Closed Range Integral Operators Between $L^p_0$ and $L^q_0$, Between $L^p_0$ and the Hardy Space $H^2$, Between $L^p_0$ and Besov Space

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Abstract

We study the relation between the integral operators $S_g$ with closed range on the weighted Bloch spaces and $S_g$ with closed range on the weighted Dirichlet spaces $D^p_0$. In particular, we study the integral operators $S_g$ with closed range between $L^p_0$ and $L^q_0$, between $L^p_0$ and Hardy space $H^2$, and between $L^p_0$ and Besov space.

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1. Introduction

For $g$ analytic function on the open unit disk $D$, the integral operator $S_g$ is defined by

$$S_g f(z) = \int_0^z f'(w)g(w)dw.$$
For \( z, w \in D \), let \( \varphi_z(w) = \frac{z - w}{1 - \overline{z}w} \) and, \( dA(z) \) the area measure on \( D \).

For \( p > 0, \alpha > -1 \), the weighted Dirichlet space \( D_\alpha^p \) is defined to be the space of analytic functions \( f \) on \( D \) such that

\[
|f(0)| + \left( \int_D (1 - |z|^2)^\alpha |f'(z)|^p dA(z) \right)^{\frac{1}{p}} < \infty.
\]

In case \( \alpha = 1 \) and \( p = 2 \), then \( D_1^2 = H^2 \) is the classical Hardy space. Furthermore, in case \( \alpha = p \) and \( 1 \leq p < \infty \), then \( D_p^p = L^p_\alpha \) is the usual Bergman space. In particular, \( D_0^2 = D \) is called the Dirichlet space. (See [19].)

For \( \alpha > 0 \), the weighted Bloch space \( B_\alpha \) is defined to be the space of analytic functions \( f \) on \( D \) such that

\[
|f(0)| + \sup_{z \in D} (1 - |z|^2)^\alpha |f'(z)| < \infty.
\]

Note that \( B_1 = B \) is the usual Bloch space.

The amount \( \sup_{z \in D} (1 - |z|^2)^\alpha |f'(z)| \) is a pseudonorm, which coincides with the \( B_\alpha \)-norm on the closed subspace of functions that vanish at the origin. So it coincides with the quotient norm on \( B_\alpha / C \) where \( C \) denotes the closed subspace of constant functions. The space \( BMOA \) is defined to be the space of analytic functions \( f \) on \( D \) such that

\[
|f(0)| + \sup_{a \in D} \left( \int_D (1 - |\varphi_a(z)|^2)|f'(z)|^2 dA(z) \right)^{\frac{1}{2}} < \infty.
\]

Let \( X \) be a Banach space and let \( T \) a linear operator from \( X \) into \( X \). An operator \( T \) is called bounded below on \( X \) if there exists a constant \( C > 0 \) such that \( \| Tf \| \geq C \| f \| \) for all \( f \in X \).

In [1], Austin Anderson determined the boundedness from below of integral operators on \( H^2 \), the Bloch space \( B \) and \( L^p_\alpha \). Moreover, in [4], Kostas Panteris determined the boundedness from below of integral operators on \( H^p \), the BMOA and Besov space. We characterized the boundedness of integral operators on the Bloch spaces \( B \), the weighted Bloch space \( B_\alpha \), the weighted BMOA and the weighted Bergman space \((\delta)[6][7][9])\). Furthermore, Santeri Miichinen, Jordi Pau, Antti Perälä and Maofa Wang ([3]) studied the boundedness and compactness of Volterra type integration operators \( T_g \) between Bergman spaces and Hardy spaces, where \( T_g f(z) = \int_0^z f(w)g'(w)dw \).

In this paper, we study when integral operators are bounded below on the weighted Dirichlet space \( D_\alpha^p \), the weighted Bloch spaces \( B_\alpha \) and the Bergman spaces \( L^p_\alpha \), respectively. Moreover, we study relationship between the boundedness from below of integral operators on \( D_\alpha^p \) and it on \( B_\alpha \). As a result, we...
can characterized the boundedness from below of integral operators between \( L^p_\alpha \) and \( H^2 \), between \( L^p_\alpha \) and \( B_p \), respectively.

2. Background Material

In this section, we introduce several results to prove the main theorem. Since the integral operator \( S_g \) maps every constant function to the 0 function, in the case of considering the property of being bounded below for \( S_g \), it is useful to consider spaces of analytic functions modulo the constants.

In [1] Austin Anderson proved the following result.

**Theorem A.** ([1]) Let \( Y \) be a Banach space of analytic functions on the disk. For nonconstant \( g \), \( S_g \) is bounded below on \( Y/\mathbb{C} = \{ f \in Y : f(0) = 0 \} \) if and only if it has closed range on \( Y/\mathbb{C} = \{ f \in Y : f(0) = 0 \} \).

In [1] Austin Anderson also proved the following result.

**Theorem B.** ([1]) The following are equivalent for \( g \in H^\infty \).
1. \( g = BF \) for a finite product \( B \) of interpolating Blaschke products and \( F \) such that \( F, 1/F \in H^\infty \).
2. \( S_g \) is bounded below on \( B/\mathbb{C} = \{ f \in B : f(0) = 0 \} \).
3. There exist \( r < 1 \) and \( \eta > 0 \) such that for all \( a \in D \), \( \sup_{z \in D(a,r)} |g(z)| > \eta \).
4. \( S_g \) is bounded below on \( H^2/\mathbb{C} = \{ f \in H^2 : f(0) = 0 \} \).
5. \( M_g \) is bounded below on \( L^p_\alpha \).
6. \( S_g \) is bounded below on \( L^p_\alpha/\mathbb{C} = \{ f \in L^p_\alpha : f(0) = 0 \} \).

In this paper, we characterize the relation between the integral operators \( S_g \) with closed range on the weighted Bloch spaces and \( S_g \) with closed range on the weighted Dirichlet spaces \( D^\alpha_p \). And we get several results with respect to the boundedness from below of integral operators between Bergman spaces \( L^p_\alpha \) and Bergman spaces \( L^q_\alpha \), the boundedness from below of integral operators between Bergman spaces \( L^p_\alpha \) and Hardy space, and the boundedness from below of integral operators between Bergman spaces \( L^p_\alpha \) and Besov space \( B_p \).

3. The main results

To prove the main theorem, we need the following lemma.

**Lemma 1.** Let \( \alpha, \beta > 0 \) and \( \alpha \neq \beta \). Suppose \( S_g : B_\alpha \rightarrow B_\beta \) is bounded. Then the following hold.
1. If \( \alpha > \beta \), then \( S_g : B_\alpha/\mathbb{C} \rightarrow B_\beta/\mathbb{C} \) is not bounded below.
2. If \( \alpha < 1 \leq \beta \), or \( 1 \leq \alpha < \beta \), and \( g \in H^\infty \), then \( S_g : B_\alpha/\mathbb{C} \rightarrow B_\beta/\mathbb{C} \) is not bounded below.
Proof. Let $\alpha > \beta$. If $1 < s < \infty$, $f_a(z) := (1 - |az|^{1-\alpha} - 1$, then we have

$$\sup_{a \in D}(1 - |a|^2)^{2(\beta - \alpha)}|g(a)|^2$$

$$= \sup_{a \in D}(1 - |a|^2)^{2(\beta - \alpha - 1)}(1 - |a|^2)|g(a)|^2$$

$$\leq C \sup_{a \in D}(1 - |a|^2)^{2(\beta - \alpha - 1)} \int_{D(a,r)} |g(z)|^2 dA(z)$$

$$\leq C \sup_{a \in D} \int_{D(a,r)} \frac{(1 - |z|^2)^{2(\beta - 1)}}{1 - |az|^{2\alpha}} |g(z)|^2(1 - |\varphi_a(z)|^2)^s dA(z)$$

$$\leq C \sup_{a \in D} \int_D (1 - |z|^2)^{2(\beta - 1)} |f'(z)|^2 |g(z)|^2(1 - |\varphi_a(z)|^2)^s dA(z)$$

$$= C \sup_{a \in D} \int_D (1 - |z|^2)^{2(\beta - 1)} (Sg f_a)'(z)^2(1 - |\varphi_a(z)|^2)^s dA(z)$$

$$\approx \left\{ \sup_{z \in D} (1 - |z|^2)^{\beta} |(Sg f_a)'(z)| \right\}^2 < \infty.$$ 

which implies $g \equiv 0$ as $\alpha > \beta$. Thus $S_g : \mathcal{B}_\alpha/\mathcal{C} \to \mathcal{B}_\beta/\mathcal{C}$ is not bounded below as $\alpha > \beta$.

Next, suppose that $\alpha < 1 \leq \beta$, or $1 \leq \alpha < \beta$, and $g \in H^\infty$. Let $f_{w_n}(z) := 1 - \frac{|w_n|^2}{w_n z} \left\{ \frac{1}{(1 - \frac{|w_n|^2}{w_n z})^{\alpha+1}} - 1 \right\}$. Then $f'_{w_n}(z) = \frac{1 - |w_n|^2}{(1 - \frac{|w_n|^2}{w_n z})^{\alpha+1}}$. Let $w_n \to \partial D$. Then it is clear that $f_{w_n} \in \mathcal{B}_\alpha$ and

$$\sup_{z \in D} |f'_{w_n}(z)||1 - |z|^2|^\alpha = \sup_{z \in D} \frac{(1 - |w_n|^2)(1 - |z|^2)^\alpha}{|1 - \frac{|w_n|^2}{w_n z}|^{\alpha+1}} \geq 1.$$ 

On the other hand, there exists a constant $C > 0$ such that

$$\sup_{z \in D} |(Sg f_{w_n})'(z)|(1 - |z|^2)^\beta$$

$$\leq \| g \|_\infty \sup_{z \in D} |f'_{w_n}(z)|(1 - |z|^2)^\beta$$

$$= \| g \|_\infty \sup_{z \in D} \frac{(1 - |w_n|^2)(1 - |z|^2)^{\alpha+(\beta-\alpha)}}{|1 - \frac{|w_n|^2}{w_n z}|^{\alpha+(\beta-\alpha)+1}}$$

$$\leq C \| g \|_\infty \sup_{z \in D} \frac{1 - |w_n|^2}{|1 - \frac{|w_n|^2}{w_n z}|^{-(\beta-\alpha)+1}} \to 0 (n \to \infty).$$

Thus $S_g : \mathcal{B}_\alpha/\mathcal{C} \to \mathcal{B}_\beta/\mathcal{C}$ is not bounded below as $\alpha < 1 \leq \beta$, or $1 \leq \alpha < \beta$, $g \in H^\infty$. □

Let $\alpha > -1$. For $\forall a \in D$, the following estimate is standard ([10]).
Closed range composition operators

\[ \int_D (1 - |z|^2)^{\alpha} \frac{dA(z)}{|1 - \overline{a}z|^{\lambda}} \sim \begin{cases} (1 - |a|^2)^{\lambda + 2 - \lambda} & (\lambda > \alpha + 2) \\ \log \frac{2}{1 - |a|^2} & (\lambda = \alpha + 2) \\ 1 & (\lambda < \alpha + 2). \end{cases} \]

Using the estimate (\@), we have the following result.

**Theorem 2.** Let $0 < p, q < +\infty$, and $\alpha, \gamma > 0$. Suppose that $S_g : D_p^{\alpha} \to D_q^{\gamma}$ is bounded. If $S_g : D_p^{\alpha} / C \to D_q^{\gamma} / C$ is bounded below, then there exists a constant $K > 0$ such that

\[ \sup_{z \in D} |(S_g f)'(z)| (1 - |z|^2)^\gamma \geq K S_{p,q,\alpha}(f) \]

for all $f \in B_{\alpha}$, where

\[ S_{p,q,\alpha}(f) := \begin{cases} \sup_{z \in D} |f'(z)|(1 - |z|^2)^{\alpha + 2(\frac{1}{p} - \frac{1}{q})} & (1 < q \leq p) \\ \sup_{z \in D} |f'(z)|(1 - |z|^2)^{\alpha + 2(\frac{1}{p} - 1)} \left( \log \frac{2}{1 - |z|^2} \right)^{-1} & (q = 1 < p) \\ \sup_{z \in D} |f'(z)|(1 - |z|^2)^{\alpha + 2(\frac{1}{p} - 1)} & (0 < q < 1 \leq p) \end{cases} \]

**Proof.** For $a \in D$ and $\forall f \in B_{\alpha}$, we see that

\[ F(z) = \int_0^z f'(\zeta) \varphi_a'(\zeta) d\zeta \in D_p^{\alpha}. \] (2.1)

In fact, using the evaluation (\@), it holds that

\[ \left\{ \int_D (1 - |z|^2)^{\alpha} |F'(z)|^p dA(z) \right\}^{\frac{1}{p}} = \left\{ \int_D |f'(z)|^p |\varphi_a'(z)|^p (1 - |z|^2)^\alpha dA(z) \right\}^{\frac{1}{p}} \leq \sup_{z \in D} (1 - |z|^2)^{\alpha} |f'(z)| \left\{ \int_D |\varphi_a'(z)|^p dA(z) \right\}^{\frac{1}{p}}, \]

and

\[ \left\{ \int_D |\varphi_a'(z)|^p dA(z) \right\}^{\frac{1}{p}} = \left\{ \int_D \frac{(1 - |a|^2)^p}{|1 - \overline{a}z|^{2p}} dA(z) \right\}^{\frac{1}{p}} \sim \begin{cases} (1 - |a|^2)^{\frac{p}{2} - 1} & (p > 1) \\ (1 - |a|^2) \log \frac{2}{1 - |a|^2} & (p = 1) \\ (1 - |a|^2)^{\frac{p}{2}} & (0 < p < 1). \end{cases} \]

Hence $F(z) = \int_0^z f'(\zeta) \varphi_a'(\zeta) d\zeta \in D_p^{\alpha}$.

Let $p \geq q > 1$ and $f \in B_{\alpha}$, then (2.1) implies that $F \in D_p^{\alpha}$. Since $S_g$
: $\mathcal{D}_p^{\alpha_\gamma} / \mathcal{C} \to \mathcal{D}_q^{\alpha_\gamma} / \mathcal{C}$ is bounded below, for any $a \in D$, using subharmonicity of $|f \circ \varphi_a|^p$, there exists a constant $K > 0$ such that

$$
\left\{ |f'(a)|^p (1 - |a|^2)^{p(\alpha - 1)} \right\}^{\frac{1}{p}} \\
\leq K \left\{ \int_D (1 - |z|^2)^{p(\alpha - 1)} |f'(z)|^p (1 - |\varphi_a(z)|^2)^p |dA(z)| \right\}^{\frac{1}{p}} \\
= K \left\{ \int_D |f'(z)|^p |\varphi_a'(z)|^p (1 - |z|^2)^{p|a|} |dA(z)| \right\}^{\frac{1}{p}} \\
= K \left\{ \int_D (1 - |z|^2)^{|a|} |F'(z)|^p |dA(z)| \right\}^{\frac{1}{p}} \\
\leq K \left\{ \int_D (1 - |z|^2)^{|a|} |(S_g f)'(z)|^q |dA(z)| \right\}^{\frac{1}{q}} \\
= K \left\{ \sup_{z \in D} |(S_g f)'(z)| |(1 - |z|^2)|^\gamma \right\} \left\{ \int_D |\varphi_a'(z)|^q |dA(z)| \right\}^{\frac{1}{q}}.
$$

Since $S_g$ is bounded on $L^q_a$ and that $\varphi_a' \in L^q_a$, for any $a \in D$, using the evaluation (@),

$$
\left\{ \int_D |\varphi_a'(z)|^q |dA(z)| \right\}^{\frac{1}{q}} \approx (1 - |a|^2)^{(2-q)\gamma} < \infty.
$$

Hence there exists a constant $K' > 0$ such that

$$
\sup_{z \in D} |f'(z)| (1 - |z|^2)^{\alpha + 2(\frac{1}{p} - \frac{1}{q})} \leq K' \sup_{z \in D} |(S_g f)'(z)| (1 - |z|^2)^\gamma \quad (\forall f \in \mathcal{B}_\alpha).
$$

Let $p > q = 1$ and $f \in \mathcal{B}_\alpha$, then (2.1) implies that $F \in \mathcal{D}_p^{\alpha_\gamma}$. So we can also prove that there exists a constant $K' > 0$ such that

$$
\sup_{z \in D} |f'(z)| (1 - |z|^2)^{\alpha + 2(\frac{1}{p} - \frac{1}{q})} \leq K' \sup_{z \in D} |(S_g f)'(z)| (1 - |z|^2)^\gamma \quad (\forall f \in \mathcal{B}_\alpha).
$$

Let $p \geq 1 > q > 0$ and $f \in \mathcal{B}_\alpha$, then (2.1) implies that $F \in \mathcal{D}_p^{\alpha_\gamma}$. Thus we can also prove that there exists a constant $K' > 0$ such that

$$
\sup_{z \in D} |f'(z)| (1 - |z|^2)^{2(\frac{1}{p} - \frac{1}{q})} \leq K' \sup_{z \in D} |(S_g f)'(z)| (1 - |z|^2)^\gamma \quad (\forall f \in \mathcal{B}_\alpha).
$$

This completes the proof of theorem. \hfill \Box

**Remark 3.** Let $1 < p < \infty$. If $S_g : L^q_B / \mathcal{C} \to L^p_B / \mathcal{C}$ is bounded below, then $S_g : B / \mathcal{C} \to B / \mathcal{C}$ is bounded below. In fact, when $1 < p < \infty$, applying
\( \alpha = \gamma = 1 \) and \( q = p > 1 \) in Theorem 2 and using the property (\( \oplus \)), we can prove that \( S_g : \mathcal{B}/\mathcal{C} \rightarrow \mathcal{B}/\mathcal{C} \) is bounded below. Of course, this implication can be obtained directly from Theorem B without using Theorem 2.

**Remark 4.** If \( S_g : H^2/\mathcal{C} \rightarrow H^2/\mathcal{C} \) is bounded below, then \( S_g : \mathcal{B}^\frac{1}{2}/\mathcal{C} \rightarrow \mathcal{B}^\frac{1}{2}/\mathcal{C} \) is bounded below. In fact, applying \( \alpha = \gamma = \frac{1}{2} \) in Theorem 2 and using the property (\( \oplus \)), we can prove that \( S_g : \mathcal{B}^\frac{1}{2}/\mathcal{C} \rightarrow \mathcal{B}^\frac{1}{2}/\mathcal{C} \) is bounded below. On the other hand, we also have the following.

**Theorem 5.** If \( S_g : H^2/\mathcal{C} \rightarrow H^2/\mathcal{C} \) is bounded below, then \( S_g : BMOA/\mathcal{C} \rightarrow BMOA/\mathcal{C} \) is bounded below.

**Proof.** In fact, for any \( a \in D \) and \( f \in BMOA \), we have that

\[
F(z) = \int_0^z f'(w) (\varphi_a'(w))^{\frac{1}{2}} \, dw \in H^2.
\]

In fact, we have

\[
\int_D (1 - |z|^2)|F'(z)|^2 \, dA(z)
\]
\[
= \int_D (1 - |z|^2)|f'(z)|^2 |\varphi_a'(z)| \, dA(z)
\]
\[
= \int_D (1 - |\varphi_a(z)|^2)|f'(z)|^2 \, dA(z)
\]
\[
\leq \sup_{a \in D} \int_D (1 - |\varphi_a(z)|^2)|f'(z)|^2 \, dA(z) < \infty.
\]

Hence we have that \( F \in H^2 \).

On the other hand, let \( Pf(z) = \int_0^z f(w)h'(w) \, dw \). Then there exists a constant \( k > 0 \) such that \( \int_D |f(z)|^2|h'(z)|^2(1 - |z|^2) \, dA(z) \leq k \int_D |f'(z)|^2(1 - |z|^2) \, dA(z) \) if and only if \( |h'(z)|(1 - |z|^2) \, dA(z) \) is a Carleson measure (see [8]). Furthermore, the above inequality shows that the operator norm \( \| P \|_{H^2} \) is comparable to \( \| h \|_{BMOA} \).

If \( S_g : H^2/\mathcal{C} \rightarrow H^2/\mathcal{C} \) is bounded below, then for any \( a \in D \) and \( f \in BMOA \), for some constants \( C, C' > 0 \), since \( \int_D \left| \left( (\varphi_a'(z))^{\frac{1}{2}} \right) \right|^2 (1 - |z|^2) \, dA(z) < \infty \), we have

\[
\int_D (1 - |\varphi_a(z)|^2)|f'(z)|^2 \, dA(z)
\]
\[
= \int_D (1 - |z|^2)|F'(z)|^2 \, dA(z)
\]
\[
\leq C \int_D (1 - |z|^2)|(S_gF)'(z)|^2 \, dA(z)
\]
= C \int_D (1 - |z|^2) \left| g(z) \right|^2 |f'(z)|^2 |\varphi'_a(z)| \, dA(z) \\
= C \int_D \left| (\varphi'_a(z))^{\frac{1}{2}} (1 - |z|^2) (Sg f)'(z) \right|^2 dA(z) \\
\leq C C' \| Sg f \|_{BMO}^2 \int_D \left| (\varphi'_a(z))^{\frac{1}{2}} \right|^2 (1 - |z|^2) dA(z) < \infty.

Hence \( S_g : BMOA/\mathcal{C} \to BMOA/\mathcal{C} \) is bounded below. \( \square \)

**Remark 6.** The above implication can be obtained directly from both Kostas Panteris’ result in [4] and Theorem B without using Theorem 5.

If \( \gamma = 1 \), then \( L^p_a \to \mathcal{D}^{\mathcal{D}_p/\mathcal{C}} \) and that there exists a symbol \( g \) such that \( S_g : L^p_a/\mathcal{C} \to L^p_a/\mathcal{C}(= \mathcal{D}^{\mathcal{D}_p/\mathcal{C}}) \) is bounded below. If \( \gamma \neq 1 \), then there is no symbol \( g \) such that \( S_g : L^p_a/\mathcal{C} \to \mathcal{D}^{\mathcal{D}_p/\mathcal{C}} \) is bounded below.

**Corollary 7.** Let \( 1 < p < \infty \). If \( \gamma > 1 \) and \( g \in H^\infty \), then \( S_g : L^p_a \to \mathcal{D}^{\mathcal{D}_p/\mathcal{C}} \) is bounded, while there is no symbol \( g \) such that \( S_g : L^p_a/\mathcal{C} \to \mathcal{D}^{\mathcal{D}_p/\mathcal{C}} \) is bounded below. If \( \gamma < 1 \), supposing that \( S_g : L^p_a \to \mathcal{D}^{\mathcal{D}_p/\mathcal{C}} \) is bounded, then there is no symbol \( g \) such that \( S_g : L^p_a/\mathcal{C} \to \mathcal{D}^{\mathcal{D}_p/\mathcal{C}} \) is bounded below.

**Proof.** Let \( p > 1 \). If \( \gamma > 1 \), since \( g \in H^\infty \), the boundedness of \( S_g : L^p_a \to \mathcal{D}^{\mathcal{D}_p/\mathcal{C}} \) is trivial. Let \( p > 1 \). Theorem 2 and (2) of Lemma 1 imply that \( S_g : L^p_a/\mathcal{C} \to \mathcal{D}^{\mathcal{D}_p/\mathcal{C}} \) is not bounded below. If \( \gamma < 1 \), supposing the boundedness of \( S_g : L^p_a \to \mathcal{D}^{\mathcal{D}_p/\mathcal{C}} \), then Theorem 2 and (1) of Lemma 1 imply that \( S_g : L^p_a/\mathcal{C} \to \mathcal{D}^{\mathcal{D}_p/\mathcal{C}} \) is not bounded below. \( \square \)

If \( p > q > 1 \) and \( S_g : L^p_a \to \mathcal{D}^{\mathcal{D}_q/\mathcal{C}} \), then we have the following.

**Corollary 8.** Let \( 1 < q < p \) and \( g \in H^\infty \). If \( \gamma \geq 1 \), then \( S_g : L^p_a \to \mathcal{D}^{\mathcal{D}_q/\mathcal{C}} \) is bounded, while there is no symbol \( g \) such that \( S_g : L^p_a/\mathcal{C} \to \mathcal{D}^{\mathcal{D}_q/\mathcal{C}} \) is bounded below.

**Proof.** If \( \gamma \geq 1 \), since \( g \in H^\infty \), the boundedness of \( S_g : L^p_a \to \mathcal{D}^{\mathcal{D}_q/\mathcal{C}} \) follows from Hölder’s inequality. If \( \gamma \geq 1 \), then Theorem 2 and (2) of Lemma 1 imply that \( S_g : L^p_a/\mathcal{C} \to \mathcal{D}^{\mathcal{D}_q/\mathcal{C}} \) is not bounded below.

If \( 0 < q < p = 1 \), if \( \gamma \neq 1 \), then there is no symbol \( g \) such that \( S_g : L^1_a/\mathcal{C} \to \mathcal{D}^{\mathcal{D}_q/\mathcal{C}} \) is bounded below.

**Corollary 9.** Let \( 0 < q < 1 \). If \( \gamma > 1 \) and \( g \in H^\infty \), then \( S_g : L^1_a \to \mathcal{D}^{\mathcal{D}_q/\mathcal{C}} \) is bounded, while there is no symbol \( g \) such that \( S_g : L^1_a/\mathcal{C} \to \mathcal{D}^{\mathcal{D}_q/\mathcal{C}} \) is bounded below. If \( \gamma < 1 \), supposing that \( S_g : L^1_a \to \mathcal{D}^{\mathcal{D}_q/\mathcal{C}} \) is bounded, then there is no symbol \( g \) such that \( S_g : L^1_a/\mathcal{C} \to \mathcal{D}^{\mathcal{D}_q/\mathcal{C}} \) is bounded below.

**Proof.** Let \( 0 < q < 1 \). If \( \gamma > 1 \), since \( g \in H^\infty \), the boundedness of \( S_g : L^1_a \to \mathcal{D}^{\mathcal{D}_q/\mathcal{C}} \) follows from Hölder’s inequality. If \( \gamma > 1 \), supposing that the
boundedness of \( S_g : L^1_a \to \mathcal{D}_q^{r'}/L^q_a \), then Theorem 2 and (2) of Lemma 1 imply that \( S_g : L^1_a/C \to \mathcal{D}_q^{r'}/C \) is not bounded below. If \( \gamma < 1 \), supposing that \( S_g : L^1_a \to \mathcal{D}_q^{r'}/C \) is bounded, Theorem 2 and (1) of Lemma 1 imply that \( S_g : L^1_a/C \to \mathcal{D}_q^{r'}/C \) is not bounded below. \( \square \)

With respect to the integral operator \( S_g : \mathcal{D}_p^{\alpha} \to L^q_a \), then we have the following.

**Corollary 10.** Suppose \( \alpha < 1 \) and \( g \in H^\infty \). If \( p \geq q > 1 \) or \( p > 1 = q \) or \( p \geq 1 > q > 0 \), supposing that \( S_g : \mathcal{D}_p^{\alpha} \to L^q_a \) is bounded, then there is no symbol \( g \) such that \( S_g : \mathcal{D}_p^{\alpha}/C \to L^q_a/C \) is bounded below.

**Proof.** Since \( \alpha < 1 \) and \( g \in H^\infty \), the boundedness of \( S_g : \mathcal{D}_p^{\alpha} \to L^q_a \) follows from Hölder’s inequality. If \( p \geq q > 1 \) or \( p > 1 = q \) or \( p \geq 1 > q > 0 \), supposing that \( S_g : \mathcal{D}_p^{\alpha} \to L^q_a \) is bounded, then Theorem 2 and (2) of Lemma 1 imply that \( S_g : \mathcal{D}_p^{\alpha}/C \to L^q_a/C \) is not bounded below. \( \square \)

The following result has never been proven so far.

**Corollary 11.** Suppose \( \frac{4}{3} < q \leq 2 \) and \( g \in H^\infty \). Then \( S_g : H^2 \to L^q_a \) is bounded, while there is no symbol \( g \) such that \( S_g : H^2/C \to L^q_a/C \) is bounded below.

**Proof.** Since \( g \in H^\infty \), the boundedness of \( S_g : H^2 \to L^q_a \) \((q \leq 4)\) follows from Hölder’s inequality and the fact \( H^2 \subset L^q_a(q \leq 4) \) (see [2]). Applying \( \alpha = \frac{1}{2} \), \( \gamma = 1 \) and \( p = 2, \frac{4}{3} < q \leq 2 \) in Theorem 2, it follows from (2) of Lemma 1 that \( S_g : H^2/C \to L^q_a/C \) is not bounded below.

The following result has never been proven so far.

**Corollary 12.** Suppose \( 2 \leq p < 4 \). If \( S_g : L^p_a \to H^2 \) is bounded, then there is no symbol \( g \) such that \( S_g : L^p_a/C \to H^2/C \) is bounded below.

**Proof.** Suppose that \( 2 \leq p < 4 \), and that \( S_g : L^p_a \to H^2 \) is bounded. Applying \( \alpha = 1 \), \( \gamma = \frac{1}{2} \), \( 2 \leq p < 4 \) and \( q = 2 \) in Theorem 2, it follows from (1) of Lemma 1 and the fact \( 1 + 2\left(\frac{1}{p} - \frac{1}{2}\right) > \frac{1}{2} = \gamma \), that \( S_g : L^p_a/C \to H^2/C \) is not bounded below. \( \square \)

The following result characterizes the boundedness from below of the integral operator \( S_g : L^p_a/C \to L^q_a/C \).

**Corollary 13.** Suppose \( 1 < q < p \), or \( 0 < q \leq 1 < p < 2 \) and \( g \in H^\infty \). Then \( S_g : L^p_a \to L^q_a \) is bounded, while there is no symbol \( g \) such that \( S_g : L^p_a/C \to L^q_a/C \) is bounded below.

**Proof.** If \( 1 < q < p \), or \( 0 < q \leq 1 < p < 2 \), since \( g \in H^\infty \), the boundedness of \( S_g : L^p_a \to L^q_a \) follows from Hölder’s inequality. Applying
\( \alpha = \gamma = 1 \) and \( 1 < q < p \), or \( 0 < q \leq 1 < p < 2 \) in Theorem 2, it follows from (2) of Lemma 1 that \( S_g : L^p_a/C \to L^q_a/C \) is not bounded below. \qed

The following result characterizes the boundedness from below of the integral operator \( S_g : B_p/C \to B_q/C \) which has never been proven so far.

**Corollary 14.** Suppose \( 2 < q \leq p < \infty \) and that \( S_g : B_p \to B_q \) is bounded. If \( S_g : B_p/C \to B_q/C \) is bounded below, then \( S_g : B_{1-\frac{2}{q}}/C \to B_{1-\frac{2}{q}}/C \) is bounded below.

**Proof.** Applying \( \alpha = 1 - \frac{2}{p} \) and \( \gamma = 1 - \frac{2}{q} \) in Theorem 2, we can prove that \( S_g : B_{1-\frac{2}{q}}/C \to B_{1-\frac{2}{q}}/C \) is bounded below. \qed

The following results characterize the boundedness from below of integral operators \( S_g : B_p/C \to L^q_a/C \) and \( S_g : L^p_a/C \to B_q/C \) which have never been proven so far.

**Corollary 15.** Suppose \( 2 < q \leq p < \infty \) and \( g \in H^\infty \). Then \( S_g : B_p \to L^q_a \) is bounded, while there is no symbol \( g \) such that \( S_g : B_p/C \to L^q_a/C \) is bounded below.

**Proof.** Since \( 2 < q \leq p \) and \( g \in H^\infty \), the boundedness of \( S_g : B_p \to L^q_a \) follows from Hölder’s inequality. Since \( 2 < q \leq p < \infty \), applying \( \alpha = 1-\frac{2}{p} \), \( \gamma = 1 \) in Theorem 2, it follows from (2) of Lemma 1 that \( S_g : B_p/C \to L^q_a/C \) is not bounded below. \qed

**Corollary 16.** Suppose \( 2 < q \leq p < \infty \). If \( S_g : L^p_a \to B_q \) is bounded, then there is no symbol \( g \) such that \( S_g : L^p_a/C \to B_q/C \) is bounded below.

**Proof.** Since \( 2 < q \leq p < \infty \), applying \( \alpha = 1 \), \( \gamma = 1-\frac{2}{q} \) in Theorem 2, it follows from (1) of Lemma 1 that \( S_g : L^p_a/C \to B_q/C \) is not bounded below. \qed

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**References**

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