

# On Continuity of Generalized Riesz Potentials

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**Abstract.** This study establishes the theorem on continuity of generalized Riesz potentials with non-isotropic kernels depending on  $(\beta, \gamma)$ -distance.

**Mathematics Subject Classification:** 31B10, 42B20, 47B06

**Keywords:** Riesz potentials, Non-isotropic distance

## 1. INTRODUCTION

The classical Riesz potentials and their generalizations have important applications in different problems of functions theory [1], [2], [3].

In this paper we consider the Riesz potentials with kernel depending on non-isotropic distance.

First we define this distance. Suppose that  $\beta_k$ ,  $k = 1, 2, \dots, n$  and  $\gamma$  are positive numbers and define for  $x \in R^n$  the generalized  $(\beta, \gamma)$  -distance from origin by formula

$$\|x\|_{\beta, \gamma} = \left( |x_1|^{\frac{2\beta_1}{\gamma}} + \dots + |x_n|^{\frac{2\beta_n}{\gamma}} \right)^{\frac{\gamma}{2n} \left| \frac{1}{\beta} \right|}$$

where  $\left| \frac{1}{\beta} \right| = \frac{1}{\beta_1} + \dots + \frac{1}{\beta_n}$ .

In the case of  $\gamma = 2$   $\beta_k = \frac{1}{\lambda_k}$  we have the well known  $\lambda$ -distance [3], [4] and in the case of  $\gamma = 2$   $\beta_k = 2$ ,  $k = 1, 2, \dots, n$ .  $\|x\|_{\beta, \gamma}$  is the Euclidean distance in  $R^n$ .

Note also that for any  $t > 0$

$$\left( \left| t^{\frac{\gamma}{2\beta_1}} x_1 \right|^{\frac{2\beta_1}{\gamma}} + \dots + \left| t^{\frac{\gamma}{2\beta_n}} x_n \right|^{\frac{2\beta_n}{\gamma}} \right)^{\frac{\gamma}{2n} \left| \frac{1}{\beta} \right|} = t^{\frac{\gamma}{2n} \left| \frac{1}{\beta} \right|} \|x\|_{\beta, \gamma}$$

that is the generalized  $(\beta, \gamma)$ -distance has the homogeneity properties in the form

$$\left\| t^{\frac{\gamma}{2\beta}} x \right\|_{\beta, \gamma} = t^{\frac{\gamma}{2n} \left| \frac{1}{\beta} \right|} \|x\|_{\beta, \gamma}$$

where  $t^{\frac{\gamma}{2\beta}} = \left( t^{\frac{\gamma}{2\beta_1}}, \dots, t^{\frac{\gamma}{2\beta_n}} \right)$ .

The value  $\|x - y\|_{\beta, \gamma}$  we call the generalized using the inequality  $(a + b)^m \leq 2^m (a^m + b^m)$ ,  $m > 1$ . We obtain

$$\|x + y\|_{\beta, \gamma} \leq 2^{\frac{2+\gamma}{\beta_{\max}} \left| \frac{1}{\beta} \right|} \left( \|x\|_{\beta, \gamma} + \|y\|_{\beta, \gamma} \right),$$

where  $\beta_{\max} = \max(\beta_1, \dots, \beta_n)$  there for the generalized  $(\beta, \gamma)$ -distance has the properties

- (a)  $\|x\|_{\beta, \gamma} = 0 \iff x = 0$
- (b)  $\left\| p^{\frac{\gamma}{2\beta}} \right\|_{\beta, \gamma} = p^{\frac{\gamma}{2n} \left| \frac{1}{\beta} \right|} \|x\|_{\beta, \gamma}$  where  $p^{\frac{\gamma}{2\beta}} x = \left( p^{\frac{\gamma}{2\beta_1}} x_1, \dots, p^{\frac{\gamma}{2\beta_n}} x_n \right)$  and  $p > 0$ ,
- (c)  $\|x + y\|_{\beta, \gamma} \leq 2^{\frac{2+\gamma}{\beta_{\max}} \left| \frac{1}{\beta} \right|} \left( \|x\|_{\beta, \gamma} + \|y\|_{\beta, \gamma} \right)$  and so gives a non-isotropic quasi-distance in  $\mathbb{R}^n$ .

Now for  $0 < \alpha < n$ , let us define the generalized Riesz potential.

$$\Lambda_{\beta, \gamma} f(x) = \int_{\mathbb{R}^n} \|x - y\|_{\beta, \gamma}^{\alpha-n} f(y) dy.$$

We will study the conditions of finiteness almost every where of these potentials. Note that the conditions of these type for classical Riesz potentials were studied by [1], [2]. To formulate the basic condition on  $f$ , we introduce the function  $w$ , a positive, increasing on  $(0, \infty)$  and satisfying the following two conditions:

1.  $\int_1^\infty w(t)^{-1/(p-1)} t^{-1} dt < \infty$ ,  $1 < p < \infty$
2. for any  $r > 0$ , there is a constant  $A > 0$  such that  $w(2r) < A(w(r))$  holds.

**Lemma 1.1.** *There exists a positive constant  $M$  such that*

$$\begin{aligned} & \int_{\{y: f(y) \geq a\}} \Lambda_{\beta, \gamma}(x - y) f(y) dy \\ & \leq M \left( \int_{\{y: f(y) \geq a\}} f(y)^p w(f(y)) dy \right)^{1/p} \left( \int_a^\infty w(t)^{-1/(p-1)} t^{-1} dt \right)^{1/p'} \end{aligned}$$

for any  $a > 0$  and any non-negative measurable function  $f$  on  $\mathbb{R}^n$ , where  $\alpha p = n$ ,  $\frac{1}{p} + \frac{1}{p'} = 1$  and  $\Lambda_{\beta, \gamma}(x - y) = \|x - y\|_{\beta, \gamma}^{\alpha-n}$ .

*Proof.* Define  $H_j = \{y \in \mathbb{R}^n : 2^{j-1}a \leq f(y) \leq 2^j a\}$  for each positive integer  $j$ , and take  $r_j \geq 0$  such that  $|H_j| = |B_{\beta, \gamma}(0, r_j)|$ , where  $|G|$  denotes the Lebesgue

measure of a set  $G \subset \mathbb{R}^n$ . Then we note that

$$\begin{aligned} \int_{\{y:f(y)\geq a\}} \Lambda_{\beta,\gamma}(x-y)f(y)dy &= \sum_{j=1}^{\infty} \int_{H_j} \|x-y\|_{\beta,\gamma}^{\alpha-n} f(y)dy \\ &\leq \sum_{j=1}^{\infty} 2^j a \int_{H_j} \|x-y\|_{\beta,\gamma}^{\alpha-n} dy \leq M_1 \sum_{j=1}^{\infty} 2^j a \int_{B_{\beta,\gamma}(x,r_j)} \|x-y\|_{\beta,\gamma}^{\alpha-n} dy \\ &= M_1 \sum_{j=1}^{\infty} 2^j a |H_j|^{\frac{\alpha}{n}\gamma|\frac{1}{\beta}|}. \end{aligned}$$

Applying the Holder inequality

$$\begin{aligned} &\int_{\{y:f(y)\geq a\}} \Lambda_{\beta,\gamma}(x-y)f(y)dy \\ &\leq M_2 \left( \sum_{j=1}^{\infty} (2^{j-1}a)^p w(2^{j-1}a)|H_j| \right)^{1/p} \left( \sum_{j=1}^{\infty} w(2^j a)^{-1/(p-1)} \right)^{1/p'} \\ &\leq M_3 \left( \int_{\{y:f(y)\geq a\}} f(y)^p w(f(y))dy \right)^{1/p} \left( \int_a^{\infty} w(t)^{-1/(p-1)} t^{-1} dt \right)^{1/p'}, \end{aligned}$$

where  $M_1$ ,  $M_2$  and  $M_3$  are positive constants independent of  $f$ ,  $x$  and  $a$ . Thus Lemma is proved.  $\square$

**Theorem 1.2.** *Let  $p = \frac{n}{\alpha}$  and  $f$  be a non-negative measurable function  $\mathbb{R}^n$  satisfying the conditions*

$$\begin{aligned} \int_{\mathbb{R}^n} (1 + \|y\|_{\beta,\gamma})^{\alpha-n} f(y)dy &< \infty \\ \int_{\mathbb{R}^n} f(y)^p w(f(y))dy &< \infty. \end{aligned}$$

*Then the generalized  $(\beta, \gamma)$ -Riesz potential is a continuous function in  $\mathbb{R}^n$ .*

*Proof.* For a positive  $r$  and any  $x_0 \in \mathbb{R}^n$  we denote the open  $(\beta, \gamma)$ -ball by  $B_{\beta,\gamma}(x_0, r)$  with radius  $r$  and a center  $x_0$  as

$$B_{\beta,\gamma}(x, r) = \{y \in \mathbb{R}^n : \|x - y\|_{\beta,\gamma} < r\}.$$

Then

$$\begin{aligned} \Lambda_{\beta,\gamma} f(x) &= \left\{ \int_{B_{\beta,\gamma}(x_0,r)} \frac{f(y)}{\|x-y\|_{\beta,\gamma}^{n-\alpha}} dy + \int_{\mathbb{R}^n - B_{\beta,\gamma}(x_0,r)} \frac{f(y)}{\|x-y\|_{\beta,\gamma}^{n-\alpha}} dy \right\} \\ &= E'_r(x) + E''_r(x). \end{aligned}$$

Firstly, we will prove the accessory inequality for the following integral

$$(1.1) \quad \int_{\{y:f(y)\geq a\}} \frac{f(y)}{\|x-y\|_{\beta,\gamma}^{n-\alpha}} dy,$$

where  $a$  is a positive number. Passing to spherical coordinates by formulas the following

$$y_1 = x_1 + (\rho \cos \theta_1)^{\frac{\gamma}{\beta_1}}, \dots, y_n = x_n + (\rho \sin \theta_1 \dots \sin \theta_{n-1})^{\frac{\gamma}{\beta_n}}.$$

We obtained  $\|x - y\|_{\beta, \gamma} = \rho^{\frac{\gamma}{n} |\frac{1}{\beta}|}$ . Denoting the Jacobian of this transformation by  $I_{\beta, \gamma}(\rho, u_1, \dots, u_n)$  and the Jacobian of transformation  $y_k = \rho u_k$  by  $I(\rho, u_1, \dots, u_n)$  we obtain

$$I_{\beta, \gamma} = \rho^{\gamma |\frac{1}{\beta}| - n} \gamma^n \frac{1}{\beta_1} \dots \frac{1}{\beta_n} u_1^{\frac{\gamma}{\beta_1}} \dots \frac{1}{\beta_n} u_n^{\frac{\gamma}{\beta_n}} I$$

or

$$I_{\beta, \gamma} = \rho^{\gamma |\frac{1}{\beta}| - 1} \Omega_{\beta, \gamma}(\theta)$$

where  $\Omega_{\beta, \gamma}(\theta)$  is bounded function, which depend only on angles  $\theta_1, \dots, \theta_{n-1}$ .

Now consider  $E'_r(x)$ . Separating this term in two integrals on the domains  $\{B_{\beta, \gamma}(x_0, r) : f(y) \leq 1\}$  and  $\{B_{\beta, \gamma}(x_0, r) : f(y) > 1\}$  respectively, we have that the first of these two integrals can easily calculate. Applying the inequality (1.1) with  $a = 1$  to the second integral we obtained

$$(1.2) \quad |E'_r(x)| \leq M \left\{ r^{\frac{\gamma}{n} |\frac{1}{\beta}| \alpha} + \left( \int_{B_{\beta, \gamma}(x_0, r)} f^p(y) w(f(y)) dy \right)^{\frac{1}{p}} \right\}$$

where  $M > 1$  is a constant. Since the right hand side of the equation is independent from  $x$ , then Lemma holds for any  $x \in \mathbb{R}^n$ . This is for any  $x \in \mathbb{R}^n$

$$(1.3) \quad \int_{B_{\beta, \gamma}(x_0, r)} \frac{f(y)}{\|x - y\|_{\beta, \gamma}^{n-\alpha}} dy < \infty.$$

Now we see that the inequality

$$\begin{aligned} E''_r(x_0) &= \int_{\mathbb{R}^n - B_{\beta, \gamma}(x_0, r)} \frac{f(y)}{(1 + \|y\|_{\beta, \gamma})^{n-\alpha}} \left( \frac{1 + \|y\|_{\beta, \gamma}}{\|x_0 - y\|_{\beta, \gamma}} \right)^{n-\alpha} dy \\ &\leq \left( \frac{1 + \|x_0\|_{\beta, \gamma}}{r^{\frac{\gamma}{n} |\frac{1}{\beta}|}} \right) \int_{\mathbb{R}^n} \frac{f(y)}{(1 + \|y\|_{\beta, \gamma})^{n-\alpha}} dy \end{aligned}$$

holds. From this and (1.2) it follows that  $\Lambda_{\beta, \gamma} f(x_0)$  is finite. Setting  $r = 2 \|x - x_0\|_{\beta, \gamma}$  we obtained from lemma

$$\lim_{x \rightarrow x_0} E'_{2\|x-x_0\|_{\beta, \gamma}}(x) = 0.$$

Now we are considering  $E''_r(x)$  where

$$r = 2 \|x - x_0\|_{\beta, \gamma}$$

$$y \in \mathbb{R}^n - B_{\beta, \gamma}(x_0, 2 \|x - x_0\|_{\beta, \gamma})$$

then

$$\begin{aligned} \|x_0 - y\|_{\beta,\gamma} &= \|x_0 - x + x - y\|_{\beta,\gamma} \leq \|x - y\|_{\beta,\gamma} + \|x - x_0\|_{\beta,\gamma} \\ &\leq \|x - y\|_{\beta,\gamma} + \frac{\|y - x_0\|_{\beta,\gamma}}{2}. \end{aligned}$$

Since

$$\|x - y\|_{\beta,\gamma} \geq \frac{1}{2} \|x_0 - y\|_{\beta,\gamma}.$$

We obtain

$$|E''_{2\|x-x_0\|_{\beta,\gamma}}(x)| \leq 2 \int_{\mathbb{R}^n} \frac{f(y)}{\|x_0 - y\|_{\beta,\gamma}^{n-\alpha}} dy$$

From this we have obtained

$$|\Lambda_{\beta,\gamma} f(x) - \Lambda_{\beta,\gamma} f(x_0)| \leq \left| E'_{2\|x-x_0\|_{\beta,\gamma}}(x) \right| + \left| E''_{2\|x-x_0\|_{\beta,\gamma}}(x) - \int_{\mathbb{R}^n} \frac{f(y)}{\|x_0 - y\|_{\beta,\gamma}^{n-\alpha}} dy \right|$$

By (1.3), the first term converge to zero as  $x \rightarrow x_0$  and the second term tends to zero by Lebesgue dominated theorem.

The proof is completed.  $\square$

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**Received: October 2, 2006**