

# The Schröder Equation and Some Elementary Functions

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## Abstract

We survey the Schröder equation  $f(\varphi(x)) = pf(x)$  and a set of additional assumptions on  $f : (0, \infty) \rightarrow \mathbb{R}$  in such special cases that result in a unique power function, logarithm function or exponential function. Two of our results will be slight improvements of the previously known characterizations of these functions.

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## 1 Introduction

On positive real numbers, power function and exponential function and logarithm function all are monotone, continuous, differentiable and integrable. A natural question arises: are there any other analytical properties in terms of

which it is possible to sort out these functions from each others? The answer: yes, for example, those that can be expressed using functional equations. For instance, *the Schröder equation*

$$f(\varphi(x)) = pf(x),$$

where  $\varphi$  is a function and  $p$  is a constant, is one of the most studied equations in functional analysis and also suitable for our purpose.

In this note, we shall survey some recent results on the characterizing of elementary functions using functional equations and slightly improve a couple of them. More precisely, given a monotone or continuous function  $f : (0, \infty) \rightarrow \mathbb{R}$ , we shall give necessary and sufficient conditions on  $f$  in terms of the Schröder and Böttcher equations so that  $f$  results in a unique power function, logarithm function or exponential function, respectively.

In [5], the authors show that if a strictly monotone function  $f : (0, \infty) \rightarrow \mathbb{R}$  satisfies for all  $x > 0$  the equations

$$f(px) = af(x), \quad f(qx) = bf(x),$$

where  $a, b, p, q > 0$  are constants so that  $r = \log q / \log p$  is irrational, then  $f(x) = x^r$ . We shall generalize this result by establishing in this note the following theorem.

**Theorem A.** *Let  $f : (0, \infty) \rightarrow \mathbb{R}$  be monotone or continuous, and let  $p, q \neq 1$  be positive real numbers such that  $p^m \neq q^n$  for all nonzero integers  $m$  and  $n$ , and let  $r$  be a fixed real number. If*

$$f(px) = p^r f(x) \quad \text{and} \quad f(qx) = q^r f(x) \tag{1}$$

*whenever  $x > 0$ , then  $f(x) = f(1)x^r$  for all  $x > 0$ .*

The continuous part of Theorem A also follows from a more general, and non-elementary, result of Matkowski (Theorem 2 in [7]).

Choosing  $r = 1$  or  $r = -1$ , Theorem A provides us with the conditions under which two quantities are directly or inversely proportional, respectively. A very recent and thorough survey on a set of different elementary conditions guaranteeing the direct or inverse proportionality is given in [3]. For example, it holds that if  $f : (0, \infty) \rightarrow \mathbb{R}$  is a differentiable function so that  $\lim_{x \rightarrow 0} f'(x)$  exists and  $f(px) = pf(x)$  for all  $x > 0$  and some positive constant  $p \neq 1$ , then

$f(x) = f(1)x$  for all  $x > 0$ . The same is true also if, instead of differentiability,  $f$  is supposed to be continuous and, in addition to that, subadditive or superadditive.

In our approach, Theorem A will play an essential role also in defining exponential and logarithm functions. By replacing (1) with the functional equations

$$h(x^p) = ph(x), \quad h(x^q) = qh(x) \quad (2)$$

and

$$h(px) = h(x)^p, \quad h(qx) = h(x)^q \quad (3)$$

and assuming again that  $p, q \neq 1$  are positive real numbers such that  $p^m \neq q^n$  for all nonzero integers  $m$  and  $n$ , we shall derive from Theorem A the following two consequences.

**Corollary B.** *If a monotone or continuous function  $h : (1, \infty) \rightarrow \mathbb{R}$  satisfies (2) whenever  $x > 1$ , then  $h(x) = h(e) \ln x$  for every  $x > 1$ .*

**Corollary C.** *If a monotone or continuous function  $h : (0, \infty) \rightarrow (0, \infty)$  satisfies (3) for all  $x > 0$ , then  $h(x) = h(1)^x$  for every  $x > 0$ .*

Also Corollary B provides us with a slight improvement of a certain classical result. The functional equations, given by Cauchy,  $h(xy) = h(x) + h(y)$  and  $h(x+y) = h(x)h(y)$  associated to the logarithm function and the exponential function, respectively, turn into (2) and (3) with  $p = q = 2$  by setting here  $y = x$ . Presumably it was Fubini, cf. [2], who first showed that the only function  $h : (1, \infty) \rightarrow \mathbb{R}$  that satisfies (2) with  $p = q = 2$  and the condition

$$\lim_{x \rightarrow 1^+} \frac{h(x)}{x-1} = 1$$

is the logarithm function. Our result is better in the sense that the analytical property which we assume is more elementary than in the one of Fubini.

In Kuczma's and Choczewski's and Ger's characterization of the exponential function (Theorem 10.2.2 and Remark 10.2.3 in [5]) the number of conditions on  $h$  is the same as in our Corollary C but two of them are different from those in our result. Also, our proof is a somewhat more elementary than theirs.

## 2 The proofs of A – C

*Proof Theorem A.* Let us begin by writing  $S = \{p^m q^n \mid m, n \in \mathbb{Z}\}$ . Now, if  $x \in S$ , then by (1) we have

$$f(x) = f(p^m q^n) = p^{mr} f(q^n) = p^{mr} q^{nr} f(1) = (p^m q^n)^r f(1) = f(1)x^r.$$

Hence, if  $r = 0$  and  $f$  is continuous or monotone, or if  $r \neq 0$  and  $f$  is continuous, then the claim follows if  $S$  is dense in  $(0, \infty)$ .

To that end, by taking logarithms of the elements of  $S$ , we notice that  $S$  is dense in  $(0, \infty)$  if and only if  $T = \{m + n \ln q / \ln p \mid m, n \in \mathbb{Z}\}$  is dense in the set of real numbers. Because  $p^m \neq q^n$  for all  $m, n \in \mathbb{Z} \setminus \{0\}$ , the number  $\ln q / \ln p$  must be irrational and, thus,  $T$  is dense in  $\mathbb{R}$ , see the Dirichlet's theorem in [1].

Let us suppose now that  $r \neq 0$  and  $f$  is monotone. Write  $c = f(1)$ . If  $c = 0$ , then  $f(x) = 0$  in the dense set  $S$ . Thus, by monotonicity,  $f(x) = 0$  for every  $x > 0$ , and the claim follows.

Next, assume that  $c, r > 0$ . Then  $f$  is increasing. Also, let us presume that there exists  $x_0 > 0$  such that  $f(x_0) = cx_0^r + d$  where  $d > 0$ . Since  $S$  is dense, there exist integers  $m$  and  $n$  such that  $x_1 = p^m q^n > x_0$  and

$$x_1 - x_0 < \frac{(d/c)^{\frac{1}{r}}}{C}, \quad (4)$$

where the constant  $C > 1$  will be determined later. But now also

$$\begin{aligned} f(x_1) &= cx_1^r = c(x_0 + (x_1 - x_0))^r < c\left(x_0 + \frac{(d/c)^{\frac{1}{r}}}{C}\right)^r \\ &= cx_0^r + d/C^r + C' \end{aligned}$$

where  $C'$  is the constant depending on  $c, C, r$  and  $x_0$  so that  $C \rightarrow \infty$  implies  $C' \rightarrow 0$  for any fixed  $x_0$ .

Therefore, by choosing  $C$  in (4) large enough we get  $f(x_1) < f(x_0)$  from the above calculation. This contradicts with the fact that  $f$  is increasing.

In the case  $d < 0$ , we choose  $x_1 < x_0$  and proceed analogously. If  $c, r < 0$ , then  $f$  is again increasing and the analogous argument holds. The cases  $c < 0 < r$  and  $r < 0 < c$  imply that  $f$  is decreasing but the similar argument as above can once again be constructed. Theorem A is now proven.

*Proof of Corollary B.* Let  $h : (1, \infty) \rightarrow \mathbb{R}$  be a monotone (or continuous) function satisfying (2) for  $x > 1$ . Define now the function  $f : (0, \infty) \rightarrow \mathbb{R}$  by setting  $f(x) = h \circ \exp(x) = h(e^x)$  for every  $x > 0$ . Then  $f$  is monotone (or, respectively, continuous) and, moreover,

$$h((e^x)^p) = ph(e^x) \Leftrightarrow f(px) = pf(x)$$

and

$$h((e^x)^q) = qh(e^x) \Leftrightarrow f(qx) = qf(x)$$

for every  $x > 0$ . Hence,  $h(e^x) = f(x) = f(1)x = h(e)x$  for every  $x > 0$ . This implies that  $h(x) = h(e) \ln x$  for every  $x > 1$ . The proof of Corollary B is complete.

*Proof of Corollary C.* Let  $h : (0, \infty) \rightarrow (0, \infty)$  be a monotone (or continuous) function satisfying (3) for  $x > 0$ . Again, we define a monotone (or, respectively, continuous) function  $f : (0, \infty) \rightarrow \mathbb{R}$  by writing  $f(x) = \ln \circ h(x) = \ln(h(x))$  for every  $x > 0$ . Then, for every  $x > 0$ ,

$$h(px) = h(x)^p \Leftrightarrow \ln(h(px)) = p \ln(h(x)) \Leftrightarrow f(px) = pf(x).$$

Similarly,  $h(qx) = h(x)^q \Leftrightarrow f(qx) = qf(x)$  whenever  $x > 0$ . Therefore, by Theorem A,

$$\ln(h(x)) = f(x) = f(1)x = \ln(h(1))x,$$

and thus,  $h(x) = h(1)^x$  for every  $x > 0$ . Corollary C is verified.

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