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# Dynamical Properties for a Fourth Order Rational Difference Equation[1]

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### Abstract

In this paper, the rule of trajectory structure of a fourth-order rational difference equation is clearly drawn out. The lengths of positive and negative semicycles of nontrivial solutions to successively occur are found to occur periodically with prime period 15, i.e., the rule is $\cdots$, $4^-, 3^+, 1^-, 2^+, 2^-$, $1^+, 1^-, 1^+$, $4^-, 3^+, 1^-, 2^+, 2^-$, $1^+, 1^-, 1^+$, $4^-, 3^+, 1^-$, $2^+, 2^-$, $1^+, 1^-, 1^+$, $\cdots$. By utilizing the rule, the positive equilibrium point of the equation is verified to be globally asymptotically stable. rational difference equations.

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## 1. INTRODUCTION

It is extremely difficult to understand thoroughly the global behaviors of solutions of rational difference equations although they have simple forms . One can refer to [1-12], especially [1, 2] for examples to illustrate this.

The study of rational difference equations of order greater than one is quite challenging and rewarding because some prototypes for the development of the basic theory of the global behavior of nonlinear difference equations of order greater than one come from the results for rational difference equations. For this, see, for example, the papers in the journal of " Advances in Difference Equations " and the references cited therein. Furthermore, there have not been any effective general methods to deal with the global behavior of rational difference equations of order greater than one so far. Therefore, the study of rational difference equations of order greater than one is worth further consideration.

Recently, S. Kalabu*ăić* and M. R. S. Kulenovi *ć* [3] considered the rate of convergence of solutions of the following second - order rational difference equation

$$x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-1}}{A+Bx_n+Cx_{n-1}},\quad n=0,1,\cdots, \tag{$E_1$}$$

with nonnegative parameters $\alpha,\beta,\gamma,A,B,C$ and nonnegative initial conditions $x_{-1},x_0$.

M. R. S. Kulenovi*ć* et al [4] investigated the global behavior of solutions of the following second - order rational difference equation

$$x_{n+1}=\frac{\alpha+\beta x_n}{A+Bx_n+Cx_{n-1}},\quad n=0,1,\cdots, \tag{$E_2$}$$

where the parameters $\alpha,\beta,A,B,C$ and the initial conditions $x_{-1},x_0$ are nonnegative.

Tim Nesemann [5] utilized the **Strong Negative Feedback Property** of [6] to study the global asymptotic stability of the following third - order rational difference equation

$$x_{n+1} = \frac{x_{n-1} + x_n x_{n-2}}{x_n x_{n-1} + x_{n-2}}, \quad n = 0, 1, \cdots, \tag{$E_3$}$$

where the initial values $x_{-2}, x_{-1}, x_0 \in (0,\ \infty)$.

Youhui Su and Wantong Li [7] investigated the global behavior of a higher order rational difference equation.

Xiaofan Yang, et al, [8, 9] also studied global asymptotic stability of rational difference equations with higher order.

In the present paper we consider the following fourth-order rational difference equation

$$x_{n+1} = \frac{x_{n-2}x_{n-3} + a}{x_{n-2} + x_{n-3}}, \quad n = 0, 1, 2, \cdots, \tag{1}$$

where $a \in [0, \infty)$ and the initial values $x_{-3}, x_{-2}, x_{-1}, x_0 \in (0,\ \infty)$. Our main aim is to use the different methods from the known literature, such as [1-9], to investigate the rule of trajectory structure of solutions and global asymptotic stability of positive equilibrium point of equation (1).

Obviously, every solution $\{x_n\}_{n=-3}^{\infty}$ of Eq.(1) is positive, that is, $x_n > 0, n = -3, -2, \cdots$.

When $a = 0$, it is clear from Eq.(1) that $0 < x_{n+1} < x_{n-2}, n = 0, 1, \cdots$. So, three subsequences $\{x_{3n}\}_{n=0}^{\infty}$, $\{x_{3n+1}\}_{n=-1}^{\infty}$ and $\{x_{3n+2}\}_{n=-1}^{\infty}$ of solution $\{x_n\}_{n=-3}^{\infty}$ of Eq.(1) are monotonically decreasing with lower bound 0. Hence, the limits $\lim_{n\to\infty} x_{3n}$, $\lim_{n\to\infty} x_{3n+1}$ and $\lim_{n\to\infty} x_{3n}$ exit and are finite. Denote by $L$, $M$ and $N$ respectively. Taking limit on both sides of $x_{3n+1}(x_{3n-2} + x_{3n-3}) = x_{3n-2}x_{3n-3}$, $x_{3n+2}(x_{3n-1} + x_{3n-2}) = x_{3n-1}x_{3n-2}$ and $x_{3n+3}(x_{3n} + x_{3n-1}) = x_{3n}x_{3n-1}$, we have

$$M(L + M) = LM,\ N(M + N) = MN \ \text{ and } \ L(L + N) = LN. \tag{2}$$

Solving the three equations gives rise to $L = M = N = 0$. That is to say, $\lim_{n\to\infty} x_n = 0$. Thus, in the sequel, we shall assume that $a > 0$.

The change of variables $x_n = \sqrt{a}\ y_n$ reduces Eq. (1) to the difference equation

$$y_{n+1} = \frac{y_{n-2}y_{n-3} + 1}{y_{n-2} + y_{n-3}}, \quad n = 0, 1, 2, \cdots, \tag{3}$$

where the initial values $y_{-3}, y_{-2}, y_{-1}, y_0 \in (0,\ \infty)$.

Here, for convenience of readers, we give some corresponding definitions, also review some results which will be useful in our investigation of the behavior of solutions of Eq.(3). Let $I$ be some interval of real numbers and let $f : I \times I \to I$ be a continuously differentiable function.

Then for every group of initial conditions $x_{-3}, x_{-2}, x_{-1}, x_0 \in I$, the difference equation

$$x_{n+1} = f(x_{n-2},\ x_{n-3}), \quad n = 0, 1, 2, \cdots, \tag{4}$$

has a unique solution $\{x_n\}_{n=-3}^{\infty}$.

A point $\bar{x}$ is called an equilibrium point of (4) if $\bar{x} = f(\bar{x},\ \bar{x})$, i.e., $x_n = \bar{x}$, for $n \geq 0$, is a solution of Eq.(4), or, equivalently, $\bar{x}$ is a fixed point of $f$.

**Definition 1.1.** Let $\bar{x}$ be an equilibriun point of Eq. (4).

(a) The equilibrium $\bar{x}$ is called stable if, for every $\epsilon > 0$, there exists $\delta > 0$ such that if $x_{-3}, x_{-2}, x_{-1}, x_0 \in I$ and $|x_{-3}-\bar{x}|+|x_{-2}-\bar{x}|+|x_{-1}-\bar{x}|+|x_0-\bar{x}| < \delta$, then $|x_n - \bar{x}| < \epsilon$ for all $\geq 1$.

(b) The equilibrium $\bar{x}$ is called locally asymptotically stable if it is stable and if there exists $\gamma > 0$ such that if $x_{-3}, x_{-2}, x_{-1}, x_0 \in I$ and $|x_{-3} - \bar{x}| + |x_{-2} - \bar{x}| + |x_{-1} - \bar{x}| + |x_0 - \bar{x}| < \gamma$, then $\lim_{n\to\infty} x_n = \bar{x}$.

(c) The equilibrium $\bar{x}$ is called a global attractor if

$$\lim_{n\to\infty} x_n = \bar{x} \quad \text{for any} \quad x_{-3}, x_{-2}, x_{-1},\ x_0 \in I.$$

(d) The equilibrium $\bar{x}$ is called globally asymptotically stable if it is stable and is a global attractor.

(e) The equilibrium $\bar{x}$ is called unstable if it is not stable.

(f) The equilibrium $\bar{x}$ is called a repeller if there exists $\gamma > 0$ such that for $x_{-3}, x_{-2}, x_{-1}, x_0 \in I$ and $|x_{-3} - \bar{x}| + |x_{-2} - \bar{x}| + |x_{-1} - \bar{x}| + |x_0 - \bar{x}| < \gamma$, there exists $N \geq -3$ such that $|x_N - \bar{x}| \geq \gamma$.

Clearly, a repeller is an unstable equilibrium.

Let

$$p = \frac{\partial f(\bar{x},\ \bar{x})}{\partial u} \qquad \text{and} \qquad q = \frac{\partial f(\bar{x},\ \bar{x})}{\partial v},$$

where $f(u,\ v)$ is the function in Eq. (4) and $\bar{x}$ is an equilibrium of the equation. Then the equation

$$y_{n+1} = py_{n-2} + qy_{n-3}, \qquad n = 0, 1, \cdots \tag{5}$$

is called the linearized equation associated with Eq. (4) about the equilibrium point $\bar{x}$.

DEFINITION 1.2. The length of a semicycle is the number of the total terms contained in it.

DEFINITION 1.3. A solution $\{x_n\}_{n=-3}^{\infty}$ of equation (1) is said to be eventually trivial if $x_n$ is eventually equal to $\bar{x} = 1$. Otherwise, the solution is said to be nontrivial.

For the other concepts in this paper, see [1, 2].

It is easy to see that Eq.(3) has a unique positive equilibrium $\bar{x} = 1$. Eq.(3) is interesting in its own right. To the best of our knowledge, whereas, Eq.(3) has not been investigated so far. Therefore, theoretically, it is meaningful to study the qualitative properties of Eq.(3).

Moreover, if we rewrite Eq.(3) as Eq.(4), then, correspondingly, $f(x,\ y) = \frac{xy+1}{x+y}$, which, obviously, does not satisfy the monotonic nature with respect to $x$ or $y$. Thus, it is very difficult to apply the known results obtained in the literature, such as [1-9], to Eq.(3). Just as this, it is necessary to study the qualitative properties of Eq.(3).

## 2. SEVERAL LEMMAS

In order to simplifying the proof of our main result, we need the following lemmas.

LEMMA 2.1 A positive solution $\{x_n\}_{n=-3}^{\infty}$ of Eq.(3) is eventually equal to 1 if and only if

$$(x_{-3} - 1)(x_{-2} - 1)(x_{-1} - 1)(x_0 - 1) = 0. \tag{6}$$

**Proof.** Assume that (6) holds. Then according to Eq.(3), it is easy to see that the following conclusion is true:

(a) if $x_{-3} = 1$, then $x_n = 1$ for $n \geq 7$;

(b) if $x_{-2} = 1$, then $x_n = 1$ for $n \geq 4$;

(c) if $x_{-1} = 1$, then $x_n = 1$ for $n \geq 5$;

(d) if $x_0 = 1$, then $x_n = 1$ for $n \geq 6$.

Conversely, assume that

$$(x_{-3} - 1)(x_{-2} - 1)(x_{-1} - 1)(x_0 - 1) \neq 0. \tag{7}$$

Then we show $x_n \neq 1$ for any $n \geq 1$. For the sake of contradiction, assume that for some $N \geq 1$ ,

$$x_N = 1 \qquad \text{and that} \quad x_n \neq 1 \quad \text{for} \quad -3 \leq n \leq N-1. \tag{8}$$

Clearly,

$$1 = x_N = \frac{x_{N-2}x_{N-3}+1}{x_{N-2}+x_{N-3}},$$

which implies $x_{N-2} = 1$ or $x_{N-3} = 1$. This contradicts (8).

**Remark 2.1** If the initial conditions do not satisfy the equality (6), then $x_n \neq 1$ for $n \geq -3$ and $x_n \neq x_{n-3}$ for $n \geq 0$.

LEMMA 2.2 Let $\{x_n\}_{n=-3}^{\infty}$ be a positive solution of Eq.(3) which is not eventually equal to 1. Then the following conclusions are valid:

(a) $(x_{n+1} - x_{n-2})(x_{n-2} - 1) < 0$, for $n \geq 0$ ;

(b) $(x_{n+1} - x_{n-3})(x_{n-3} - 1) < 0$, for $n \geq 0$ ;

(c) $(x_{n+1} - 1)(x_{n-2} - 1)(x_{n-3} - 1) > 0$, for $n \geq 0$.

**Proof** In view of Eq.(3), we obtain

$$x_{n+1} - x_{n-2} = \frac{(1-x_{n-2})(1+x_{n-2})}{(x_{n-2}+x_{n-3}}, \quad n = 0, 1, 2, \cdots ,$$

and

$$(x_{n+1}-1)(x_{n-2}-1)(x_{n-3}-1) = \frac{(x_{n-2}-1)^2(x_{n-3}-1)^2}{x_{n-2}+x_{n-3}}, \quad n = 0, 1, 2, \cdots .$$

From this, the inequalities (a) and (c) follow. The proof of (b) is similar to that of (a). The proof is over.

## 3. MAIN RESULTS AND THEIR PROOFS

First we analyze the structure of the semicycles of nontrivial solutions of equation (1). Here, we confine us to consider the situation of the strictly oscillatory solution of equation (3).

THEOREM 3.1 Let $\{x_n\}_{n=-3}^{\infty}$ be a strictly oscillatory solution of equation (3). Then the rule for the lengths of positive and negative semicycles of this solution to successively occur is $\cdots, 4^-, 3^+, 1^- 2^+, 2^-, 1^+, 1^-, 1^+, 4^-, 3^+, 1^-, 2^+, 2^-, 1^+, 1^-, 1^+, 4^-, 3^+, 1^-, 2^+, 2^-, 1^+, 1^-, 1^+, \cdots$.

**Proof** By Lemma 2.2 (c), one can see that the length of a negative semicycle is not larger than 4, whereas, the length of a positive semicycle is at

most 3. Based on the strictly oscillatory character of the solution, we see, for some integer $p \geq 0$ , one of the following four cases must occur:
Case 1: $x_{p-3} > 1, x_{p-2} < 1, x_{p-1} < 1$ and $x_p < 1$;
Case 2: $x_{p-3} > 1, x_{p-2} < 1, x_{p-1} < 1$ and $x_p > 1$;
Case 3: $x_{p-3} > 1, x_{p-2} < 1, x_{p-1} > 1$ and $x_p > 1$;
Case 4: $x_{p-3} > 1, x_{p-2} < 1, x_{p-1} > 1$ and $x_p < 1$.

If Case 1 occurs, it follows from Lemma 2.2 (c) that $x_{p+1} < 1$, $x_{p+2} > 1$, $x_{p+3} > 1$, $x_{p+4} > 1, x_{p+5} < 1$, $x_{p+6} > 1$, $x_{p+7} > 1$, $x_{p+8} < 1, x_{p+9} < 1$, $x_{p+10} > 1$, $x_{p+11} < 1$, $x_{p+12} > 1$, $x_{p+13} < 1$, $x_{p+14} < 1$, $x_{p+15} < 1$, $x_{p+16} < 1$, $x_{p+17} > 1$, $x_{p+18} > 1$, $x_{p+19} > 1$, $x_{p+20} < 1$, $x_{p+21} > 1$, $x_{p+22} > 1$, $x_{p+23} < 1$, $x_{p+24} < 1$, $x_{p+25} > 1$, $x_{p+26} < 1$, $x_{p+27} > 1$, $x_{p+28} < 1$, $x_{p+29} < 1$, $x_{p+30} < 1$, $x_{p+31} < 1$, $x_{p+32} > 1$, $x_{p+33} > 1$, $x_{p+34} > 1$, $x_{p+35} < 1$, $x_{p+36} > 1$, $x_{p+37} > 1$, $x_{p+38} < 1$, $x_{p+39} < 1$, $x_{p+40} > 1$, $x_{p+41} < 1$, $x_{p+42} > 1$, $x_{p+43} < 1$, $x_{p+44} < 1$, $x_{p+45} < 1$, $x_{p+46} < 1$, $x_{p+47} > 1$, $x_{p+48} > 1$, $x_{p+49} > 1$, $x_{p+50} < 1$, $x_{p+51} > 1$, $x_{p+52} > 1$, $x_{p+53} < 1$, $x_{p+54} < 1$, $x_{p+55} > 1$, $x_{p+56} < 1$ , $x_{p+57} > 1, \cdots$.

It means that the rule for the lengths of positive and negative semicycles of the solution of equation (1) to successively occur is $\cdots$, $4^-, 3^+, 1^-, 2^+, 2^-$, $1^+, 1^-, 1^+$, $4^-, 3^+, 1^-, 2^+, 2^-$, $1^+, 1^-, 1^+$, $4^-, 3^+, 1^-, 2^+, 2^-$, $1^+, 1^-, 1^+, \cdots$.

If Case 2 occurs, then Lemma 2.2 (c) implies that $x_{p+1} < 1$, $x_{p+2} > 1$, $x_{p+3} < 1$, $x_{p+4} < 1$, $x_{p+5} < 1$, $x_{p+6} < 1$, $x_{p+7} > 1$, $x_{p+8} > 1, x_{p+9} > 1$, $x_{p+10} < 1$, $x_{p+11} > 1$, $x_{p+12} > 1$, $x_{p+13} < 1$, $x_{p+14} < 1$, $x_{p+15} > 1$, $x_{p+16} < 1$, $x_{p+17} > 1$, $x_{p+18} < 1$, $x_{p+19} < 1$, $x_{p+20} < 1$, $x_{p+21} < 1$, $x_{p+22} > 1$, $x_{p+23} > 1$, $x_{p+24} > 1$, $x_{p+25} < 1$, $x_{p+26} > 1$, $x_{p+27} > 1$, $x_{p+28} < 1$, $x_{p+29} < 1$, $x_{p+30} > 1$, $x_{p+31} < 1$, $x_{p+32} > 1$, $x_{p+33} < 1$, $x_{p+34} < 1$, $x_{p+35} < 1$, $x_{p+36} < 1$, $x_{p+37} > 1$, $x_{p+38} > 1$, $x_{p+39} > 1$, $x_{p+40} < 1$, $x_{p+41} > 1$, $x_{p+42} > 1$, $x_{p+43} < 1$, $x_{p+44} < 1$, $x_{p+45} > 1$, $x_{p+46} < 1$, $x_{p+47} > 1$, $\cdots$. This shows the rule for the numbers of terms of positive and negative semicycles of the solution of equation (1) to successively occur still is $\cdots$, $4^-, 3^+, 1^-, 2^+, 2^-$, $1^+, 1^-, 1^+$, $4^-, 3^+, 1^-, 2^+, 2^-$, $1^+, 1^-, 1^+$, $4^-, 3^+, 1^-, 2^+, 2^-$, $1^+, 1^-, 1^+, \cdots$.

When Case 3 or Case 4 happens, a similar deduction leads to that $x_{p+1} < 1$, $x_{p+2} < 1$, $x_{p+3} > 1$, $x_{p+4} < 1, x_{p+5} > 1$, $x_{p+6} < 1$, $x_{p+7} < 1$, $x_{p+8} < 1, x_{p+9} < 1$, $x_{p+10} > 1$, $x_{p+11} > 1$, $x_{p+12} > 1$, $x_{p+13} < 1$, $x_{p+14} > 1$, $x_{p+15} > 1$, $x_{p+16} < 1$, $x_{p+17} < 1$, $x_{p+18} > 1$, $x_{p+19} < 1$, $x_{p+20} < 1$, $x_{p+21} < 1$, $x_{p+22} < 1$, $x_{p+23} < 1$, $x_{p+24} < 1$, $x_{p+25} > 1$, $x_{p+26} > 1$ , $x_{p+27} > 1, \cdots$, or $x_{p+1} < 1$, $x_{p+2} < 1$,

$x_{p+3} < 1$, $x_{p+4} > 1, x_{p+5} > 1$, $x_{p+6} > 1$, $x_{p+7} < 1$, $x_{p+8} > 1, x_{p+9} > 1$, $x_{p+10} < 1$, $x_{p+11} < 1$, $x_{p+12} > 1$, $x_{p+13} < 1$, $x_{p+14} > 1$, $x_{p+15} < 1$, $x_{p+16} < 1$, $x_{p+17} < 1$, $x_{p+18} < 1$, $x_{p+19} > 1$, $x_{p+20} > 1$, $x_{p+21} > 1$, $x_{p+22} < 1$, $x_{p+23} > 1$, $x_{p+24} > 1$, $x_{p+25} < 1$, $x_{p+26} < 1$ , $x_{p+27} > 1$, $x_{p+28} < 1$, $x_{p+29} > 1$, $x_{p+30} < 1$, $x_{p+31} < 1$, $x_{p+32} < 1$, $x_{p+33} < 1$, $x_{p+34} > 1$, $x_{p+35} > 1$, $x_{p+36} > 1$, $x_{p+37} < 1$, $x_{p+38} > 1$, $x_{p+39} > 1$, $x_{p+40} < 1$, $x_{p+41} < 1$, $x_{p+42} > 1$, $x_{p+43} < 1, x_{p+44} > 1$, $\cdots$.

Thus, the same regulation is valid for the lengths of positive and negative semicycles which occur successively. The proof is complete.

**Remark 3.1** It is known to all that the four cases in the proof of Theorem 3.1 are caused by the perturbation of the initial around the equilibrium point. So, the theorem 3.1 actually indicates that the perturbation of the initial values may lead to the variation of the trajectory structure rule for the solutions of equation (3).

We are now in a position to state the second main result in this note.

**Theorem 3.2** Assume that $a \in [0, \infty)$. Then the positive equilibrium of equation (3) is globally asymptotically stable.

**Proof** We must prove that the positive equilibrium point $\bar{x}$ of equation (3) is both locally asymptotically stable and globally attractive. The linearized equation of equation(3) about the positive equilibrium $\bar{x} = 1$ is

$$y_{n+1} = 0 \cdot y_n + 0 \cdot y_{n-1} + 0 \cdot y_{n-2} + 0 \cdot y_{n-3}, \quad n = 0, 1, \cdots .$$

By virtue of [2, Remark 1. 3. 7 ], $\bar{x}$ is locally asymptotically stable. It remains to verify that every positive solution $\{x_n\}_{n=-3}^{\infty}$ of equation (3) converges to 1 as $n \to \infty$. Namely, we want to prove

$$\lim_{n\to\infty} x_n = \bar{x} = 1. \tag{9}$$

If the initial values of the solution satisfy (6), then Lemma 1 says the solution is eventually equal to 1 and of course, (9) holds. Therefore, we assume in the following that the initial values of the solution do not satisfy (6). Then, by Remark 2.1 we know, for any solution $\{x_n\}$ of equation (3), $x_n \neq 1$ for $n \geq -3$ and $x_n \neq x_{n-3}$ for $n \geq 0$.

If the solution is nonoscillatory about the positive equilibrium point $\bar{x}$ of equation (1), then we know from Lemma 2.2 (c) that the solution is actually an eventually positive one. According to Lemma 2.2 (a), we see that $\{x_{3n}\}$, $\{x_{3n-1}\}$ and $\{x_{3n-2}\}$ are eventually decreasing and bounded from the below by the constant 1. So, the limits

$$\lim_{n\to\infty} x_{3n} = L, \quad \lim_{n\to\infty} x_{3n+1} = M \quad \text{and} \quad \lim_{n\to\infty} x_{3n+2} = N$$

exist and are finite. Noting

$$x_{3n+1} = \frac{x_{3n-2} + x_{3n-3} + 1}{x_{3n-2} + x_{3n-3}}, \quad x_{3n+2} = \frac{x_{3n-1} + x_{3n-2} + 1}{x_{3n-1} + x_{3n-2}}$$

and

$$x_{3n+3} = \frac{x_{3n} + x_{3n-1} + 1}{x_{3n} + x_{3n-1}},$$

taking the limits on both sides of the above equations, we obtain

$$M = \frac{ML+1}{L+M}, \quad N = \frac{NM+1}{N+M}, \quad \text{and} \quad L = \frac{LN+1}{N+L}.$$

Solving these equations gives rise to $L = M = N = 1$, which shows (5) is true.

Thus, it suffices to prove that (5) holds for the solution to be strictly oscillatory.

Consider now $\{x_n\}$ to be strictly oscillatory about the positive equilibrium point $\bar{x}$ of equation (1). From Theorem 3.1, one understands that the rule for the lengths of positive and negative semicycles which occur successively is $\cdots$, $4^-, 3^+, 1^-, 2^+, 2^-,\ 1^+, 1^-, 1^+,\ 4^-, 3^+, 1^-, 2^+, 2^-,\ 1^+, 1^-, 1^+,\ 4^-, 3^+, 1^-, 2^+, 2^-,$ $1^+, 1^-, 1^+, \cdots$.

For simplicity, we denote by $\{x_p, x_{p+1}, x_{p+2}, x_{p+3}\}^-$ the terms of a negative semicycle of length four, followed by $\{x_{p+4}, x_{p+5}, x_{p+6}\}^+$ a positive semicycle with length three, then a negative semicycle $\{x_{p+7}\}^+$ and a positive semicycle $\{x_{p+8}, x_{p+9}\}^-$, and so on. Namely, the rule for the positive and negative semicycles to occur successively can be periodically expressed as follows: $\{x_{p+15n}, x_{p+15n+1}, x_{p+15n+2}, x_{p+15n+3}\}^-$, $\{x_{p+15n+4}, x_{p+15n+5}, x_{p+15n+6}\}^+$, $\{x_{p+15n+7}\}^-$, $\{x_{p+15n+8}, x_{p+15n+9}\}^+$, $\{x_{p+15n+10}, x_{p+15n+11}\}^-$, $\{x_{p+15n+12}\}^+$, $\{x_{p+15n+13}\}^-$, $\{x_{p+15n+14}\}^+$, $n = 0, 1, \cdots$.

Then the following results can be easily observed:

(i) $x_{p+15n} < x_{p+15n+3} < x_{p+15n+7} < x_{p+15n+11} < x_{p+15n+15}$;

(ii) $x_{p+15n} < x_{p+15n+3} < x_{p+15n+7} < x_{p+15n+10} < x_{p+15n+13} < x_{p+15n+16} < 1$;

(iii) $x_{p+15n} < x_{p+15n+13} < x_{p+15n+17} < 1$;

(iv) $x_{p+15n+12} < x_{p+15n+8} < x_{p+15n+4}$; $x_{p+15n+12} < x_{p+15n+9} < x_{p+15n+6}$;

(v) $x_{p+15n+6}x_{p+15n+3} < 1$; $x_{p+15n+4}x_{p+15n} < 1$.

Actually, the inequalities (i), (ii),(iii)and (iv) can be directly obtained from Lemma 2.2 (a) and (b). The observations

$$x_{p+15n+6} = \frac{x_{p+15n+3}x_{p+15n+2}+1}{x_{p+15n+3}+x_{p+15n+2}} < \frac{1}{x_{p+15n+3}}$$

and

$$x_{p+15n+4} = \frac{x_{p+15n+1}x_{p+15n}+1}{x_{p+15n+1}+x_{p+15n}} < \frac{1}{x_{p+15n}}$$

indicate that (v) is true.

Now, it follows from (i) that $\{x_{p+15n}\}_{n=0}^{\infty}$ is increasing with upper bound 1. So, the limit $\lim_{n\to\infty} x_{p+15n} = L$ exists and is finite. Furthermore, $0 < L \leq 1$. Accordingly, by using (i) again, we obtain $\lim_{n\to\infty} x_{p+15n+3} = \lim_{n\to\infty} x_{p+15n+7} = \lim_{n\to\infty} x_{p+15n+11} = L$. Combining (iv) and (v), we have

$$1 < x_{p+15n+12} < x_{p+15n+8} < x_{p+15n+4} < \frac{1}{x_{p+15n}} \tag{10}$$

and

$$1 < x_{p+15n+12} < x_{p+15n+9} < x_{p+15n+6} < \frac{1}{x_{p+15n+3}}. \tag{11}$$

It is easy to know from Eq.(3) that

$$x_{p+15n+12}(x_{p+15n+15} - x_{p+15n+11}) = 1 - x_{p+15n+15}x_{p+15n+11}. \tag{12}$$

(10) implying the boundedness of $x_{p+15n+12}$ and taking limits on both sides of (12) produce $1 - L^2 = 0$, which reads $L = 1$.

Hence, one can see from (ii) and (iii) that

$$\lim_{n\to\infty} x_{p+15n+16} = \lim_{n\to\infty} x_{p+15n+17} = \lim_{n\to\infty} x_{p+15n+10} = \lim_{n\to\infty} x_{p+15n+13} = 1.$$

Thereout, $\lim_{n\to\infty} x_{p+15n+14} = \frac{x_{p+15n+11}x_{p+15n+10}+1}{x_{p+15n+11}+x_{p+15n+10}} = 1$ and $\lim_{n\to\infty} x_{p+15n+16} = \lim_{n\to\infty} x_{p+15n+2} = 1$. This, together with $x_{p+15n+5} = \frac{x_{p+15n+2}x_{p+15n+1}+1}{x_{p+15n+2}+x_{p+15n+5}}$, shows that $\lim_{n\to\infty} x_{p+15n+5} = 1$.

It is easy to derive from (10) and (11) that

$$\lim_{n\to\infty} x_{p+15n+12} = \lim_{n\to\infty} x_{p+15n+9} = \lim_{n\to\infty} x_{p+15n+8} = \lim_{n\to\infty} x_{p+15n+6} = \lim_{n\to\infty} x_{p+15n+4} = 1.$$

Up to this, we have shown $\lim_{n\to\infty} x_{p+15n+k} = 1, k = 1, 2 \cdots, 15$. So, the proof for Theorem 3.2 is complete.

# REFERENCES

[1] R. P. Agarwal, Difference Equations and Inequalities, Marcel Dekker, New York, 1992(1st edition), 2000(2nd edition).

[2] V. L. Kocic, G.Ladas, Global behavior of nonlinear difference equations of higher order with applications, Kluwer Academic Publishers, Dordrecht, 1993.

[3] S. Kalabu*ǎić* and M. R. S. Kulenovi*ć*, Rate of convergence of solutions of rational difference equation of second order, Advances in Difference Equations, 2004, Vol.2004(2), 121-139.

[4] M. R. S. Kulenovi*ć*, G. Ladas, L. F. Martins and I. W.Rodrigues, The Dynamics of $x_{n+1} = \frac{\alpha+\beta x_n}{A+Bx_n+Cx_{n-1}}$: Facts and Conjectures, Computers Math. Appl., 2003, 45: 1087-1099.

[5] Tim Nesemann, Positive nonlinear difference equations: some results and apllications, Nonlinear Analysis, 2001, Vol.47: 4707-4717.

[6] A. M. Amleh, N. Kruse, G. Ladas, On a class of difference equations with strong negative feedback, J. Difference Equa. Appl., 1999, Vol.5(4), 497-515.

[7] Youhui Su, Wantong Li, Global attractivity of a higher order nonlinear difference equation. J. Difference Equa. Appl., 2005, Vol.11(10), 947-958.

[8] Xiaofan Yang, Hongjian Lai, David J. Evans, Graham M. Megson, Global asymptotic stability in a rational recursive sequence. Appl.Math. Comput., 2004, 158(3), 703-716.

[9] Xiaofan Yang, Jianqiu Cao,Graham M. Megson, Global asymptotic stability in a class of Putnam-type equations. Nonlinear Anal., 2006, 64(1), 42-50.

[10] Xianyi Li, Deming Zhu, Global asymptotic stability of a nonlinear recursive sequence, Appl. Math. Letters, 2004, Vol.17(7), 833-838.

[11] Xianyi Li, Deming Zhu, Global asymptotic stability in a rational equation, J. Difference Equa. Appl., 2003, Vol.9 (9), 833-839.

[12] Xianyi Li, Deming Zhu, Global asymptotic stability for two recursive difference equations, Appl. Math. Comput., 2004, Vol.150 (2), 481-492.

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