On Continuity of Generalized Riesz Potentials

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Abstract. This study establishes the theorem on continuity of generalized
Riesz potentials with non-isotropic kernels depending on \((\beta, \gamma)\)-distance.

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1. INTRODUCTION

The classical Riesz potentials and their generalizations have important ap-
plications in different problems of functions theory [1], [2], [3].
In this paper we consider the Riesz potentials with kernel depending on non-isotropic distance.
First we define this distance. Suppose that \(\beta_k, k = 1, 2, ..., n\) and \(\gamma\) are
positive numbers and define for \(x \in \mathbb{R}^n\) the generalized \((\beta, \gamma)\)-distance form
origin by formula

\[
\|x\|_{\beta, \gamma} = \left( |x_1|^{\frac{2\beta_1}{\gamma}} + ... + |x_n|^{\frac{2\beta_n}{\gamma}} \right)^{\frac{\gamma}{2n}}
\]

where \(\frac{1}{\beta} = \frac{1}{\beta_1} + ... + \frac{1}{\beta_n}\).
In the case of \(\gamma = 2, \beta_k = \frac{1}{\lambda_k}\) we have the well known \(\lambda\)-distance [3], [4] and
in the case of \(\gamma = 2, \beta_k = 2, k = 1, 2, ..., n\). \(\|x\|_{\beta, \gamma}\) is the Euclidean distance in
\(\mathbb{R}^n\).
Note also that for any \(t > 0\)

\[
\left( t^{\frac{2}{2\beta_1}} |x_1|^{\frac{2\beta_1}{\gamma}} + ... + t^{\frac{2}{2\beta_n}} |x_n|^{\frac{2\beta_n}{\gamma}} \right)^{\frac{\gamma}{2n}} = t^{\frac{\gamma}{2n}} \|x\|_{\beta, \gamma}
\]

that is the generalized \((\beta, \gamma)\)-distance has the homogeneity properties in the form
\[ \| t^{\frac{\alpha}{p}} x \|_{\beta, \gamma} = t^{\frac{\beta}{p}} \| x \|_{\beta, \gamma} \]

where \( t^{\frac{\alpha}{p}} = \left( t^{\frac{\alpha_{1}}{p_{1}}}, ..., t^{\frac{\alpha_{n}}{p_{n}}} \right). \)

The value \( \| x - y \|_{\beta, \gamma} \) we call the generalized using the inequality \((a + b)^m \leq 2^m (a^m + b^m), \ m > 1\). We obtain

\[ \| x + y \|_{\beta, \gamma} \leq 2^{\frac{2m}{\beta_{\max}}} \| x \|_{\beta, \gamma} + \| y \|_{\beta, \gamma}, \]

where \( \beta_{\max} = \max(\beta_1, ..., \beta_n) \) there for the generalized \((\beta, \gamma)\)-distance has the properties
(a) \( \| x \|_{\beta, \gamma} = 0 \iff x = 0 \)
(b) \( \| p^\frac{\beta}{p} \|_{\beta, \gamma} = p^\frac{\beta}{p} \| x \|_{\beta, \gamma} \) where \( p^\frac{\beta}{p} x = \left( p^\frac{\beta_{1}}{p_{1}} x_{1}, ..., p^\frac{\beta_{n}}{p_{n}} x_{n} \right) \) and \( p > 0 \),
(c) \( \| x + y \|_{\beta, \gamma} \leq 2^{\frac{2m}{\beta_{\max}}} \| x \|_{\beta, \gamma} + \| y \|_{\beta, \gamma} \) and so gives a non-isotropic quasi-distance in \( \mathbb{R}^n \).

Now for \( 0 < \alpha < n \), let us define the generalized Riesz potential.

\[ \Lambda_{\beta, \gamma} f(x) = \int_{\mathbb{R}^n} \| x - y \|_{\beta, \gamma}^{\alpha-n} f(y) dy. \]

We will study the conditions of finiteness almost everywhere of these potentials. Note that the conditions of these type for classical Riesz potentials were studied by [1], [2]. To formulate the basic condition on \( f \), we introduce the function \( w \), a positive, increasing on \((0, \infty)\) and satisfying the following two conditions:

1. \( \int_{1}^{\infty} w(t)^{\frac{1}{p-1}} t^{-1} dt < \infty, \ 1 < p < \infty \)
2. for any \( r > 0 \), there is a constant \( A > 0 \) such that \( w(2r) < A(w(r)) \) holds.

**Lemma 1.1.** There exists a positive constant \( M \) such that

\[ \int_{\{y : f(y) \geq a\}} \Lambda_{\beta, \gamma} (x - y) f(y) dy \]

\[ \leq M \left( \int_{\{y : f(y) \geq a\}} (f(y)^p w(f(y))) dy \right)^{\frac{1}{p}} \left( \int_{A}^{\infty} w(t)^{-\frac{1}{p-1}} t^{-1} dt \right)^{1/p'} \]

for any \( a > 0 \) and any non-negative measurable function \( f \) on \( \mathbb{R}^n \), where \( \alpha p = n, \ \frac{1}{p} + \frac{1}{p'} = 1 \) and \( \Lambda_{\beta, \gamma} (x - y) = \| x - y \|_{\beta, \gamma}^{\alpha-n}. \)

**Proof.** Define \( H_j = \{ y \in \mathbb{R}^n : 2^j a \leq f(y) \leq 2^{j+1} a \} \) for each positive integer \( j \), and take \( r_j \geq 0 \) such that \( |H_j| = |B_{\beta, \gamma}(0, r_j)| \), where \( |G| \) denotes the Lebesque
measure of a set $G \subset \mathbb{R}^n$. Then we note that

$$\int_{\{y : f(y) \geq a\}} \Lambda_{\beta, \gamma}(x-y) f(y) dy = \sum_{j=1}^{\infty} \int_{H_j} \|x-y\|_{\beta, \gamma}^{\alpha-n} f(y) dy$$

$$\leq \sum_{j=1}^{\infty} 2^j a \int_{H_j} \|x-y\|_{\beta, \gamma}^{\alpha-n} dy \leq M_1 \sum_{j=1}^{\infty} 2^j a \int_{B_{\beta, \gamma}(x,r_j)} \|x-y\|_{\beta, \gamma}^{\alpha-n} dy$$

$$= M_1 \sum_{j=1}^{\infty} 2^j a |H_j|^{|\beta|/\hat{\beta}|}.$$

Applying the Holder inequality

$$\int_{\{y : f(y) \geq a\}} \Lambda_{\beta, \gamma}(x-y) f(y) dy$$

$$\leq M_2 \left( \sum_{j=1}^{\infty} (2^j a)^p w(2^j a |H_j|) \right)^{1/p} \left( \sum_{j=1}^{\infty} w(2^j a)^{-1/(p-1)} \right)^{1/p'}$$

$$\leq M_3 \left( \int_{\{y : f(y) \geq a\}} f(y)^p w(f(y)) dy \right)^{1/p} \left( \int_{a}^{\infty} w(t)^{-1/(p-1)} t^{-1} dt \right)^{1/p'},$$

where $M_1$, $M_2$ and $M_3$ are positive constants independent of $f$, $x$ and $a$. Thus Lemma is proved. \hfill \Box

**Theorem 1.2.** Let $p = \frac{n}{\alpha}$ and $f$ be a non-negative measurable function $\mathbb{R}^n$ satisfying the conditions

$$\int_{\mathbb{R}^n} (1 + \|y\|_{\beta, \gamma})^{\alpha-n} f(y) dy < \infty$$

$$\int_{\mathbb{R}^n} f(y)^p w(f(y)) dy < \infty.$$

Then the generalized $(\beta, \gamma)$-Riesz potential is a continuous function in $\mathbb{R}^n$.

**Proof.** For a positive $r$ and any $x_0 \in \mathbb{R}^n$ we denote the open $(\beta, \gamma)$-ball by $B_{\beta, \gamma}(x_0, r)$ with radius $r$ and a center $x_0$ as

$$B_{\beta, \gamma}(x, r) = \{y \in \mathbb{R}^n : \|x-y\|_{\beta, \gamma} < r\}.$$

Then

$$\Lambda_{\beta, \gamma} f(x) = \left\{ \int_{B_{\beta, \gamma}(x_0, r)} \frac{f(y)}{\|x-y\|_{\beta, \gamma}^{\alpha-n}} dy + \int_{\mathbb{R}^n - B_{\beta, \gamma}(x_0, r)} \frac{f(y)}{\|x-y\|_{\beta, \gamma}^{\alpha-n}} dy \right\}$$

$$= E'_r(x) + E''_r(x).$$

Firstly, we will prove the accessory inequality for the following integral

$$\int_{\{y : f(y) \geq a\}} \|x-y\|_{\beta, \gamma}^{\alpha-n} dy.$$
where $a$ is a positive number. Passing to spherical coordinates by formulas the following

$$y_1 = x_1 + (\rho \cos \theta_1)\frac{x}{|x|}, \ldots, y_n = x_n + (\rho \sin \theta_1 \ldots \sin \theta_{n-1})\frac{x}{|x|}.$$ 

We obtained $\|x - y\|_{\beta,\gamma} = \rho^{\frac{1}{|\beta|}} |x|$. Denoting the Jacobian of this transformation by $I_{\beta,\gamma}(\rho, u_1, \ldots, u_n)$ and the Jacobian of transformation $y_k = \rho u_k$ by $I(\rho, u_1, \ldots, u_n)$ we obtain

$$I_{\beta,\gamma} = \rho^{\frac{1}{|\beta|} - n\gamma} \frac{1}{\beta_1} \ldots \frac{1}{\beta_n} u_1^{\frac{1}{\beta_1}} \ldots u_n^{\frac{1}{\beta_n}} I$$

or

$$I_{\beta,\gamma} = \rho^{\frac{1}{|\beta|} - 1} - \Omega_{\beta,\gamma}(\theta)$$

where $\Omega_{\beta,\gamma}(\theta)$ is bounded function, which depend only on angles $\theta_1, \ldots, \theta_{n-1}$.

Now consider $E_r'(x)$. Separating this term in two integrals on the domains $\{B_{\beta,\gamma}(x_0, r) : f(y) \leq 1\}$ and $\{B_{\beta,\gamma}(x_0, r) : f(y) > 1\}$ respectively, we have that the first of these two integrals can easily calculate. Applying the inequality (1.1) with $a = 1$ to the second integral we obtained

$$|E_r'(x)| \leq M \left\{ \int_{B_{\beta,\gamma}(x_0, r)} f^p(y) w(f(y)) dy \right\}^{\frac{1}{p}}$$

where $M > 1$ is a constant. Since the right hand side of the equation is independent from $x$, then Lemma holds for any $x \in \mathbb{R}^n$. This is for any $x \in \mathbb{R}^n$

$$\int_{B_{\beta,\gamma}(x_0, r)} \frac{f(y)}{\||x - y||_{\beta,\gamma}^{n-\alpha}} dy < \infty.$$ 

Now we see that the inequality

$$E_r''(x_0) = \int_{\mathbb{R}^n - B_{\beta,\gamma}(x_0, r)} \frac{f(y)}{1 + \|y\|_{\beta,\gamma}} \frac{1 + \|y\|_{\beta,\gamma}}{\|x_0 - y\|_{\beta,\gamma}^{n-\alpha}} dy$$

$$\leq \left( \frac{1 + \|x_0\|_{\beta,\gamma}}{r^{\frac{|\beta|}{|\beta|}}} \right) \int_{\mathbb{R}^n} \frac{f(y)}{1 + \|y\|_{\beta,\gamma}} \frac{1}{\|x_0 - y\|_{\beta,\gamma}^{n-\alpha}} dy$$

holds. From this and (1.2) it follows that $\Lambda_{\beta,\gamma} f(x_0)$ is finite. Setting $r = 2\|x - x_0\|_{\beta,\gamma}$ we obtained from lemma

$$\lim_{x \to x_0} E_r'(x) = 0.$$ 

Now we are considering $E_r''(x)$ where

$$r = 2\|x - x_0\|_{\beta,\gamma},$$

$$y \in \mathbb{R}^n - B_{\beta,\gamma}(x_0, 2\|x - x_0\|_{\beta,\gamma})$$
then

$$\|x_0 - y\|_{\beta, \gamma} = \|x_0 - x + x - y\|_{\beta, \gamma} \leq \|x - y\|_{\beta, \gamma} + \|x - x_0\|_{\beta, \gamma}$$

$$\leq \|x - y\|_{\beta, \gamma} + \frac{\|y - x_0\|_{\beta, \gamma}}{2}.$$ 

Since

$$\|x - y\|_{\beta, \gamma} \geq \frac{1}{2} \|x_0 - y\|_{\beta, \gamma}.$$ 

We obtain

$$|E''_{2\|x-x_0\|_{\beta, \gamma}}(x)| \leq 2 \int_{\mathbb{R}^n} \frac{f(y)}{\|x_0 - y\|_{\beta, \gamma}^{n-\alpha}} dy$$

From this we have obtained

$$|\Lambda_{\beta, \gamma} f(x) - \Lambda_{\beta, \gamma} f(x_0)| \leq \left| E'_{2\|x-x_0\|_{\beta, \gamma}}(x) \right| + \left| E''_{2\|x-x_0\|_{\beta, \gamma}}(x) - \int_{\mathbb{R}^n} \frac{f(y)}{\|x_0 - y\|_{\beta, \gamma}^{n-\alpha}} dy \right|$$

By (1.3), the first term converge to zero as $x \longrightarrow x_0$ and the second term tends to zero by Lebesque dominated theorem.

The proof is completed.

REFERENCES


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