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Dense Subspaces of the Fock Space[1]

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Abstract. Let f, g be in the entire function ring $Hol(\mathbb{C})$. We say that $f \preceq g$ if there exist $M > 0$ and $r > 0$ such that $|f(z)| \leq M|g(z)|$ whenever $|z| > r$. Let F denote the Fock space. For $f \in F$, define a subspace Δ_f of F by $\Delta_f = \{g \in F|\ g \preceq f\}$. In this note, we study the basic properties of Δ_f, especially, we proved that if the order of f is less than 2 and the zero set of f is infinite then Δ_f is dense in F. Some applications to the structure of separable Hilbert spaces are also given.

1. Introduction

The Fock space, or so-called the Segal-Bargmann space, is the analog of the Bergman space in the context of the complex plane $\mathbb{C}$. It is a Hilbert space consisting of entire functions in $\mathbb{C}$. Let

$$d\mu(z) = \frac{1}{2\pi} e^{-|z|^2} dv(z)$$

be the Gaussian measure on $\mathbb{C}$ (here dv is the ordinary Lebesgue measure). The Fock space F by definition, is the space of all μ-square-integrable entire functions on $\mathbb{C}$. It is easy to check that F is a closed subspace of $L^2(\mathbb{C})$ with the reproducing kernel function $K_\lambda(z) = e^{\bar{\lambda} z}$.

For general background on the Fock space one may consult [DG] and the references therein. We mention the work in [Jan1, Jan2, JS], where Toeplitz operators on the Fock space were investigated and the work in [CH, G, GH, GZ, HH] where the structure of quasi-invariant subspaces of Fock space were studied. We also mention the work of Cou hei Izuchi [Izu] where a complete description of cyclic vectors of the Fock space was given.

In order to study the structure of reproducing Hilbert space over the complex plane, a conception was introduced in the entire function ring $Hol(\mathbb{C})$, that is the so called partial order relation " $\preceq$ " [CGH]. Let f, g be in the entire

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function ring $Hol(\mathbb{C})$. We say that $f \preceq g$ if there exist $M > 0$ and $r > 0$ such that $|f(z)| \leq M|g(z)|$ whenever $|z| > r$.

For each $f \in F$, we define

$$\Delta_f = \{g | g \in F, \, g \preceq f\},$$

Just as we show in section 2, Δ_f is a linear subspace of F. We concentrate this subspace for two reasons. In one hand we expect it helps one understand function theoretic properties of f and structure of reproducing kernel function space. On the other hand, we expect it shine some light on the structure of general Hilbert spaces.

In this note, we study the elementary properties of Δ_f. It is well know that the polynomial ring is dense subspace of F for $\{z^n\}_{n=0}^{\infty}$ forms an orthogonal basis of F. Since $K_\lambda(z) = e^{\overline{\lambda}z}$, $\lambda \in \mathbb{C}$ are reproducing kernels of F. For $f \in F$, if $\langle f, k_\lambda \rangle = 0$, then $f = 0$. Thus $\{e^{\overline{\lambda}z} | \lambda \in \mathbb{C}\}$ is dense in F. It turns out that if f has order less than 2 and the zero set of f is infinite, then Δ_f is dense subspace of F. Two applications are given to exhibit the properties of dense subspaces of separable Hilbert spaces.

2. Preliminaries and basic properties of Δ_f

We begin with a lemma which comes from proposition 2.5 in [CGH].

Lemma 2.1 *Let f and g be in $Hol(\mathbb{C})$. Then $f \preceq g$ if and only if there exist polynomial p and q with $deg p \leq deg q$ such that*

$$\frac{f}{g} = \frac{p}{q},$$

where "deg" denote the degree of the polynomial.

The following proposition exhibits some distinctive properties of Δ_f.

Proposition 2.1 *Let $f \in F$ and $\Delta_f = \{g | \, g \in F, g \preceq f\}$. Then the following holds:*

1. *Δ_f is a linear subspace of F;*
2. *if both f and g are not equal to 0, then $\Delta_f \cap \Delta_g \neq \{0\}$ if and only if either $f \preceq g$ or $g \preceq f$ if and only if either $\Delta_f \subseteq \Delta_g$ or $\Delta_g \subseteq \Delta_f$;*
3. *if the zero set of f is finite, then Δ_f is closed.*

Proof. By the definition of $g \preceq f$, it is obvious that $g \in \Delta_f$ implies $\alpha f \in \Delta_f$, $\alpha \in \mathbb{C}$. To complete the proof of (1), it is enough to show that if $g_1, g_2 \in \Delta_f$ then $g_1 + g_2 \in \Delta_f$. Since $g_i \in \Delta_f$ implies there exists $r_i > 0$ and $M_i > 0$ such that $|g_i(z)| \leq M_i|f(z)|$ whenever $|z| > r_i$, $(i = 1, 2)$. Set $r = \max\{r_1, r_2\}$ and $M = \max\{M_1, M_2\}$. We get $|g(z) + h(z)| \leq 2M|f(z)|$ whenever $|z| > r$. Thus $g_1 + g_2 \in \Delta_f$.

We note that $f \preceq g$ if and only if $\Delta_f \subseteq \Delta_g$. If either $f \preceq g$ or $g \preceq f$ holds, it is obvious that $\Delta_f \cap \Delta_g \neq \{0\}$.

Assume that $0 \neq h(z) \in \Delta_f \cap \Delta_g$, then by Lemma 2.1 we have

$$\frac{h(z)}{f(z)} = \frac{p_1(z)}{q_1(z)} \text{ and } \frac{\mathrm{h(z)}}{\mathrm{g(z)}} = \frac{\mathrm{p_2(z)}}{\mathrm{q_2(z)}}$$

where p_1, p_2, q_1 and q_2 are polynomials such that $\deg p_1 \leq \deg q_1$ and $\deg p_2 \leq \deg q_2$. Hence we get

$$\frac{f(z)}{g(z)} = \frac{p_2(z)q_1(z)}{p_1(z)q_2(z)}.$$

By using Lemma 2.1 again, we get either $f \preceq g$ or $g \preceq f$, and this complete the proof of (2).

If the zero set of f is finite, then there exists $f_1 \in F$ without zero points in the complex plane and a polynomial p such that $f = f_1 p$. Assume that $g \preceq f$, then g necessarily has the form $g = qf_1$ where q is a polynomial such that $\deg q \leq \deg p$. Thus Δ_f has finite dimensional and hence is closed. This complete the proof. $\square$

From the above proposition, we know that when $Z(f) = \{z | f(z) = 0\}$ is finite, then Δ_f is closed. However, when $Z(f) = \{z | f(z) = 0\}$ is infinite, our result shows that Δ_f is not closed in general. In fact, Δ_f is dense in F when $\rho(f) < 2$.

We also need some conceptions and results concerning entire functions.

Let us recall that the order ρ of an entire function f is defined by

$$\rho = \limsup_{r \to \infty} \frac{\log \log M_f(r)}{\log r}$$

where $M_f(r) = \max_{|z|=r} |f(z)|$.

For $\rho < 1$, by the Hadamard theorem an entire function of order ρ has the form

$$f(z) = Cz^m \prod_{n=1}^{\omega} (1 - \frac{z}{a_n}), \ \ \omega \leq \infty$$

with

$$\sum_{n=1}^{\omega} \frac{1}{|a_n|} < \infty.$$

For $1 \leq \rho < 2$, by the Hadamard theorem, the Borel theorem (Theorem 3 of page 30 in [Lev]) and the theorem 1 of page 31 in [Lev] an entire function of order ρ has the form

$$f(z) = Ce^{az+b} z^m \prod_{n=1}^{\omega} (1 - \frac{z}{a_n}) e^{\frac{z}{a_n}}, \ \ \omega \leq \infty$$

with

$$\sum_{n=1}^{\omega} \frac{1}{|a_n|^2} < \infty.$$

We also note that if $f \in F$, then $\rho(f) \leq 2$. Let $\sigma(g)$ denote the type of an entire function g. Then the Fock space F contains all the entire function f with $\rho(f) < 2$ and all entire function g with $\rho(g) = 2$ and $\sigma(g) < \frac{1}{2}$. When $\rho(g) = 2$ and $\sigma(g) < \frac{1}{2}$ then g can be in F or opposite. We will use this facts without saying in this note.

We need the following lemma [Lev] to prove our main results.

Lemma 2.2 (the Borel estimate). *Let $G(u,p)$ be the Weierstrass primary factors. That is*

$$G(u,p) = \begin{cases} 1-u, & p=0, \\ (1-u)exp[u + \frac{u^2}{2} + \cdots + \frac{u^p}{p}], & p>0. \end{cases}$$

For $u \in \mathbb{C}$ the estimates

$$\log|G(u,p)| \leq 3e(2+\log p)\frac{|u|^{p+1}}{1+|u|}, \quad p>0$$

$$\log|G(u,0)| \leq \log(1+|u|)$$

are valid.

3. Main results and its application

Let $Z(f)$ denote zero set of f. The following is the main result of this note.

Theorem 3.1 *Let f be in F such that $\rho(f) < 2$ and $Z(f)$ is infinite. Then Δ_f is dense in F.*

Proof. We divide the proof into two cases according to the value of $\rho(f)$.

If $\rho(f) < 1$, then

$$f(z) = Cz^m \prod_{n=1}^{\infty}(1 - \frac{z}{a_n}) \text{ with } \sum_{n=1}^{\infty}\frac{1}{|a_n|} < \infty.$$

Let $g_n(z) = \prod_{k=n}^{\infty}(1-\frac{z}{a_k})$. Since

$$\frac{g_n(z)}{f(z)} = \frac{1}{Cz^m\prod_{k=1}^{n-1}(1-\frac{z}{a_k})},$$

we have $g_n \preceq f$ and hence $g_n \in \Delta_f$ by lemma 2.1. Since $(1+\frac{|z|}{|a_k|}) \leq e^{\frac{|z|}{|a_k|}}$, we get

$$\begin{aligned} |g_n(z)| &= |\prod_{k=n}^{\infty}(1-\frac{z}{a_k})| \\ &\leq \prod_{k=n}^{\infty}(1+\frac{|z|}{a_k}) \\ &\leq exp\{\sum_{k=n}^{\infty}\frac{|z|}{|a_k|}\}. \end{aligned}$$

Since $\sum_{n=1}^{\infty}\frac{1}{|a_n|}<\infty$, we set $\sum_{n=1}^{\infty}\frac{1}{|a_n|}=\gamma$. Then

$$|g_n(z)|\leq exp(\gamma|z|)$$

for all n and hence $|g_n(z)-1|^2\leq[(exp(2\gamma|z|)+2exp(\gamma|z|)+1]$. It is obvious that $exp(2\gamma|z|)+2exp(\gamma|z|)+1$ is μ-integrable in $\mathbb{C}$. Thus by the Lebesgue Control Convergence Theorem, we have

$$\begin{aligned}\lim_{n\to\infty}||g_n(z)-1||^2 &= \lim_{n\to\infty}\int_{\mathbb{C}}|g_n(z)-1|^2e^{-|z|^2}d\mu\\ &= \int_{\mathbb{C}}\lim_{n\to\infty}|g_n(z)-1|^2e^{-|z|^2}d\mu\\ &= 0\end{aligned}$$

Therefore we have proved that $1\in\overline{\Delta_f}$.

Let $g_n^1(z)=z\prod_{k=n+1}^{\infty}(1-\frac{z}{a_k})$. Similarly, one can show that $g_n^1(z)\in\Delta_f$ and $g_n^1(z)\overset{||\cdot||}{\to}z$ and hence $z\in\overline{\Delta_f}$. One can set $g_n^l(z)=z^l\prod_{k=n+l}^{\infty}(1-\frac{z}{a_k})$ to prove that $g_n^l(z)\overset{||\cdot||}{\to}z^l$ for $l=1,2,3,\cdots$. Since the linear span of $\{z^l\}_{l=0}^{\infty}$ are dense in F. We get that Δ_f is dense subspace of F.

If $1\leq\rho(f)<2$, then

$$f(z)=Ce^{az+b}z^m\prod_{n=1}^{\infty}(1-\frac{z}{a_n})e^{\frac{z}{a_n}}\ \text{ with }\ \sum_{n=1}^{\infty}\frac{1}{|a_n|^2}<\infty.$$

By lemma 2.1, we have

$$|(1-\frac{z}{a_n})e^{\frac{z}{a_n}}|\leq exp\{6e\frac{|\frac{z}{a_n}|^2}{1+|\frac{z}{a_n}|}\}.$$

Thus

$$|f(z)|\leq|Ce^{az+b}|exp\{6e\sum_{n=1}^{\infty}\frac{|z|^2}{|a_n|^2}\}.$$

Since $\sum_{n=1}^{\infty}\frac{1}{|a_n|^2}<\infty$, there exists $N>0$ such that $6e\sum_{n=N+1}^{\infty}\frac{1}{|a_n|^2}<\frac{1}{8}$. Let

$$g_n(z)=e^{az+b}\prod_{k=N+n}^{\infty}(1-\frac{z}{a_k})e^{\frac{z}{a_k}}.$$

Then $|g_n(z)|\leq|e^{az+b}|exp\{\frac{1}{8}|z|^2\}$ and

$$\frac{g_n(z)}{f(z)}=\frac{1}{z^m\prod_{k=1}^{N+n-1}(1-\frac{z}{a_k})}.$$

So $g_n\preceq f$ and hence $g_n\in\Delta_f$ by lemma 2.1. Thus

$$|g_n(z)-e^{az+b}|^2\leq|e^{2(az+b)}|exp\{\frac{1}{4}|z|^2\}+|e^{az+b}|exp\{\frac{1}{8}|z|^2\}+|e^{2(az+b)}|.$$

It is easy to check that the integration (with respect to the Gauss measure) of the right side of above inequality is finite. Again by the Lebesgue Control Convergence Theorem, we have

$$\begin{aligned}\lim_{n\to\infty}||g_n(z)-e^{az+b}||^2 &= \lim_{n\to\infty}\int_{\mathbb{C}}|g_n(z)-e^{az+b}|^2e^{-|z|^2}d\mu(z)\\ &= \int_{\mathbb{C}}\lim_{n\to\infty}|g_n(z)-e^{az+b}|^2e^{-|z|^2}d\mu(z)\\ &= 0.\end{aligned}$$

Therefore $e^{az+b}\in\overline{\Delta_f}$, Similar to the proof of the first part one can show that $e^{az+b}z^k\in\overline{\Delta_f}$ for $k=1,2,\cdots$. Since e^{az+b} is a cyclic vector in F [Izu]. We have proved that Δ_f is dense in F and this complete the proof. □

Remark *Let* $f(z)=\prod_{k=2}^{\infty}(1-\frac{z}{\sqrt{k}\log k})$. *Since* $\sum_{k=2}^{\infty}\frac{1}{(\sqrt{k}\log k)^{2-\delta}}$ *is divergence for each* $\delta>0$ *while* $\sum_{k=1}^{\infty}\frac{1}{(\sqrt{k}\log k)^2}$ *is convergence, we have* $\rho(f)=2$ *by the Borel Theorem. Let* M *be the integer such that* $\sum_{k=M}^{\infty}\frac{1}{(\sqrt{k}\log k)^2}<\frac{1}{4}$ *and* $h(z)=\prod_{k=M}^{\infty}(1-\frac{z}{k\log k})$. *Then the type of* h *less than* $1/4$ *and hence* $h\in F$. *Similar to the proof of the last part of theorem 3.1, one can show that* Δ_h *is a dense subspace of* F *such that* $\rho(h)=2$

We end our note with two applications of our results.

Corollary 3.1 *Let* H *be a separable Hilbert space. Then there exist a sequences of subspace* H_i *with* $H_{i+1}\subseteq H_i\subseteq H$ $(i=1,2,\cdots)$ *such that*

1. H_i *is dense in* H*,*
2. *dim* $H_i/H_{i+1}=1$.

Proof. Without loss of generality, we may consider H as F. For $n\in N$ let $f_n(z)=\prod_{k=n}^{\infty}(1-\frac{z}{k^2})$. Then the convergence exponent of the zero set of f_n is 0. So $\rho(f_n)=0$. Thus $f_n\in F$ and Δ_{f_n} is dense in F by theorem 3.1. Since $f_{n+1}\preceq f_n$ we have $\Delta_{f_{n+1}}\subseteq\Delta_{f_n}$ by proposition 2.1 and it is obvious that $dim\,\Delta_{f_n}/\Delta_{f_{n+1}}=1$. □

Corollary 3.2 *Let* H *be a separable Hilbert space. Then there exist dense subspaces* H_α $(\alpha\in[0,1])$ *of* H *such that* $H_\alpha\cap H_\beta=0$ *for* $\alpha,\beta\in[0,1]$ *and* $\alpha\neq\beta$.

Proof. For $\alpha\in[0,1]$ let $g_\alpha(z)=\prod_{k=1}^{\infty}(1-\frac{z}{n^2+\alpha})$. Then Δ_{g_α} is dense in F by theorem 3.1. and $\Delta_{g_\alpha}\cap\Delta_{g_\beta}=0$ by proposition 2.1. This complete the proof. □

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