

# Banach Spaces with Small Local Diameter

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## Abstract

The main goal of this paper is to prove that any Banach space  $X$  with weakly compact and sliceable unit sphere admits an equivalent Gâteaux differentiable norm.

In this paper we concentrate mainly on the problem of differentiability of a Banach space. The present work shows that if the unit sphere of a Banach space is weakly compact and  $\varepsilon$ -sliceable then the Banach space admits an equivalent Gâteaux differentiable norm.

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## 1 Introduction

A norm  $\|\cdot\|$  of Banach space  $X$  is called *strictly convex* if  $x, y \in X$  satisfy  $2\|x\|^2 + 2\|y\|^2 - \|x + y\|^2 = 0$ , then  $x = y$ .

Let  $f$  be a real valued function defined on a Banach space  $X$ . We say that  $f$  is Gâteaux differentiable at  $x \in X$ , if for each  $h \in X$ ,

$$\lim_{t \rightarrow 0} \frac{f(x + th) - f(x)}{t} = f'(x)(h)$$

exist and is a linear continuous function in  $h$ , that is  $f'(x) \in X^*$ . we say that the Banach space  $(X, \|\cdot\|)$  is Gâteaux differentiable if the norm  $\|\cdot\|$  of Banach space is Gâteaux differentiable at all  $0 \neq x \in X$ .

Let  $X$  be a Banach space with dual  $X^*$ ,  $A \subseteq X$ , and  $B \subseteq X^*$ . By *polar of*  $A$  in  $X$  is, the set  $A^\circ = \{f \in X^* : f(x) \leq 1, \text{ for all } x \in A\}$  and *polar of*  $B$  in  $X^*$  is the set  $B_\circ = \{x \in X : f(x) \leq 1 \text{ for all } f \in B\}$ . By Bipolar theorem, in

every Banach space if  $A$  is weakly closed, convex, and  $0 \in A$  then  $A = (A^\circ)_\circ$ . similarly, if  $B$  is  $w^*$ -closed, convex, and  $0 \in B$ , then  $(B_\circ)^\circ = B$ .

In 1992 J.E. Jayne, I. Nomioka, and C. A. Rogers introduced and studied the class of  $\sigma$ -fragmented topological spaces and its applications in Banach Spaces see [1, 2], arriving at the concept of spaces having a countable cover by sets of small local diameter.

On the other hand A. Molat6, J. Orihuela, and S. Troyanski see [3] considered the above property in the particular case and called it JNR property that if  $(X, w)$  is a Banach space with its weak topology and  $\rho$  a metric on  $X$ , then the topological space  $(X, w)$  is said to be a countable cover by sets of small local  $\rho$ -diameter ( $\rho$ -SLD for short) if for every  $\varepsilon > 0$ , it is possible to write  $X = \cup_{n=1}^\infty X_n^\varepsilon$  such that for each  $n \in \mathbb{N}$  and every point  $x \in X_n^\varepsilon$  there is a  $w$ -neighbourhood  $V$  of  $x$  such that  $\rho - \text{diam}(V \cap X_n^\varepsilon) < \varepsilon$ .

Then, they introduce the concept of sJNR which is a particular case of JNR. The subset  $L$  of normed space  $X$  has sJNR property, if for every  $\varepsilon > 0$ ,  $L$  has a countable cover by sets which are a union of slices of diameter less than  $\varepsilon$  if  $L = \cup_{n=1}^\infty L_n$  in such a way that for every  $x \in L_n, n \in \mathbb{N}$ , there exist a slice  $S(L_n)$  containing  $x$  and diameter less than  $\varepsilon$ .

By aid of this property they prove that a unit sphere  $S_X$  of Banach space  $X$  has sJNR if and only if  $X$  admits an equivalent locally uniformly rotund norm see [3] and M. Raja in see [4] express that,  $X^*$ , the dual of Banach space  $X$  admits a dual locally uniformly rotund norm if and only if  $(X^*, w^*)$  has  $\|\cdot\| - SLD$  and  $X^*$  admits a dual strictly convex norm.

Recall that, if  $E$  is a nonempty subset of a Banach space, then the  $\|\cdot\| - \text{diam}(E)$  is the supremum of  $\|x - y\|$  for all  $x, y \in E$ . An *open affine half space* of Banach space  $X$  is a weakly open set of the form  $H = \{x \in X : f(x) > \lambda\}$  for some  $f \in X^* \setminus \{0\}$  and  $\lambda \in \mathbb{R}$ . Also, let  $C$  be a subset of  $X$ . An *open slice of  $C$*  is a nonempty intersection of  $C$  with an open half space of  $X$ . In other word, an open slice of  $C$  is a nonempty subset of  $C$  which is relatively weak open in  $C$ . we denote the unit sphere of Banach space  $X$  by  $S_X = \{x \in X : \|x\| = 1\}$ .

**Definition 1.1** *A subset  $C$  of Banach Space  $X$  is called  $\varepsilon$ -sliceable, if for every  $\varepsilon > 0$  and  $x \in C$  there exists an open slice  $S(x)$  of  $C$  containing  $x$  such that  $\|\cdot\| - \text{diam}(S(x)) < \varepsilon$ .*

To show that a Banach space is Gâteaux differentiable, we need to use a classical result of the well known Šmulian theorem, as follow:

**Corollary 1.2 (Šmulian)** *The  $\|\cdot\|$  of Banach space  $(X, \|\cdot\|)$  is Gâteaux differentiable at  $x \in S_X$  if and only if there is a unique  $f \in S_{X^*}$  such that  $f(x) = 1$*

## 2 Separability

To approach differentiability of Banach space we start by proving the following theorem to show that under our conditions the Banach space is separable. Then we will show that every separable Banach space has an equivalent Gâteaux differentiable norm.

**Theorem 2.1** *Let the unit sphere  $S_X$  of Banach space  $X$  be weakly compact and  $\varepsilon$  – sliceable for every  $\varepsilon > 0$ . Then  $X$  is separable.*

Proof: Let  $\varepsilon > 0$  and  $x \in S_X$ . Since  $S_X$  is  $\varepsilon$ –sliceable, there exists an open slice  $S(x)$  of  $S_X$  such that  $\|\cdot\| - \text{diam}(S(x)) < \varepsilon$ . We know  $S(x) = H(x) \cap S_X$  where  $H(x)$  is an open affine half space containing  $x$ . Clearly  $S_X \subseteq \cup_{x \in S_X} H(x)$ . As  $H(x)$  is  $w$ –open and  $S_X$  is  $w$ –compact, There is a finitely many  $x_i \in S_X$ ,  $i = 1, 2, 3, \dots, k$  such that  $S_X \subseteq \cup_{i=1}^k H(x_i)$ . Therefore to each positive integer  $m$  there exist an positive integer  $k_m$ ,  $x_{im} \in S_X$ ,  $i = 1, 2, \dots, k_m$  and corresponding to each  $x_{im}$  a slice  $S(x_{im})$  of  $S_X$  such that  $\|\cdot\| - \text{diam}(S(x_{im})) < \frac{1}{m}$ .

Now consider the countable set  $A = \cup_{m=1}^{\infty} \{x_{im} : x_{im} \in S_X, i = 1, 2, \dots, k_m\}$ . We will show that  $\overline{A} = S_X$ .

Let  $x \in S_X$  and  $\varepsilon > 0$ , there exists  $m \in \mathbb{N}$  such that  $\frac{1}{m} < \varepsilon$ . Corresponding to each  $m \in \mathbb{N}$  there is a slice  $S(x_{im}) = H(x_{im}) \cap S_X$  of  $S_X$  such that  $S_X \subseteq \cup_{i=1}^{k_m} H(x_{im})$ . Therefore,  $x \in H(x_{im})$  for some  $1 \leq i \leq k_m$  and consequently  $x \in H(x_{im}) \cap S_X = S(x_{im})$  with  $\|\cdot\| - \text{diam}(S(x_{im})) < \frac{1}{m}$ . Hence  $\|x - x_{im}\| < \frac{1}{m} < \varepsilon$  and  $A$  is norm dense in  $S_X$ . Therefore  $S_X$  is separable and consequently  $X$  is separable.

In the following we assert and prove a lemma which we need it in the proof of last theorem. We call  $\|\cdot\|^*$  on  $X^*$  is a *dual norm* of some norm  $\|\cdot\|$  on  $X$ , if for every  $f \in X^*$ ,  $\|f\|^* = \sup_{x \in B_X} |f(x)|$  where  $B_X = \{x \in X : \|x\| \leq 1\}$  is the closed unit ball of  $X$ .

**Lemma 2.2** *A norm on  $X^*$  is a dual norm of some norm on  $X$  if and only if the closed unit ball  $B_{X^*}$  of  $X^*$  is  $w^*$  – closed in  $X^*$ .*

Proof: Let  $\phi: X \rightarrow X^{**}$  be the canonical isomertic isomorphism of  $X$  onto closed subspace  $\phi(X) \subset X^{**}$  which is defined by  $(\phi x)f = f(x)$ , for every  $x \in X$ . Thus for every  $x \in X$  the set  $W(x) = \{f \in X^* : |f(x)| \leq 1\} = \{f \in X^* : |(\phi x)f| \leq 1\} = (\phi x)^{-1}[-1, 1]$  is  $w^*$ –closed.

Let  $\|\cdot\|$  be The dual norm of  $(X, \|\cdot\|)$ , we will show that the norm closed unit ball  $B = \{f \in X^* : \|f\| \leq 1\}$  is  $w^*$ –closed. Since  $\|\cdot\|$  is a dual norm we have

$$\begin{aligned} B &= \{f \in X^* : \sup_{x \in B_X} |f(x)| \leq 1\} \\ &= \{f \in X^* : |f(x)| \leq 1, \text{ for all } x \in B_X\} \\ &= \cap_{x \in B_X} W(x). \end{aligned}$$

As we have shown at above, for each  $x \in B_X$  the set  $W(x)$  is  $w^*$ -closed and consequently  $B$ , the closed unit ball of  $(X^*, \|\cdot\|)$ , is  $w^*$ -closed.

Vice versa, Let  $B$  be  $w^*$ -closed. Then by Bipolar theorem we have  $B = (B_\circ)^\circ$ . Since  $B$  is convex, balance, and  $0 \in B$ , so is the polar set  $B_\circ$  of  $B$ . Therefore the Minkowski functional  $\mu_{B_\circ}(x)$ , is a norm on  $X$  which is equivalent to original norm of  $X$ . we will show that  $\|\cdot\| = \mu_{B_\circ}^*$ , where  $\mu_{B_\circ}^*$  is the dual norm of Minkowski functional  $\mu_{B_\circ}$  on  $X$ . Because,

$$\begin{aligned} B = (B_\circ)^\circ &= \{f \in X^* : f(x) \leq 1 \text{ for all } x \in B_\circ\} \\ &= \{f \in X^* : |f(x)| \leq 1 \text{ for all } x \in B_\circ\} \\ &= \{f \in X^* : \sup_{x \in B_\circ} |f(x)| \leq 1\} \\ &= \left\{f \in X^* : \sup_{\mu_{B_\circ}(x) \leq 1} |f(x)|\right\} \\ &= \{f \in X^* : \mu_{B_\circ}^*(x) \leq 1\}. \end{aligned}$$

Hence the two closed unit balls of  $(X^*, \|\cdot\|)$  and  $(X^*, \mu_{B_\circ}^*)$  are equal.

### 3 Differentiability

**Lemma 3.1** *Let  $(X, \|\cdot\|)$  be a Banach space and let  $\|\cdot\|^*$  be the dual norm on  $X^*$ . If  $\|\cdot\|^*$  is strictly convex, then  $\|\cdot\|$  is Gâteaux differentiable*

Proof: By corollary 1.2, we need only to show that for  $x \in S_X$ , there is a unique  $f \in S_{X^*}$  such that  $f(x) = 1$ . Let  $x \in S_X$  and  $f, g \in S_{X^*}$  be such that  $f(x) = g(x) = 1$ . Then  $2 \geq \|f + g\| \geq (f + g)(x) = 2$ . Therefore  $\|f + g\|^* = 2$  and from the strictly convex of the  $\|\cdot\|^*$  we obtain  $f = g$ .

**Theorem 3.2** *Every separable Banach space admits an equivalent Gâteaux differentiable norm.*

Proof: Let  $X$  be a separable Banach space. Thus  $S_X$  is separable, that is, there is a sequence  $\{x_n\}$  in  $S_X$  which is norm dense in  $S_X$ . If  $(X^*, \|\cdot\|)$  is dual of  $X$ , for every  $g \in X^*$  define a  $\|\cdot\|$  on  $X^*$  by

$$\|\|f\|\|^2 = \|f\|^2 + \sum_{i=1}^{\infty} 2^{-i} f^2(x_i)$$

where  $f \in X^*$  and  $\|\cdot\|$  is the dual norm of  $X^*$ . The norm  $\|\cdot\|$  is equivalent to the origin norm of  $X^*$ , because

$$\begin{aligned} \|f\|^2 \leq \|\|f\|\|^2 &= \|f\|^2 + \sum_{i=1}^{\infty} 2^{-i} f^2(x_i) \\ &\leq \|f\|^2 + \sum_{i=1}^{\infty} 2^{-i} \|f\|^2 \|x_i\|^2 \\ &= \|f\|^2 (1 + \sum_{i=1}^{\infty} 2^{-i}) = 3 \|f\|^2. \end{aligned}$$

Therefore,  $B = \{f \in X^* : \|\|f\|\| \leq 1\}$  is  $w^*$ -closed and by lemma 2.2 this new norm is a dual norm of some norm on  $X$ .

To show that The  $\|\|\cdot\|\|$  is Gâteaux differentiable, by lemma 3.1 it is enough to show that  $\|\|\cdot\|\|$  is strictly convex, that is, for every  $f, g \in X^*$  if

$$2 \|\|f\|\|^2 + 2 \|\|g\|\|^2 - \|\|f + g\|\|^2 = 2 \|f\|^2 + 2 \sum_{i=1}^{\infty} 2^{-i} f^2(x_i) + 2 \|g\|^2 + 2 \sum_{i=1}^{\infty} 2^{-i} g^2(x_i) - 2 \|f + g\|^2 - 2 \sum_{i=1}^{\infty} 2^{-i} (f + g)^2(x_i) = 0$$

then  $f = g$ .

Since  $0 \leq (\|f\| - \|g\|)^2 + \|f + g\|^2 \leq 2 \|f\|^2 + 2 \|g\|^2 - \|f + g\|^2$ . We have  $0 \leq 2 \|f\|^2 + 2 \|g\|^2 - \|f + g\|^2$ . Also  $2f^2(x_i) + 2g^2(x_i) - (f + g)^2(x_i) \geq 0$  for every  $i \in \mathbb{N}$ . Hence from  $f, g \in X^*$  and  $2 \|\|f\|\|^2 + 2 \|\|g\|\|^2 - \|\|f + g\|\|^2 = 0$  we get  $2f^2(x_i) + 2g^2(x_i) - (f + g)^2(x_i) = (f(x_i) - g(x_i))^2 = 0$ . Thus  $f = g$  on a dense subset of  $S_X$ , but  $f, g$  are continuous functional and consequently  $f = g$  on  $S_X$ . Therefore the norm  $\|\|\cdot\|\|$  is strictly convex and by lemma 3.1 it is Gâteaux differentiable.

## References

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