Closed Range Composition Operators on the Besov Space $B_p$, the Besov Type Space $B_{p,p-1}$

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Abstract

We study the composition operators $C_\phi$ with closed range on the Besov space $B_p$, the Besov type space $B_{p,p-1}$

Keywords: composition operator, weighted Dirichlet space, Besov space, Besov type space, Bloch space, Bergman space, Hardy space, closed range, bounded below

1. Introduction

For $z,w \in D$, let $\varphi_z(w) = \frac{z - w}{1 - \overline{z}w}$ and, $dA(z)$ the area measure on $D$. For $\varphi$ analytic self-map of the open unit disk $D$, the composition operator $C_\varphi$ is defined by $C_\varphi(f) = f \circ \varphi$.

For $p > 0$, $\alpha > -1$, the weighted Dirichlet space $D^\alpha_p$ is defined to be the space of analytic functions $f$ on $D$ such that $\| f \|_{D^\alpha_p} = |f(0)| + \left( \int_D (1 - |z|^2)^\alpha |f'(z)|^p dA(z) \right)^{\frac{1}{p}} < \infty$. In case $\alpha = 1$ and $p = 2$, then $D^1_2 = H^2$ is the classical Hardy space. Furthermore, in case $\alpha = p$ and $1 \leq p < \infty$, then $D^p_p = L^p_\alpha$ is the usual Bergman space. Also, in case $\alpha = p - 2$ and $1 < p < \infty$, $D^{p-2}_p = B_p$ is called the Besov space. In particular, $D^0_2 = D$ is called the Dirichlet space. Also, in case $\alpha = p - 1$ and $0 < p < \infty$, $D^{p-1}_p = B_{p,p-1}$ is
called the Besov type space. In particular, $\mathcal{D}_2^1 = B_{2,1} = H^2$ is the classical Hardy space. It is trivial that $B_p \subset B_{p,p-1}$ ($p > 1$).

For $\alpha > 0$, the weighted Bloch space $B_\alpha$ is defined to be the space of analytic functions $f$ on $D$ such that $\|f\|_{B_\alpha} = |f(0)| + \sup_{z \in D} (1 - |z|^2)^\alpha |f'(z)| < \infty$. Note that $B_1 = B$ is the usual Bloch space. In the case of $1 < p < q$, it is known that $B_p \subset B_q \subset B$.

The amount $\sup_{z \in D} (1 - |z|^2)^\alpha |f'(z)|$ is a pseudonorm, which coincides with the $B_\alpha$-norm on the closed subspace of functions that vanish at the origin. So it coincides with the quotient norm on $B_\alpha/C$ where $C$ denotes the closed subspace of constant functions. By Schwarz-Pick lemma, the operator $C_\varphi$ is bounded on the Bloch space $B$. Furthermore, it follows from Littlewood's subordination theorem that $C_\varphi$ is bounded on the Bergman space $L^p_a$ for all $1 \leq p < \infty$.

To state our investigations, we give some definitions. Let $X$ be a Banach space and let $T$ a linear operator from $X$ into $X$. An operator $T$ is called bounded below on $X$ if there exists a constant $C > 0$ such that $\|Tf\| \geq C \|f\|$ for all $f \in X$. (Clearly, when a composition operator $C_\varphi$ is defined on a space of analytic functions on $D$, $C_\varphi$ is bounded below on the space if and only if $C_\varphi$ is closed range.) Furthermore, a subset $H$ of $D$ is called a sampling set for the space $B_\alpha$ if there exists a constant $C > 0$ such that $\sup_{z \in D} (1 - |z|^2)^\alpha |f'(z)| \leq C \sup_{z \in H} (1 - |z|^2)^\alpha |f'(z)|$ for all $f \in B_\alpha$. For $\epsilon > 0$, let $G_\epsilon = \varphi \left( \left\{ z \in D, \frac{1 - |z|^2 |\varphi'(z)|}{1 - |\varphi(z)|^2} \geq \epsilon \right\} \right)$. In [6], P.Ghatage, D.Zheng and N.Zorboska determined the boundedness from below of composition operators on the Bloch space using a sampling set $G_\epsilon$ for the Bloch space. Moreover, N.Zoroska ([21], [22]) characterized the boundedness from below of composition operators on the Bergman spaces. Also, H.Chen and P.Gauthier characterized the boundedness from below of composition operators on $B_\alpha$ in [4]. Furthermore, W.Smith ([11]) studied the boundedness and compactness of composition operators between Bergman spaces and Hardy spaces. M Tjani ([12]) studied the compactness of composition operators on the Besov spaces. M Tjani ([13]) also studied closed range composition operators on Besov spaces and Besov type spaces. In this paper, we study when composition operators are bounded below on the $B_p$ and $B_{p,p-1}$, respectively.

2. Preliminary notes

In this section, we introduce several results to prove the main theorem. In [1] J.R.Akeroyd and P.G.Ghatage proved the following result.

Theorem 1. ([1]) Let $\varphi$ be a univalent, analytic self-map of $D$. Then $C_\varphi$ is closed range on $L^p_a$ if and only if $\varphi$ is an automorphism of $D$.

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In [17] we proved the following result.

**Theorem 2.** ([17]) Let \( \alpha > 1 \). Suppose \( \varphi \) is a univalent self-map of \( D \). Then the following are equivalent.

1. \( C_\varphi : B_\alpha \rightarrow B_\alpha \) is bounded below.
2. \( C_\varphi : L_2^a(= \mathcal{D}_2^a) \rightarrow L_2^a(= \mathcal{D}_2^a) \) is bounded below.
3. \( C_\varphi : H^2(= \mathcal{D}_1^a) \rightarrow H^2(= \mathcal{D}_1^a) \) is bounded below.
4. \( C_\varphi : \mathcal{D}_2^a \rightarrow \mathcal{D}_2^a \) is bounded below.
5. \( \varphi \) is an automorphism of \( D \).

In [13] M Tjani proved the following results.

**Theorem 3.** ([13]) Let \( p > 2 \). Then if \( C_\varphi \) is closed range on \( B_{p,p-1} \), then \( C_\varphi \) is closed range on \( H^2 \).

**Theorem 4.** ([13]) Let \( \varphi \) be a boundedly valent, analytic self-map of \( D \) and \( p > 2 \). Then \( C_\varphi \) is closed range on \( B_p \) if and only if \( C_\varphi \) is closed range on \( B \).

In [18] we proved the following result.

**Theorem 5.** ([18]) Let \( 0 < \alpha < 1 \) and \( 1 < p < \infty \). Suppose \( \varphi \) is a univalent self-map of \( D \). Furthermore, suppose that \( C_\varphi : B_\alpha \rightarrow B_\alpha \) is bounded (i.e. \( \sup_{z \in D} (1 - |z|^2)^\alpha (1 - |\varphi(z)|^2)^{-\alpha} |\varphi'(z)| < \infty \)), and that \( C_\varphi : \mathcal{D} \rightarrow \mathcal{D} \) is bounded. Then, the following are equivalent.

1. \( C_\varphi : L_2^a \rightarrow L_2^a \) is bounded below.
2. \( C_\varphi : \mathcal{D}_p^\gamma \rightarrow \mathcal{D}_p^\gamma \) is bounded below for some \( \gamma > 1 \).
3. \( C_\varphi : \mathcal{D}_p^\gamma \rightarrow \mathcal{D}_p^\gamma \) is bounded below for all \( \gamma > 1 \).
4. \( C_\varphi : B_\alpha \rightarrow B_\alpha \) is bounded below for some \( 0 < \alpha < 1 \).
5. \( C_\varphi : B_\alpha \rightarrow B_\alpha \) is bounded below for all \( 0 < \alpha < 1 \).
6. \( C_\varphi : B_\gamma \rightarrow B_\gamma \) is bounded below for some \( \gamma > 1 \).
7. \( C_\varphi : B_\gamma \rightarrow B_\gamma \) is bounded below for all \( \gamma > 1 \).
8. \( C_\varphi : \mathcal{D} \rightarrow \mathcal{D} \) is bounded below.
9. \( C_\varphi : B \rightarrow B \) is bounded below.
10. \( \varphi \) is an automorphism of \( D \).

In [18] we also proved the following result.

**Theorem 6.** ([18]) Let \( 0 < p, q < +\infty \), and \( \alpha, \gamma > 0 \). Suppose that \( C_\varphi : \mathcal{D}_p^{\alpha} \rightarrow \mathcal{D}_q^{\gamma} \) is bounded. If \( C_\varphi : \mathcal{D}_p^{\alpha} \rightarrow \mathcal{D}_q^{\gamma} \) is bounded below, then there exists a constant \( K > 0 \) such that

\[
\sup_{z \in D} |(C_\varphi f)'(z)|(1 - |z|^2)^\gamma \geq KS_{p,q,\alpha}(f)
\]
for all \( f \in B_\alpha \), where
\[
S_{p,q,\alpha}(f) := \begin{cases} 
\sup_{z \in D} |f'(z)| (1 - |z|^2)^{\alpha + 2(\frac{1}{p} - \frac{1}{q})} & (1 < q \leq p) \\
\sup_{z \in D} |f'(z)| (1 - |z|^2)^{\alpha + 2(\frac{1}{p} - 1)} \left( \log \frac{2}{1 - |z|^2} \right)^{-1} & (q = 1 < p) \\
\sup_{z \in D} |f'(z)| (1 - |z|^2)^{\alpha + 2(\frac{1}{p} - 1)} & (0 < q < 1 \leq p).
\end{cases}
\]

The following is trivial using Theorem 2 and Theorem 3.

**Proposition 7.** Let \( p \geq 2 \). Suppose \( \varphi \) is a univalent self-map of \( D \).
Then the following conditions are equivalent:

1. \( C_\varphi : B_{p,p-1} \to B_{p,p-1} \) is bounded below.
2. \( \varphi \) is an automorphism of \( D \).

In this context, we study the following natural problem.

**Problem A.** Let \( p \geq 2 \). Suppose \( \varphi \) is a univalent self-map of \( D \).
Then the following conditions are equivalent:

1. \( C_\varphi : B_p \to B_p \) is bounded below.
2. \( \varphi \) is an automorphism of \( D \).

In this paper, we get several results with respect to the boundedness from below of composition operators \( B_p, B_{p,p-1} \)

### 3. The main results and the univalent case

If \( \varphi(0) = a \) and \( \psi = \varphi \circ \varphi \), then \( C_{\varphi} \) is bounded below on \( B_\alpha \) (or \( D_\alpha \)) to \( B_\alpha \) (or \( D_\alpha \)) if and only if \( C_{\psi} \) is bounded below on \( B_\alpha \) (or \( D_\alpha \)) to \( B_\alpha \) (or \( D_\alpha \))(See [6] and [21]). So we assume from now on that \( \varphi(0) = 0 \) and that \( C_{\varphi} \) is acting on the subspace of functions that vanish at the origin.

In [18] we proved the following result.

**Lemma 8.** ([18]) Let \( 0 < \alpha < \beta < 1 \). If \( C_{\varphi} : B_\alpha \to B_\alpha \) is bounded, then \( C_{\varphi} : B_\beta \to B_\beta \) is bounded.

In [18] we also proved the following result.

**Lemma 9.** ([18]) Let \( 0 < \alpha < \beta < 1 \) Suppose \( \varphi \) is a univalent self-map of \( D \) and that \( C_{\varphi} : B_\alpha \to B_\alpha \) is bounded
(i.e. \( \sup_{z \in D} (1 - |z|^2)^{\alpha} (1 - |\varphi(z)|^2)^{-\alpha} |\varphi'(z)| < \infty \)). Then the following are equivalent.
(1) \( C_\phi : B_\alpha \to B_\alpha \) is bounded below.
(2) \( C_\phi : B_\beta \to B_\beta \) is bounded below.
(3) \( \phi \) is an automorphism of \( D \).

**Lemma 10.** ([13]) Let \( p \geq 2 \) and \( \phi \) a holomorphic self-map of \( D \).
(1) If \( \alpha \geq p - 2 \) and \( \phi \) is boundedly valent, then \( C_\phi \) is a bounded operator on \( B_{p,\alpha} \).
(2) If \( \alpha \geq p - 1 \), then \( C_\phi \) is a bounded operator on \( B_{p,\alpha} \).

**Corollary 11.** Let \( p \geq 2 \), and \( \phi \) a holomorphic self-map of \( D \). If \( C_\phi : B_{p,p-1} \to B_{p,p-1} \) is bounded below, then \( C_\phi : B_{1-\frac{1}{p}} \to B_{1-\frac{1}{p}} \) is bounded below.

**Proof.** Let \( p \geq 2 \). Then Lemma 10 implies that \( C_\phi \) is a bounded operator on \( B_{p,p-1} \). Applying \( \alpha = 1 - \frac{1}{p} \) and \( \gamma = 1 - \frac{1}{q} \) in Theorem 6, we can prove that \( C_\phi : B_{1-\frac{1}{q}} \to B_{1-\frac{1}{q}} \) is bounded below. \( \square \)

**Corollary 12.** Let \( p > 2 \), and \( \phi \) a holomorphic self-map of \( D \). If \( C_\phi : B_p \to B_p \) is bounded below, then \( C_\phi : B_{1-\frac{2}{p}} \to B_{1-\frac{2}{p}} \) is bounded below.

**Proof.** Let \( p > 2 \). Then Lemma 10 implies that \( C_\phi \) is a bounded operator on \( B_p \). Applying \( \alpha = 1 - \frac{2}{p} \) and \( \gamma = 1 - \frac{2}{q} \) in Theorem 6, we can prove that \( C_\phi : B_{1-\frac{2}{q}} \to B_{1-\frac{2}{q}} \) is bounded below. \( \square \)

Using Corollary 12, we have the following result which satisfies Problem A under certain conditions.

**Theorem 13.** Let \( p > 2 \). Suppose \( \phi \) is a univalent self-map of \( D \) and that \( C_\phi : B_{1-\frac{2}{p}} \to B_{1-\frac{2}{p}} \) is bounded
(i.e. \( \sup_{z \in D} (1 - |z|^2)^{1-\frac{2}{p}} (1 - |\phi(z)|^2)^{-(1-\frac{2}{p})} |\phi'(z)| < \infty \). Then the following are equivalent.
(1) \( C_\phi : B_p \to B_p \) is bounded below.
(2) \( C_\phi : B \to B \) is bounded below.
(3) \( C_\phi : B_{1-\frac{2}{p}} \to B_{1-\frac{2}{p}} \) is bounded below.
(4) \( C_\phi : B_{\gamma} \to B_{\gamma} \) is bounded below \( (1 - \frac{2}{p} < \gamma < 1) \).
(5) \( \phi \) is an automorphism of \( D \).

**Proof.** Let \( p > 2 \). Suppose that \( C_\phi : B_{1-\frac{2}{p}} \to B_{1-\frac{2}{p}} \) is bounded. Then Lemma 8 implies that \( C_\phi : B_{\gamma} \to B_{\gamma} \) is bounded \((1 - \frac{2}{p} < \gamma < 1) \). Since \( p > 2 \), Lemma 10 implies that \( C_\phi \) is a bounded operator on \( B_p \). Since the Besov spaces \( B_p \) are the only Möbius invariant spaces among all Besov type
spaces($\| f \circ \varphi_a \|_{B_p} = \| f \|_{B_p}$ for $a \in D$), using Theorem 4, Lemma 9 and Corollary 12, we can prove theorem. □

**Remark 14.** We will see that Theorem 13 slightly improves and extends Theorem 5.

Using Theorem 9 and Theorem 11, we have the following result which satisfies both Problem A and Problem B under certain conditions.

**Theorem 15.** Let $p > 2$. Suppose $\varphi$ is a univalent self-map of $D$ and that $C_\varphi : B_{1-p} \to B_{1-p}$ is bounded

(i.e. $\sup_{z \in D} (1 - |z|^2)^{1-p/2} (1 - |\varphi(z)|^2)^{-(1-p/2)} |\varphi'(z)| < \infty$). Then the following are equivalent.

1. $C_\varphi : B_p \to B_p$ is bounded below.
2. $C_\varphi : B_{p,p-1} \to B_{p,p-1}$ is bounded below.
3. $C_\varphi : H^2 \to H^2$ is bounded below.
4. $C_\varphi : BMOA \to BMOA$ is bounded below.
5. $C_\varphi : B \to B$ is bounded below.
6. $\varphi$ is an automorphism of $D$.

**Proof.** Let $p > 2$. Suppose that $C_\varphi : B_{1-p} \to B_{1-p}$ is bounded. Then Lemma 5 implies that $C_\varphi : B_{1-p} \to B_{1-p}$ is bounded. Lemma 7 implies that $C_\varphi$ is bounded on both $B_p$ and $B_{p,p-1}$. That (2) implies (3) follows from Theorem 3. That (3) implies (4) follows from Theorem 2.4 in [17]. That (4) implies (5) follows from Corollary 2 in [6]. Thus using Theorem 4, Proposition 7 and Theorem 13, we can prove theorem. □

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