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Non-Smooth Polynomials

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Abstract

The interpolants used classically have a simple geometric structure and in some cases they do not describe the fine microscopic pattern that some experimental signals display. The fractal methodology provides new tools for the analysis of functions whose graphs have a more sophisticated geometric complexity. In this paper, fractal variants of classical functions (in particular polynomials) are defined. This kind of maps preserve some properties, as the constitution of bases of $\mathcal{C}[a,b]$, and lose some others as, for instance, the differentiability in the interval. The fractal polynomials defined here are, in general, non-smooth and this characteristic makes the process of the complexity quantification easier.

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1 Introduction

The classical techniques of approximation constitute a great tool to model physical phenomena. However, the interpolants used classically have a simple geometric structure and in some cases they do not describe the fine microscopic pattern that some experimental signals display. The methodology of fractal sets generates new procedures for the analysis of functions whose graphs have a more sophisticated geometric structure.
The first characteristic to be removed in order to obtain fractal graphs is the smoothness of the mappings since, by Theorem of Besicovitch and Ursell ([6]), if a function defined on a compact interval is smooth, its fractal dimension

is one. In the present paper, "fractal perturbations" of classical functions (polynomials in particular) are constructed. This kind of maps preserve some properties, as the constitution of bases of $\mathcal{C}[a,b]$ and lose some others as, for instance, the differentiability in the interval ([14], [15]).
It seems suitable to approximate a given function by means of a map sharing its properties, in particular its quality of smooth or non-smooth. On the other hand, Banach ([1]) proved that the set of nowhere differentiable functions is topologically generic in $\mathcal{C}[0,1]$. Using fractal methods, we try to enlarge the field of interpolation and approximation including functions that can even be nowhere differentiable ([8]).
The study of non-differentiable continuous functions was initiated by Weierstrass ([19]), Bolzano ([7]), Hardy ([11]) et al. and continued by Mandelbrot ([13]), Kiesswetter ([12]), Berry and Lewis ([5]) and some others. Our approach uses the procedure of fractal interpolation ([2], [3], [16]) to construct this kind of approximants not necessarily smooth.

2 Fractal Functions

Let $t_0 < t_1 < ... < t_N$ be real numbers, and $I = [t_0, t_N]$ the closed interval that contains them. Let a set of data points $\{(t_n, x_n) \in I \times R : \ n = 0, 1, 2, ..., N\}$ be given. Set $I_n = [t_{n-1}, t_n]$ and let $L_n : I \to I_n, \ \ n \in \{1, 2, ..., N\}$ be contractive homeomorphisms such that:

$$L_n(t_0) = t_{n-1}, \ L_n(t_N) = t_n \tag{1}$$

$$|L_n(c_1) - L_n(c_2)| \le l \ |c_1 - c_2| \quad \forall \ c_1, c_2 \ \in I \tag{2}$$

for some $0 \le l < 1$.

Let $F = I \times R$ and N continuous mappings, $F_n : F \to R$, be given satisfying:

$$F_n(t_0, x_0) = x_{n-1}, \quad F_n(t_N, x_N) = x_n, \quad n = 1, 2, ..., N \tag{3}$$

$$|F_n(t,x) - F_n(t,y)| \le r|x-y|, \quad t \in I, \quad x, y \in R, \quad 0 \le r < 1 \tag{4}$$

Now define functions $w_n(t,x) = (L_n(t), F_n(t,x)), \ \forall \ n = 1, 2, ..., N$.

Theorem 2.1 ([2]) *The Iterated Function System (IFS)* $\{F, \ w_n : n = 1, 2, ..., N\}$ *defined above admits a unique attractor* G. G *is the graph of a continuous function* $f : I \to R$ *which obeys* $f(t_n) = x_n$ *for* $n = 0, 1, 2, ..., N$.

The previous function is called a Fractal Interpolation Function (FIF) corresponding to $\{(L_n(t), F_n(t,x))\}_{n=1}^N$ and it is unique satisfying the functional equation ([2]):

$$f(t) = F_n(L_n^{-1}(t), f \circ L_n^{-1}(t)), \quad n = 1, 2, ..., N, \ \ t \in I_n = [t_{n-1}, t_n] \tag{5}$$

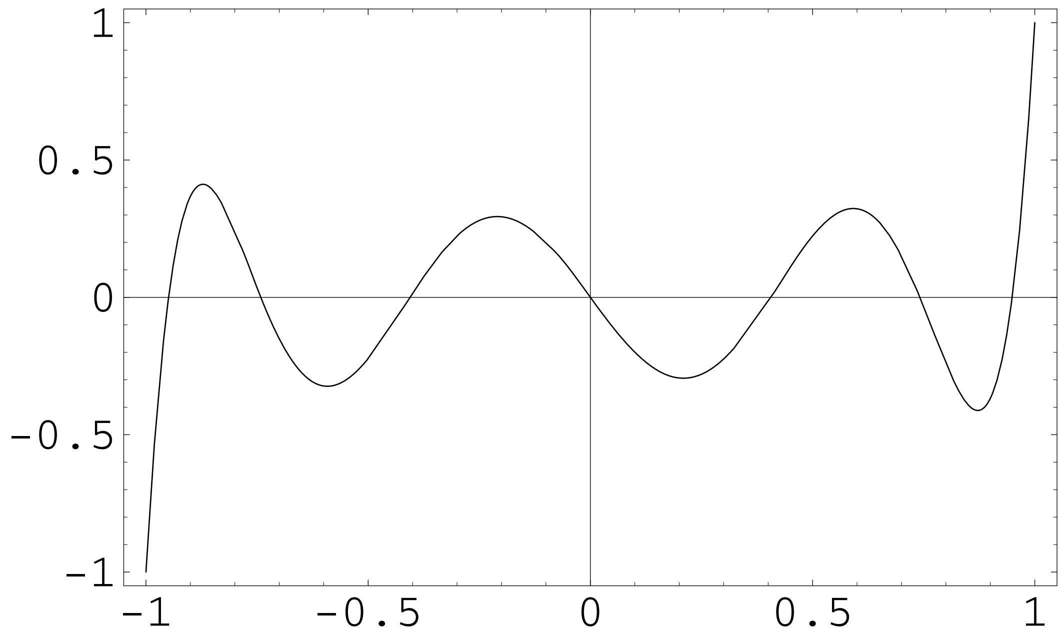

Figure 1: Seventh Legendre polynomial.

The most widely studied fractal interpolation functions so far are defined by the IFS

$$\begin{cases} L_n(t) = a_n t + b_n \\ F_n(t,x) = \alpha_n x + q_n(t) \end{cases} \tag{6}$$

where $-1 < \alpha_n < 1$. α_n is called a vertical scaling factor of the transformation w_n and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ is the scale vector of the IFS. Following the equalities (1)

$$a_n = \frac{t_n - t_{n-1}}{t_N - t_0} \qquad b_n = \frac{t_N t_{n-1} - t_0 t_n}{t_N - t_0} \tag{7}$$

Let $g \in \mathcal{C}(I)$ be a continuous function. We consider here q_n such that ([2])

$$q_n(t) = g \circ L_n(t) - \alpha_n b(t) \tag{8}$$

where b is continuous and such that

$$b(t_0) = g(t_0) = x_0, \qquad b(t_N) = g(t_N) = x_N. \tag{9}$$

The data are here $\{(t_n, g(t_n)) : n = 0, 1, \ldots, N\}$. By means of this method one can define fractal analogues of classical functions. We consider $b(t) = r_g(t)$, where $r_g(t)$ is the line passing through $(t_0, g(t_0))$ and $(t_N, g(t_N))$,

$$b(t) = r_g(t) = g(t_0) + \frac{g(t_N) - g(t_0)}{t_N - t_0}(t - t_0) \tag{10}$$

Note: The affine fractal interpolation functions ([2], [18]) are a particular case of this if we take g as a polygonal whose vertices are the data.

Definition 2.2 *Let g^α_Δ or g^α be the continuous function defined by the IFS (6), (7), (8) and (10). g^α_Δ is the α-fractal function associated to g with respect to the partition Δ.*

Sometimes we will omit the subindices in order to simplify the notation.

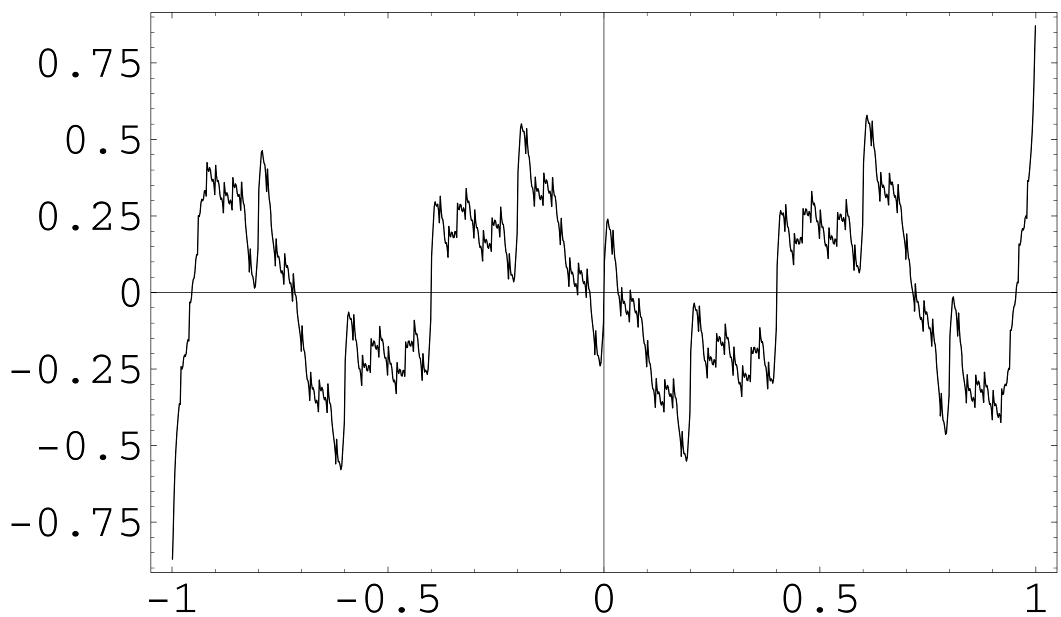

Figure 2: The graph shows the α-fractal seventh Legendre polynomial, with respect to $\Delta : -1 < -4/5 < -3/5 < ... < 4/5 < 1$ and $\alpha_n = 0.2 \quad \forall n = 1, \ldots, 10$.

Following (5) and (8), g^α satisfies the fixed point equation:

$$g^\alpha(t) = g(t) + \alpha_n(g^\alpha - r_g) \circ L_n^{-1}(t) \qquad \forall t \in I_n \tag{11}$$

g^α interpolates to g at t_n as, using (1), (9) and Theorem 2.1:

$$g^\alpha(t_n) = g(t_n) + \alpha_n(g^\alpha - r_g) \circ (t_N) = g(t_n) \qquad \forall n = 0, 1, \ldots, N \tag{12}$$

The scale vector gives a degree of freedom to the function g, allowing us to modify its properties or approach it to a given function ([14], [15]). Some classical interpolants, polynomials for instance, use only the information provided by the samples. These fractal approximants contain, however, all the information of the function. In reference [17] we have proved some properties of g^α. For instance:

Theorem 2.3 *The α-fractal function g^α of g with respect to Δ and b satisfies the inequality*

$$\|g^\alpha - g\|_\infty \leq \frac{|\alpha|_\infty}{1-|\alpha|_\infty}(\|g-b\|_\infty) \tag{13}$$

with $|\alpha|_\infty = max_{1\leq n\leq N}\{|\alpha_n|\}$.

Corollary 2.4 *Let $\alpha^m \in R^N$ a sequence of scale vectors such that $\forall m \geq 1$, $|\alpha^m|_\infty < 1$ and $\alpha^m \to 0 \in R^N$ as $m \to \infty$. The sequence g^{α^m} converges uniformly towards g.*

Proposition 2.5

$$g^\alpha = g \Leftrightarrow \alpha = 0 \text{ or } g \text{ is a line.}$$

Proof. $\Leftarrow$ The case $\alpha = 0$ is an immediate consequence of the preceding Theorem. If g is a line $r_g = g$ and the fixed point equation yields (11)

$$g^\alpha(t) = r_g(t) + \alpha_n(g^\alpha - r_g)\circ L_n^{-1}(t) \tag{14}$$

but this equation is verified by $g^\alpha = r_g = g$.

$\Rightarrow$ Let us assume that $g^\alpha = g$. If $\alpha \neq 0$, let $\alpha_n \neq 0$ for some n. For all $t \in I$, $L_n(t) \in I_n$ and applying the equation (11),

$$g\circ L_n(t) = g\circ L_n(t) + \alpha_n(g - r_g)(t)$$

and

$$g(t) = r_g(t)$$

for all $t \in I$. ◇

Consequence 2.6 *If $g(t) = k$ $\forall t \in I$, $g^\alpha(t) = g(t) = k$ $\forall t \in I$ for any scale vector.*

Proposition 2.7 *If f is not a line, $f^\alpha = f^\beta \Leftrightarrow \alpha = \beta$.*

Proof. Let us assume that $f^\alpha = f^\beta$, for any $t \in I$ we can apply (11) for $L_n(t) \in I_n$, f^α and f^β

$$f\circ L_n(t) + \alpha_n(f^\alpha(t) - r_f(t)) = f\circ L_n(t) + \beta_n(f^\alpha(t) - r_f(t))$$

then, for any $t \in I$ and n

$$(\alpha_n - \beta_n)(f^\alpha(t) - r_f(t)) = 0$$

If $\alpha_n \neq \beta_n$ for some n, $\forall t \in I$

$$f^\alpha(t) - r_f(t) = 0$$

by (11) $\forall t \in I$

$$f^\alpha(t) = f(t) = f^\beta(t)$$

but α or β is non-null and by Proposition 2.5 f is a line what contradicts the hypothesis. Consequently $\alpha = \beta$. ◇

3 Fractal Linear Operator

Let us consider the operator of $\mathcal{C}(I)$ which assigns g^α to the function g

$$\mathcal{L}^\alpha(g) = \mathcal{L}^\alpha_\Delta(g) = g^\alpha$$

This operator is linear as, by (11) $\forall t \in I_n$:

$$f^\alpha(t) = f(t) + \alpha_n(f^\alpha - r_f) \circ L_n^{-1}(t)$$
$$g^\alpha(t) = g(t) + \alpha_n(g^\alpha - r_g) \circ L_n^{-1}(t)$$

Multiplying the first equation by λ and the second by μ, and considering that

$$\lambda r_f + \mu r_g = r_{\lambda f + \mu g}$$

the function

$$\lambda f^\alpha + \mu g^\alpha$$

satisfies the equation corresponding to

$$(\lambda f + \mu g)^\alpha$$

By the uniqueness of the solution the linearity is proved. Besides, applying Theorem 2.3 one has

$$\|\mathcal{L}^\alpha(f) - f\|_\infty \leq \frac{|\alpha|_\infty}{1 - |\alpha|_\infty}\|f - r_f\|_\infty \leq \frac{2|\alpha|_\infty}{1 - |\alpha|_\infty}\|f\|_\infty \tag{15}$$

As a consequence

$$\|\mathcal{L}^\alpha\| \leq \frac{1 + |\alpha|_\infty}{1 - |\alpha|_\infty} \tag{16}$$

and thus $\mathcal{L}^\alpha$ is a linear and bounded operator.

Proposition 3.1 *If $\alpha^m \to 0$ as $m \to \infty$, $\mathcal{L}^{\alpha^m}$ converges strongly to the identity.*

Proof. It is a consequence of Corollary 2.4. $\diamond$

Lemma 3.2

$$\|f^\alpha - f\|_\infty = |\alpha|_\infty \|f^\alpha - r_f\|_\infty \tag{17}$$

Proof. Let $t_{max} \in I$ such that $|f^\alpha(t_{max}) - r_f(t_{max})| = M$ be maximum on the interval I. Let $t_{n,max} = L_n(t_{max}) \in I_n$, $\forall t \in I_n$

$$|f^\alpha(t) - f(t)| = |\alpha_n||(f^\alpha - r_f) \circ L_n^{-1}(t)|$$

and

$$max\{|f^\alpha(t) - f(t)|; t \in I_n\} = |\alpha_n||(f^\alpha - r_f) \circ L_n^{-1}(t_{n,max})|$$

$$max\{|f^\alpha(t) - f(t)|; t \in I_n\} = |\alpha_n|M$$

and

$$max\{|f^\alpha(t) - f(t)|; t \in I\} = |\alpha|_\infty M$$

◇

Theorem 3.3 $\mathcal{L}^\alpha$ *is injective.*

Proof. If $f^\alpha = 0$, applying the preceding Lemma,

$$\|f\|_\infty = |\alpha|_\infty \|r_f\|_\infty \leq |\alpha|_\infty \|f\|_\infty \tag{18}$$

but

$$|\alpha|_\infty < 1 \tag{19}$$

and

$$|\alpha|_\infty \|f\|_\infty \leq \|f\|_\infty \tag{20}$$

thus

$$|\alpha|_\infty \|f\|_\infty = \|f\|_\infty \tag{21}$$

and hence

$$\|f\|_\infty = 0 \tag{22}$$

◇

4 Fractal Polynomials

Let $\mathcal{P}_m[a,b]$ be the set of polynomials of degree lower or equal than m on $I = [a,b]$ and $\mathcal{P}[a,b] = \bigcup_{m=1}^\infty \mathcal{P}_m[a,b]$. The set $\{1, t, t^2, \ldots\}$ constitutes a basis of $\mathcal{P}[a,b]$.

Definition 4.1 *An α-fractal polynomial is an element $p^\alpha(t) \in \mathcal{C}(I)$ such that $\exists p(t) \in \mathcal{P}[a,b]$ with $\mathcal{L}^\alpha(p) = p^\alpha$. If p has degree m, p^α is an α-fractal polynomial of degree m.*

Notation: $\mathcal{P}_m^\alpha[a,b] = \mathcal{L}^\alpha(\mathcal{P}_m[a,b])$, $\mathcal{P}^\alpha[a,b] = \mathcal{L}^\alpha(\mathcal{P}[a,b])$.

By the properties of $\mathcal{L}^\alpha$, $\mathcal{P}_m^\alpha[a,b]$ is linearly spanned by $\{1, t^\alpha, (t^2)^\alpha, \ldots, (t^m)^\alpha\}$ and consequently $dim(\mathcal{P}_m^\alpha[a,b]) < \infty$. $\mathcal{P}_m^\alpha[a,b]$ is a closed and complete linear subspace of $\mathcal{C}[a,b]$.

Definition 4.2 *A fractal polynomial is a finite linear combination of the functions* $(t^i)^{\alpha^j}$*, where* $\alpha^j \neq 0$*.*

We recall that a sequence $\{x_n\}$ is closed in X if every element of X is approached by a finite linear combination of elements of the sequence ([9]).

Theorem 4.3 *Let be given a partition* Δ *of the interval* I *and a sequence of non-null scale vectors such that* $\alpha^m \to 0$ *as* $m \to \infty$*. The system*

$$S = \{1, t^{\alpha^1}, t^{\alpha^2}, \ldots, t^{\alpha^m}, \ldots, (t^2)^{\alpha^1}, (t^2)^{\alpha^2}, \ldots, (t^2)^{\alpha^m}, \ldots\} \tag{23}$$

is closed in $\mathcal{C}[a,b]$*.*

Proof.

It is a consequence of Weierstrass's Theorem on the density of polynomials in $\mathcal{C}[a,b]$ and Theorem 2.3.

$\diamond$

Consequence 4.4 *The fractal polynomials are dense in* $\mathcal{C}[a,b]$*.*

Note: If we consider the sequence $B_n(f,t)$ of Bernstein's polynomials associated to a continuous function f and $\alpha^n \to 0$, we obtain a sequence of fractal polynomials $B_n^{\alpha^n}(f,t)$ (not necessarily smooth) converging to f.

Theorem 4.5 *The system* S *is complete in* $\mathcal{C}[a,b]$*.*

Proof. By Banach's Theorem ([9]), in a normed linear space a system is fundamental if and only if it is complete. $\diamond$

The following Definition and construction can be found in any standard book on normed spaces.

Definition 4.6 *A Schauder basis for a normed linear space* E *is a (finite or infinite) sequence* $\{g_0, g_1, \ldots\}$ *such that each element* $f \in E$ *may be written uniquely in the form* $f = \sum_{m=0}^{+\infty} c_m g_m$*.*

Theorem 4.7 Schauder's Theorem*:* $\mathcal{C}[a,b]$ *possesses a basis of polygonal (piecewise linear) functions.*

The construction goes as follows. Given a dense sequence in $[a,b]$, $\{a = t_0, b = t_1, t_2, t_3, \ldots\}$ ($t_i \neq t_j$ for $i \neq j$). Let $\mathcal{L}_m f$ denote, $\forall m \geq 1$, the polygonal (or piecewise linear function) which agrees with f at the nodes $t_0, t_1, \ldots, t_m$ (conveniently ordered). Let $\mathcal{L}_0 f$ denote the constant function $\mathcal{L}_0 f = f(t_0)$. It is clear that $\forall m \geq 0$,

$$\|\mathcal{L}_m f\|_\infty \leq \|f\|_\infty$$

The base functions $g_0, g_1, \dots$ are defined in such a way that the following equality is true:

$$f = lim \mathcal{L}_m f = \mathcal{L}_0 f + \sum_{m=1}^{\infty} (\mathcal{L}_m - \mathcal{L}_{m-1}) f = \sum_{m=0}^{\infty} c_m g_m$$

As a consequence, $c_m = c_m(f)$ satisfy

$$|c_m(f)| \leq 2\|f\|_\infty \tag{24}$$

and c_m are linear.

To define a Schauder basis of fractal polynomials, we need a previous well known Lemma.

Lemma 4.8 *If L is a linear operator from a Banach space into itself such that $\|L\| < 1$, then $(I - L)^{-1}$ exists and is bounded.*

Theorem 4.9 *There exists a Schauder basis of fractal polynomials in $\mathcal{C}[a, b]$.*

Proof. Let $\{g_m\}$ be the Schauder basis of polygonal functions and

$$f = \sum_{m=0}^{+\infty} c_m g_m \tag{25}$$

By Theorem 4.3, for each g_m there exists a fractal polynomial h_m such that

$$\sum_{m=0}^{+\infty} \|g_m - h_m\|_\infty \leq \sum_{m=0}^{+\infty} \frac{1}{2^{m+3}} = \frac{1}{4} \tag{26}$$

Let us define an operator S such that

$$S(f) = \sum_{m=0}^{+\infty} c_m(f) h_m$$

where $c_m(f)$ are the coefficients of f respect to the Schauder basis g_m. Let us prove the convergence of this series. If S_n is the n-th sum and $p \in \mathbb{N}$,

$$\|S_{n+p} - S_n\|_\infty = \| \sum_{m=n+1}^{m=n+p} c_m(f) h_m \|_\infty =$$

$$= \| \sum_{m=n+1}^{m=n+p} (c_m(f) h_m - c_m(f) g_m) + \sum_{m=n+1}^{m=n+p} c_m(f) g_m \|_\infty$$

using the properties of c_m and the Definition of h_m,

$$\|S_{n+p} - S_n\|_\infty \leq \sum_{m=n+1}^{m=n+p} 2\|f\|_\infty \|h_m - g_m\|_\infty + \|\sum_{m=n+1}^{m=n+p} c_m(f) g_m\|_\infty \leq$$

$$\leq \sum_{m=n+1}^{m=n+p} 2\|f\|_\infty \frac{1}{2^{m+3}} + \|\sum_{m=n+1}^{m=n+p} c_m(f) g_m\|_\infty$$

Given $\epsilon > 0$, let us consider $\epsilon/2 > 0$. Since the series $\sum \frac{1}{2^m}$ is convergent, the Cauchy condition provides $n_1 \in \mathbb{N}$ such that $\forall n \geq n_1$,

$$\sum_{m=n+1}^{m=n+p} 2\|f\|_\infty \frac{1}{2^{m+3}} < \frac{\epsilon}{2}$$

The series $\sum c_m(f) g_m$ is convergent (25) and thus there exists $n_2 \in \mathbb{N}$ such that $\forall n \geq n_2$,

$$\|\sum_{m=n+1}^{m=n+p} c_m(f) g_m\|_\infty < \frac{\epsilon}{2}$$

Then, if $n \geq n_0 = max\{n_1, n_2\}$

$$\|S_{n+p} - S_n\|_\infty < \epsilon$$

and S_n converges.
The linearity of S is based on that of c_m. As $n \to \infty$,

$$\sum_{m=0}^{n} c_m(f) h_m + \sum_{m=0}^{n} c_m(g) h_m = \sum_{m=0}^{n} c_m(f+g) h_m \to S(f) + S(g)$$

and

$$S(f) + S(g) = \sum_{m=0}^{\infty} c_m(f+g) h_m = S(f+g)$$

Let us prove that S^{-1} exists and is continuous. $\forall f \in \mathcal{C}[a,b]$, using (25) and (26),

$$\|(I-S)(f)\|_\infty \leq \sum_{m=0}^{+\infty} |c_m(f)|\ \|g_m - h_m\|_\infty \leq 2\|f\|_\infty \sum_{m=0}^{+\infty} \|g_m - h_m\|_\infty \leq \frac{1}{2}\|f\|_\infty$$

then $\|I - S\| < 1$ and we apply Lemma 4.8, $S^{-1} = (I - (I - S))^{-1}$ exists and is bounded, then

$$f = S(S^{-1}(f)) = \sum_{m=0}^{+\infty} c_m(S^{-1}(f)) h_m$$

The expansion is unique because if

$$f = \sum_{m=0}^{+\infty} a_m h_m$$

as

$$S(g_m) = h_m$$

then

$$f = \sum_{m=0}^{+\infty} a_m S(g_m)$$

$$S^{-1}(f) = \sum_{m=0}^{+\infty} a_m g_m$$

and

$$a_m = c_m(S^{-1}(f))$$

Consequently they are unique. $\diamond$

5 Non-smooth fractal functions

5.1 Explicit representation of g^α

We consider here the particular case $I = [0, 1]$. If the nodes are equidistant the maps L_n adopt the expression

$$L_n(t) = \frac{t}{N} + \frac{n-1}{N} \tag{27}$$

and (10)

$$b(t) = x_0 + (x_N - x_0)t \tag{28}$$

In the reference [8], an explicit representation of a FIF f corresponding to a general IFS (6), (7) defined in $I = [0, 1]$ is deduced. If

$$\omega = \sum_{m=1}^{\infty} \frac{n_m - 1}{N^m} \tag{29}$$

then

$$f(\omega) = \sum_{k=0}^{\infty} (\prod_{j=0}^{k} \alpha_{n_j}) q_{n_{k+1}} (\sum_{l=1}^{\infty} \frac{n_{k+1+l} - 1}{N^l}) \tag{30}$$

The expression (29) is the N-adic representation of ω with digits n_m, where $1 \leq n_m \leq N$ for $m \geq 1$, and in (30) $\alpha_{n_0} = 1$. The value of f does not depend on the representation of ω ([4]). Let us denote

$$\omega = (n_1 n_2 n_3 \ldots) = \frac{n_1 - 1}{N} + \frac{n_2 - 1}{N^2} + \frac{n_3 - 1}{N^3} + \ldots \tag{31}$$

$$\sigma^k\omega = (n_{k+1}n_{k+2}\ldots) = \frac{n_{k+1}-1}{N} + \frac{n_{k+2}-1}{N^2} + \ldots \tag{32}$$

$$\sigma^{k+1}\omega = (n_{k+2}n_{k+3}\ldots) = \frac{n_{k+2}-1}{N} + \frac{n_{k+3}-1}{N^2} + \ldots \tag{33}$$

Using (30),

$$f(\omega) = \sum_{k=0}^{\infty}(\prod_{j=0}^{k}\alpha_{n_j})q_{n_{k+1}}(\sigma^{k+1}\omega) \tag{34}$$

In our case, by the Definition of q_n (8),

$$q_{n_{k+1}} = g \circ L_{n_{k+1}}(t) - \alpha_{n_{k+1}}b(t)$$

by (27)

$$L_{n_{k+1}}(\sigma^{k+1}\omega) = L_{n_{k+1}}(\frac{n_{k+2}-1}{N} + \frac{n_{k+3}-1}{N^2} + \ldots) =$$

$$= \frac{n_{k+1}-1}{N} + \frac{n_{k+2}-1}{N^2} + \ldots = \sigma^k\omega \tag{35}$$

and

$$g^\alpha(\omega) = \sum_{k=0}^{\infty}(\prod_{j=0}^{k}\alpha_{n_j})(g(\sigma^k\omega) - \alpha_{n_{k+1}}b(\sigma^{k+1}\omega)) \tag{36}$$

If $p_k = \alpha_{n_0}\alpha_{n_1}\ldots\alpha_{n_k}$,

$$g^\alpha(\omega) = \sum_{k=0}^{\infty}(p_k g(\sigma^k\omega) - p_{k+1}b(\sigma^{k+1}\omega))$$

In case of constant scale factors, $\alpha_{n_j} = a$ for $j = 1, 2, \ldots$, $\alpha_{n_0} = 1$

$$g^\alpha(\omega) = \sum_{k=0}^{\infty} a^k(g(\sigma^k\omega) - ab(\sigma^{k+1}\omega)) \tag{37}$$

5.2 Non-smoothness of g^α

We face here the problem of finding conditions for the non-smoothness of g^α. In order to simplify the notation we assume equidistant nodes and constant scale factors $\alpha_n = a$.

For

$$\omega = (n_1 n_2 n_3 \ldots n_r \ldots) \tag{38}$$

we consider $\omega_r \le \omega \le \omega'_r$,

$$\omega_r = (n_1 n_2 n_3 \ldots n_r 111\ldots) \tag{39}$$

$$\omega'_r = (n_1 n_2 n_3 \ldots n_r NNN\ldots) \tag{40}$$

$$\omega'_r = \sum_{k=1}^{r}\frac{n_k-1}{N^k} + \sum_{k=r+1}^{\infty}\frac{N-1}{N^k} = \omega_r + \frac{1}{N^r} \tag{41}$$

Lemma 5.1 *If $g \in \mathcal{C}^1[0,1]$, $|a| \geq \frac{1}{N}$ and g^α is differentiable at $\omega \in (0,1)$,*

$$lim_{k\to\infty} g'(\sigma^k \omega) = x_N - x_0. \tag{42}$$

Proof. $\forall k \geq r$,

$$\sigma^k \omega_r = (111\ldots) = 0 \tag{43}$$

$$\sigma^k \omega'_r = (NNN\ldots) = 1 \tag{44}$$

From (37),

$$g^\alpha(\omega_r) = \sum_{k=0}^{r-1} a^k (g(\sigma^k \omega_r) - ab(\sigma^{k+1}\omega_r)) + \sum_{k=r}^{\infty} a^k (g(\sigma^k \omega_r) - ab(\sigma^{k+1}\omega_r)) \tag{45}$$

The second term becomes ((43), (9)),

$$\sum_{k=r}^{\infty} a^k (g(\sigma^k \omega_r) - ab(\sigma^{k+1}\omega_r)) = \sum_{k=r}^{\infty} a^k x_0 (1-a) = x_0 a^r \tag{46}$$

In the same way

$$g^\alpha(\omega'_r) = \sum_{k=0}^{r-1} a^k (g(\sigma^k \omega'_r) - ab(\sigma^{k+1}\omega'_r)) + \sum_{k=r}^{\infty} a^k (g(\sigma^k \omega'_r) - ab(\sigma^{k+1}\omega'_r)) \tag{47}$$

and the second term is

$$\sum_{k=r}^{\infty} a^k (g(\sigma^k \omega'_r) - ab(\sigma^{k+1}\omega'_r)) = x_N a^r \tag{48}$$

then

$$\begin{aligned} g^\alpha(\omega'_r) - g^\alpha(\omega_r) = (x_N - x_0)a^r + \sum_{k=0}^{r-1} a^k (g(\sigma^k \omega'_r) - g(\sigma^k \omega_r)) \\ -a \sum_{k=0}^{r-1} a^k (b(\sigma^{k+1}\omega'_r) - b(\sigma^{k+1}\omega_r)) \end{aligned} \tag{49}$$

For any $0 \leq k \leq r$

$$\sigma^k \omega_r = (n_{k+1} n_{k+2} \ldots n_r 11\ldots)$$

$$\sigma^k \omega'_r = (n_{k+1} n_{k+2} \ldots n_r NN\ldots)$$

$$\sigma^k \omega'_r - \sigma^k \omega_r = \sum_{j=1}^{\infty} \frac{N-1}{N^{r-k+j}} = \frac{1}{N^{r-k}}$$

besides, for a $\xi_k^r \in (\sigma^k \omega_r, \sigma^k \omega'_r)$,

$$g(\sigma^k \omega'_r) - g(\sigma^k \omega_r) = g'(\xi_k^r) \frac{1}{N^{r-k}}$$

and (28)

$$b(\sigma^{k+1}\omega_r') - b(\sigma^{k+1}\omega_r) = (x_N - x_0)\frac{1}{N^{r-k-1}} \tag{50}$$

The second term of (49) adopts the expression

$$\sum_{k=0}^{r-1} a^k(g(\sigma^k\omega_r') - g(\sigma^k\omega_r)) = \sum_{k=0}^{r-1} a^k g'(\xi_k^r)\frac{1}{N^{r-k}} \tag{51}$$

and thus by (41), (49), (50), (51),

$$\frac{g^\alpha(\omega_r') - g^\alpha(\omega_r)}{\omega_r' - \omega_r} = (aN)^r(x_N - x_0) + \sum_{k=0}^{r-1} g'(\xi_k^r)(aN)^k - (aN)(x_N - x_0)\sum_{k=0}^{r-1}(aN)^k$$

but

$$(aN)^r(x_N - x_0) = (x_N - x_0) + \sum_{k=0}^{r-1}(x_N - x_0)(aN - 1)(aN)^k$$

$$\frac{g^\alpha(\omega_r') - g^\alpha(\omega_r)}{\omega_r' - \omega_r} = (x_N - x_0) + \sum_{k=0}^{r-1}(g'(\xi_k^r) - aN(x_N - x_0) + (x_N - x_0)(aN - 1))(aN)^k$$

and

$$\frac{g^\alpha(\omega_r') - g^\alpha(\omega_r)}{\omega_r' - \omega_r} = (x_N - x_0) + \sum_{k=0}^{r-1}(g'(\xi_k^r) - (x_N - x_0))(aN)^k$$

If g^α is differentiable at ω the limit of this quotient agrees with $(g^\alpha)'(\omega)$

$$(g^\alpha)'(\omega) = (x_N - x_0) + lim_{r\to\infty}\sum_{k=0}^{r-1}(g'(\xi_k^r) - (x_N - x_0))(aN)^k$$

As $r \to \infty$, $\sigma^k\omega_r$ and $\sigma^k\omega_r' \to \sigma^k\omega$. By hypothesis $|a| \geq 1/N$, hence

$$lim_{k\to\infty}g'(\sigma^k\omega) - (x_N - x_0) = 0$$

$\diamond$

Theorem 5.2 *If $g \in \mathcal{C}^1[0,1]$, $|a| \geq 1/N$ and $g'(t)$ does not agree with $x_N - x_0$ in a non-empty open subinterval of I, the set of points of non-differentiability of g^α is dense on $I = [0,1]$.*

Proof. Applying Lemma 5.1 , if g^α is differentiable at ω,

$$lim_{k\to\infty}g'(\sigma^k\omega) = (x_N - x_0).$$

A point $\omega \in I$ is σ-periodic if $\sigma^k\omega = \omega$ for $k \in \mathbb{N}$ ([10]). The set of σ-periodic points is dense on I because for any

$$\omega^* = (n_1 n_2 n_3 \ldots n_r n_{r+1} \ldots)$$

the point

$$\omega_r^p = (n_1 n_2 n_3 \ldots n_r n_1 n_2 n_3 \ldots n_r \ldots)$$

satisfies

$$|\omega^* - \omega_r^p| \leq \frac{1}{N^r}$$

and $\omega_r^p \to \omega^*$ as $r \to \infty$. If ω is σ-periodic and g^α is differentiable at ω $\forall k \geq 0$

$$g'(\sigma^k\omega) = (x_N - x_0)$$

Let I_ϵ be a non-empty open subinterval of I. If g^α is differentiable at any σ-periodic point of I_ϵ then $g'(\omega) = x_N - x_0$, $\forall\omega$ σ-periodic of I_ϵ. But, by the density, $g'(\omega) = x_N - x_0$ $\forall\omega \in I_\epsilon$, and this fact contradicts the hypothesis of the Theorem. As a consequence, $\exists\omega \in I_\epsilon$ σ-periodic such that g^α is not differentiable at it. As I_ϵ is arbitrary, the set of points of non-differentiability is dense on I. $\diamond$

Note: The condition for g' is verified for very general classes of interpolants like polynomials, splines, etc. In particular with this choice of scale factors we can speak about "non-smooth polynomials".

References

[1] S. Banach, Über die Baire'sche Kategorie gewisser Funktionenmengen, Studia Mathematica, **3**, (1931), 174.

[2] M.F. Barnsley, Fractal functions and interpolation, Constr. Approx., **2**, (1986), 303-329.

[3] M.F. Barnsley, *Fractals Everywhere*, Academic Press Inc., 1988.

[4] M.F. Barnsley, *Superfractals*, Cambridge University Press, 2006.

[5] M.V. Berry, Z.V. Lewis, On the Weierstrass-Mandelbrot fractal function, Proc. R. Soc. Ser. A, **370**, (1980), 459-484.

[6] A.S. Besicovitch, H.D. Ursell. On dimensional numbers of some continuous curves. In: *Classics on Fractals*, Edgar, G. A. (ed.), Addison-Wesley, 1993.

[7] B. Bolzano, Funktionenlehre, Herausgegeben und mit Anmerkungen versehen von K. Rychlik, Prague (1930).

[8] S. Chen, The non-differentiability of a class of fractal interpolation functions, Acta Math. Sci., **19(4)**, (1999), 425-430.

[9] P. J. Davis, *Interpolation and Approximation*, Dover, 1963.

[10] R. L. Devaney, *Chaotic Dynamical Systems*, Addison-Wesley, 1987.

[11] G.H. Hardy, Weierstrass's non-differentiable function, Trans. Amer. Math. Soc., **17**, (1916) 301-325.

[12] K. Kiesswetter. A simple example of a function, which is everywhere continuous and nowhere differentiable. In: *Classics on Fractals*, Edgar, G. A. (ed.), Addison-Wesley, 1993.

[13] B. Mandelbrot. *The Fractal Geometry of Nature*, Freeman, 1982.

[14] M.A. Navascués, Fractal polynomial interpolation. Z. Anal. Anwendungen, **24(2)**, (2005), 401-418.

[15] M.A. Navascués, Fractal trigonometric approximation. Electron. Trans. Numer. Anal., **20**, (2005), 64-74.

[16] M. A. Navascués, M. V. Sebastián, Generalization of Hermite functions by fractal interpolation, J. Approx. Theory, **131(1)**, (2004), 19-29.

[17] M.A. Navascués, M.V. Sebastián. Fitting curves by fractal interpolation: an application to the quantification of cognitive brain processes. In: *Thinking in Patterns: Fractals and Related Phenomena in Nature*. Novak, M.M.(ed.), World Sci., 2004.

[18] M. A. Navascués, M. V. Sebastián, Error bounds for affine fractal interpolation, Math. Ineq. Appl., **9(2)**, (2006), 273-288.

[19] K. Weierstrass. On continuous functions of a real argument that do not have a well-defined differential quotient. In: *Classics on Fractals*, Edgar, G. A. (ed.), Addison-Wesley, 1993.

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