} \geq b(v)^{q-1} \geq f(x,v), \forall x\in\Omega\backslash\Omega_{R_0},$$

then $v(x,\epsilon)$ is a sub-solution of (P). Define the function $v_m(r,\epsilon/2)$ on $[-\frac{1}{m},+\infty)$ as

$$v_m(r,\frac{\epsilon}{2}) = \begin{cases} C^*(r+\frac{1}{m})^{-s}-k^*,\ 0\leq r<R_0, \\ \\ -k^*-\int_{R_0}^{r}[C^*s(\frac{1}{m}+R_0)^{-s-1}]^{\frac{p(R_0)-1}{p(t)-1}}[\sin\varepsilon(\delta-t)]^{\frac{1}{p(t)-1}}dt \\ +C^*(\frac{1}{m}+R_0)^{-s},\ R_0<r<\delta, \\ \\ -k^*-\int_{R_0}^{\delta}[C^*s(\frac{1}{m}+R_0)^{-s-1}]^{\frac{p(R_0)-1}{p(t)-1}}[\sin\varepsilon(\delta-t)]^{\frac{1}{p(t)-1}}dt \\ \ +C^*(\frac{1}{m}+R_0)^{-s},\ r\geq\delta, \end{cases}$$

where $C^*=C^*_{\frac{\epsilon}{2}}$, $\varepsilon=\frac{\pi}{2(\delta-R_0\ )}$. We can see that $v_m(d(x),\frac{\epsilon}{2})$ is a sub-solution of (P) on Ω, according to the comparison principle, we get that $v_m(d(x),\frac{\epsilon}{2})\leq u(x)$ for any $x\in\Omega$. For any $x\in\Omega_{R_0}$, we have $v_m(d(x),\frac{\epsilon}{2})\leq v_{m+1}(d(x),\frac{\epsilon}{2})$. Then

$$u(x)\geq \lim_{m\to+\infty} v_m(d(x),\frac{\epsilon}{2}),\ \forall x\in\Omega_{R_0}.$$

When $d(x,\partial\Omega)$ is small enough, we have

$$\lim_{m\to+\infty} v_m(d(x),\frac{\epsilon}{2}) \geq C^*_{\frac{2\epsilon}{3}}(d(x))^{-s}-k^* \geq v(d(x),\epsilon).$$

According to the comparison principle, we obtain that $v(d(x),\epsilon)\leq u(x),\forall x\in\Omega$; then $\Phi_2(x)=v(d(x),\epsilon)$ is the lower control function of all of the solutions of (P). The proof is completed.

Definition 3.1 If u is a blow-up solution of (P) that satisfies

$$\lim_{d(x,\partial\Omega)\to0}\frac{u(x)}{\beta(d(x))^{-s}}=1,$$

where β and s are positive constants, then we say that the singularity of u is $\beta(d(x))^{-s}$.

Theorem 3.4 If $\lim_{d(x)\to 0} q(x) = q$, $\lim_{d(x)\to 0} p(x) = p$, and $\lim_{d(x)\to 0} \lim_{u\to+\infty} \frac{f(x,u)}{u^{q(x)-1}} = a$, where a is a positive constant, $q(x)$ is Lipschitz continuous and $q > p$, then the singularity of the blow-up solution u of (P) is $\beta(d(x))^{-s}$, where $s = \frac{p}{q-p}$, $\beta = C_0 = [\frac{1}{a}s^{p-1}(s+1)(p-1)]^{\frac{1}{q-p}}$.

Proof It can be obtained easily from Theorem 3.2 and Theorem 3.3.

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