International Journal of Mathematical Analysis
Vol. 17, 2023, no. 3, 109 - 117
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ijma.2023.912516

Long Time Behavior of a 2D Ginzburg-Landau Model with Fixed Total Magnetic Flux

Jishan Fan

Department of Applied Mathematics
Nanjing Forestry University, Nanjing, 210037, P. R. China

Tohru Ozawa [1]

Department of Applied Physics
Waseda University, Tokyo, 169-8555, Japan

Abstract

We prove the long time behavior of a 2D Ginzburg-Landau system in superconductivity with Coulomb gauge and fixed total magnetic flux. This solved a problem left open in Q.Tang [9].

Mathematics Subject Classifications: 35A05, 35A40, 35K55, 82D55

Keywords: Ginzburg-Landau model, superconductivity, asymptotic behavior

1 Introduction

We consider the long time behavior of a 2D Ginzburg-Landau model in superconductivity:

$$\partial_t\psi + i\phi\psi + (i\nabla + A)^2\psi + \frac{\lambda}{2}(|\psi|^2 - 1)\psi = 0, \tag{1.1}$$

$$\partial_t A + \nabla\phi + \operatorname{curl}^2 A + \operatorname{Re}\{(i\nabla\psi + \psi A)\overline{\psi}\} = 0, \tag{1.2}$$

[1]Corresponding author

in $Q_T := (0,T)\times\Omega$, with boundary and initial conditions

$$\nabla\psi\cdot\nu=0,\ A\cdot\nu=0,\ \int_\Omega \operatorname{curl} A dx = L,\ \operatorname{curl} A = H(t) \ \text{ on } \ (0,T)\times\partial\Omega, \tag{1.3}$$

$$(\psi, A)(\cdot,0)=(\psi_0,A_0)(\cdot) \ \text{ in } \ \Omega. \tag{1.4}$$

Here, the unknowns ψ, A, and ϕ are $\mathbb{C}$-valued, $\mathbb{R}^2$-valued, and $\mathbb{R}$-valued functions, respectively, and they stand for the order parameter, the magnetic potential, and the electric potential, respectively. $\lambda > 0$ is a Ginzburg-Landau constant, L is a given constant and $H(t)$ is the unknown applied magnetic field, and $i := \sqrt{-1}$. $\overline{\psi}$ denotes the complex conjugate of ψ, $\mathrm{Re}\psi := (\psi+\overline{\psi})/2$ is the real part of ψ, and $|\psi|^2 := \psi\overline{\psi}$ is the density of superconductivity carriers. T is any given positive constant. Ω is a simply connected and bounded domain with smooth boundary $\partial\Omega$ and ν is the outward unit normal to $\partial\Omega$.

It is well-known that the Ginzburg-Landau equations are gauge invariant, namely, if (ψ, A, ϕ) is a solution of (1.1)-(1.2), then $(\psi e^{i\chi}, A+\nabla\chi, \phi-\partial_t\chi)$ is also a solution for any real-valued smooth function χ. Accordingly, in order to obtain the well-posedness of the problem, we need to impose some gauge condition. From physical point of view, one may usually think of four types of the gauge condition:

(1) Coulomb gauge: $\operatorname{div} A = 0$ in Ω and $\int_\Omega \phi dx = 0$.

(2) Lorentz gauge: $\phi = -\operatorname{div} A$ in Ω.

(3) Lorenz gauge: $\partial_t\phi = -\operatorname{div} A$ in Ω.

(4) Temporal gauge (Weyl gauge): $\phi = 0$ in Ω.

For the initial data $(\psi_0, A_0)\in W_0 := \{(\psi_0,A_0)\big|\psi_0\in L^\infty\cap H^1, A_0\in H^1\}$, Chen et al. [1, 2], Du [3], and Fan and Ozawa [4] proved the existence and uniqueness of global strong solutions to (1.1)-(1.4) in the case of the Coulomb, Lorenz, and Lorentz as well as temporal gauges.

We denote $\operatorname{curl} A := \partial_1 A_2 - \partial_2 A_1$ for vector $A := \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$ and $\operatorname{curl} b := \begin{pmatrix} \partial_2 b \\ -\partial_1 b \end{pmatrix}$ for scalar b.

For the initial data $\psi_0, A_0 \in L^2$, under the Coulomb or Lorentz gauge, Tang and Wang (2-D) [5], Fan and Jiang (3-D) [6] proved the global existence of weak solutions. Fan and Ozawa [7] (2-D) and Fan, Gao and Guo [8] (3-D) prove the global existence and uniqueness of weak solutions for $\psi_0, A_0 \in L^d$ with $d = 2, 3$.

We will assume that the initial data (ψ_0, A_0) satisfy

$$\|\psi_0\|_{L^\infty} \le C_0, \psi_0, A_0 \in H^1(\Omega), A_0 \cdot \nu = 0, \int_\Omega \operatorname{curl} A_0 \mathrm{d}x = L. \tag{1.5}$$

Denote

$$G := \int_\Omega \left(|(i\nabla + A)\psi|^2 + |\operatorname{curl} A|^2 + \frac{\lambda}{4}(|\psi|^2 - 1)^2 \right) \mathrm{d}x, \tag{1.6}$$

then it is well-known that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}G + \int_\Omega |\partial_t \psi + i\phi\psi|^2 \mathrm{d}x + \int_\Omega (|\partial_t A|^2 + |\nabla\phi|^2)\mathrm{d}x = 0. \tag{1.7}$$

In [9], Tang assumed (1.5) holds true and showed the global existence and uniqueness of strong solutions and proved the following uniform-in-time estimate:

$$\begin{aligned} &\|\psi\|_{L^\infty(0,\infty;L^\infty)} \le \max(1, C_0), \|(\psi, A, \phi)\|_{L^\infty(0,\infty;H^1)} \le C, \\ &\int_0^\infty \int_\Omega (|\partial_t\psi|^2 + |\partial_t A|^2 + |\nabla\phi|^2)\mathrm{d}x\mathrm{d}t \le C, \end{aligned} \tag{1.8}$$

and gave some weak results on long-time behavior of the solutions and posed some problems:

Problem 1. Is $H(t)$ uniform-in-time bounded?

Problem 2. Is $\lim_{t\to\infty} \|\phi(\cdot, t)\|_{H^1} = 0$?

The aim of this paper is to solve the above two problems. We will prove

Theorem 1.1. *Let* (1.5) *hold true and we choose the Coulomb gauge. Then there exists* $0 < t_0 < \infty$ *such that*

$$\begin{aligned} &\|(\psi, A)\|_{L^\infty(t_0,\infty;H^2)} \le C, \|\partial_t(\psi, A)\|_{L^\infty(t_0,\infty;L^2)} \le C, \\ &\|\partial_t(\psi, A)\|_{L^2(t_0,\infty;H^1)} \le C, \|\phi\|_{L^\infty(t_0,\infty;H^2)} \le C, \\ &\|\partial_t\phi\|_{L^2(t_0,\infty;H^1)} \le C, \lim_{t\to\infty} \|\partial_t(\psi, A)(\cdot, t)\|_{L^2} = 0, \\ &\lim_{t\to\infty} \|\phi(\cdot, t)\|_{H^1} = 0, |H(t)| \le C \end{aligned} \tag{1.9}$$

with positive constant C *independent of time.*

It is important to study the long time behavior of the evolutionary problems because it has a unique solution and may indicate which steady state solutions are preferred in the evolution process.

The physical background of this model is that we are committed to have a mixed state inside the superconductor sample by imposing a fixed total magnetic flux. More precisely, the total magnetic flux is a physically observable quantity which is closely linked with the number of vortices. By adjusting the total magnetic flux, we should have a reasonable estimate of the number of vortices in the sample. In this case, $H(t)$ has to be adjusted in accordance and is therefore not a given quantity, which is a main difficulty.

Definition 1.1. *The quantity* (ψ, A, ϕ) *is called a strong solution if* $(\psi, A, \phi) \in L^\infty(0,T;H^1)\cap L^2(0,T;H^2)$ *and* $\partial_t\psi, \partial_t A \in L^2(0,T;L^2)$ *and the equations* (1.1) *and* (1.2) *hold true almost everywhere.*

In the following proofs, we will use the following inequality [10]:

$$\|A\|_{H^2(\Omega)} \lesssim \|A\|_{H^1(\Omega)} + \|\mathrm{div}\, A\|_{H^1(\Omega)} + \|\mathrm{curl}\, A\|_{H^1(\Omega)} + \|A\cdot\nu\|_{H^{\frac{3}{2}}(\partial\Omega)}. \quad (1.10)$$

We will also use the following lemma [11]:

Lemma 1.2. *Suppose* $y(t)$ *and* $h(t)$ *are nonnegative functions,* $y'(t)$ *is locally integrable on* $(0,\infty)$ *and* $y(t), h(t)$ *satisfy*

$$\frac{\mathrm{d}y}{\mathrm{d}t} \le C_1 y^2 + C_2 y + h(t), \quad \forall t \ge t_1, \quad (1.11)$$

$$\int_{t_1}^{\infty} y(\tau)\mathrm{d}\tau \le C_3, \int_{t_1}^{\infty} h(\tau)\mathrm{d}\tau \le C_4, \quad \forall t \ge t_1, \quad (1.12)$$

with C_1, C_2, C_3 *and* C_4 *being positive constants independent of* t*. Then for any* $r > 0$*,*

$$y(t+r) \le \left(\frac{C_3}{r} + C_2 r + C_4\right) e^{C_1 C_3}, \quad \forall t \ge t_1. \quad (1.13)$$

Moreover,

$$\lim_{t\to\infty} y(t) = 0. \quad (1.14)$$

After having proved Th.1.1, using the standard compactness principle of Aubin-Lions, it is easy to show that $(\psi(\cdot,t), A(\cdot,t), H(t))$ $\omega-$converges to the stationary solutions $(\psi_\infty, A_\infty, H_\infty)$ of the problem:

$$(i\nabla + A)^2\psi + \frac{\lambda}{2}(|\psi|^2 - 1)\psi = 0, \quad (1.15)$$

$$\mathrm{curl}^{\,2} A + \mathrm{Re}\{(i\nabla\psi + \psi A)\overline{\psi}\} = 0, \quad (1.16)$$

$$\mathrm{div}\, A = 0 \ \text{ in } \ \Omega, \quad (1.17)$$

$$\nu\cdot\nabla\psi = 0, \ \nu\cdot A = 0, \ \mathrm{curl}\, A = H_\infty, \ \int_\Omega \mathrm{curl}\, A\mathrm{d}x = L. \quad (1.18)$$

We omit the details here.

2 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1, we only need to show the a priori estimates.

First, it follows from (1.8) that there exists $t_0 \in (0, \infty)$ such that

$$(\partial_t \psi, \partial_t A)(\cdot, t_0) \in L^2(\Omega). \tag{2.1}$$

Here it should be noted that the solutions are smooth when $t > t_0$.

Applying div to (1.2) and using $\operatorname{div} A = 0$, one has

$$-\Delta \phi = \operatorname{div} \operatorname{Re}[(i\nabla\psi + \psi A)\overline{\psi}]. \tag{2.2}$$

On the other hand, let $b := \operatorname{curl} A - H(t)$, then

$$\begin{aligned}
&\int_{\partial\Omega} (\operatorname{curl}^2 A \cdot \nu)\xi \mathrm{d}S = \int_{\partial\Omega} (\operatorname{curl} b \cdot \nu)\xi \mathrm{d}S \\
&= \int_\Omega \operatorname{div}(\xi \operatorname{curl} b)\mathrm{d}x = \int_\Omega \nabla\xi \operatorname{curl} b \mathrm{d}x = -\int_\Omega \operatorname{curl}(b\nabla\xi)\mathrm{d}x \\
&= -\int_{\partial\Omega} b\nabla\xi \cdot \tau \mathrm{d}S = 0
\end{aligned} \tag{2.3}$$

due to $b = 0$ on $\partial\Omega$ for any smooth function ξ.

This shows that

$$\operatorname{curl}^2 A \cdot \nu = 0 \quad \text{on} \quad (0, T) \times \partial\Omega. \tag{2.4}$$

Now, it follows from (1.2), (1.3) and (2.4) that

$$\frac{\partial \phi}{\partial \nu} = 0 \quad \text{on} \quad (0, T) \times \partial\Omega. \tag{2.5}$$

Using $\int_\Omega \phi \mathrm{d}x = 0$, and the Poincaré inequality

$$\|\phi\|_{L^2} \lesssim \|\nabla\phi\|_{L^2}, \tag{2.6}$$

it follows from (1.8) that

$$\int_0^\infty \int_\Omega |\phi|^2 \mathrm{d}x\mathrm{d}t \le C. \tag{2.7}$$

Equation (1.1) can be written as

$$\partial_t \psi - \Delta\psi + i\phi\psi + 2iA \cdot \nabla\psi + |A|^2\psi + \frac{\lambda}{2}(|\psi|^2 - 1)\psi = 0. \tag{2.8}$$

Applying ∂_t to (2.8), testing by $\partial_t\overline{\psi}$, taking the real parts, and using (1.8), we find that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\partial_t\psi|^2\mathrm{d}x + \int_\Omega |\nabla\partial_t\psi|^2\mathrm{d}x + \int_\Omega |A|^2|\partial_t\psi|^2\mathrm{d}x \\
&\lesssim \int_\Omega |\partial_t\phi||\partial_t\psi|\mathrm{d}x + \int_\Omega |\partial_t A\cdot\nabla\psi + A\cdot\nabla\partial_t\psi||\partial_t\psi|\mathrm{d}x \\
&\quad + \int_\Omega |A||\partial_t A||\partial_t\psi|\mathrm{d}x + \int_\Omega |\partial_t\psi|^2\mathrm{d}x \\
&\lesssim \|\partial_t\phi\|_{L^2}\|\partial_t\psi\|_{L^2} + \|\partial_t A\|_{L^4}\|\partial_t\psi\|_{L^4}\|\nabla\psi\|_{L^2} + \|\nabla\partial_t\psi\|_{L^2}\|A\|_{L^4}\|\partial_t\psi\|_{L^4} \\
&\quad + \|A\|_{L^4}\|\partial_t A\|_{L^4}\|\partial_t\psi\|_{L^2} + \|\partial_t\psi\|_{L^2}^2 \\
&\lesssim \|\partial_t\phi\|_{L^2}\|\partial_t\psi\|_{L^2} + \|\partial_t A\|_{L^4}\|\partial_t\psi\|_{L^4} + \|\nabla\partial_t\psi\|_{L^2}\|\partial_t\psi\|_{L^4} \\
&\quad + \|\partial_t A\|_{L^4}\|\partial_t\psi\|_{L^2} + \|\partial_t\psi\|_{L^2}^2.
\end{aligned} \tag{2.9}
$$

We will use the Gagliardo-Nirenberg inequalities

$$
\|\partial_t A\|_{L^4}^2 \lesssim \|\partial_t A\|_{L^2}\|\partial_t A\|_{H^1}, \tag{2.10}
$$

$$
\|\partial_t \psi\|_{L^4}^2 \lesssim \|\partial_t \psi\|_{L^2}\|\partial_t \psi\|_{H^1}, \tag{2.11}
$$

and the Maxwell inequality

$$
\|A\|_{H^1} \lesssim \|\mathrm{curl}\, A\|_{L^2}. \tag{2.12}
$$

Applying ∂_t to (2.2), one has

$$
\begin{aligned}
\|\partial_t\phi\|_{L^2} &\lesssim \|\nabla\partial_t\phi\|_{L^{\frac{4}{3}}} \\
&\lesssim \|\nabla\partial_t\psi\|_{L^2} + \|\partial_t\psi\|_{L^4}\|A\|_{L^4} + \|\partial_t A\|_{L^2} + \|\partial_t\psi\|_{L^4} \\
&\lesssim \|\nabla\partial_t\psi\|_{L^2} + \|\partial_t\psi\|_{L^2} + \|\partial_t A\|_{L^2}.
\end{aligned} \tag{2.13}
$$

Inserting (2.10)-(2.13) into (2.9), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\partial_t\psi|^2\mathrm{d}x + \int_\Omega |\nabla\partial_t\psi|^2\mathrm{d}x \\
&\lesssim \epsilon\|\nabla\partial_t\psi\|_{L^2}^2 + \epsilon\|\mathrm{curl}\,\partial_t A\|_{L^2}^2 + \left(1+\frac{1}{\epsilon}\right)\|\partial_t(\psi, A)\|_{L^2}^2
\end{aligned} \tag{2.14}
$$

for any $0 < \epsilon < 1$.

Applying ∂_t to (1.2), testing by $\partial_t A$, and using (1.8), (2.10)-(2.12) and div $A = 0$, we compute

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\partial_t A|^2 \mathrm{d}x + \int_\Omega |\mathrm{curl}\,\partial_t A|^2 \mathrm{d}x + \int_\Omega |\psi|^2|\partial_t A|^2 \mathrm{d}x \\
&\lesssim \int_\Omega |\nabla\partial_t\psi||\partial_t A|\mathrm{d}x + \int_\Omega |\partial_t\psi||A||\partial_t A|\mathrm{d}x + \int_\Omega |i\nabla\psi + \psi A||\partial_t\psi||\partial_t A|\mathrm{d}x \\
&\lesssim \|\nabla\partial_t\psi\|_{L^2}\|\partial_t A\|_{L^2} + \|\partial_t\psi\|_{L^4}\|A\|_{L^2}\|\partial_t A\|_{L^4} + \|i\nabla\psi + \psi A\|_{L^2}\|\partial_t\psi\|_{L^4}\|\partial_t A\|_{L^4} \\
&\le \|\nabla\partial_t\psi\|_{L^2}\|\partial_t A\|_{L^2} + \|\partial_t\psi\|_{L^4}\|\partial_t A\|_{L^4} \\
&\lesssim \epsilon\|\nabla\partial_t\psi\|_{L^2}^2 + \epsilon\|\mathrm{curl}\,\partial_t A\|_{L^2}^2 + \left(1 + \frac{1}{\epsilon}\right)\|\partial_t(\psi, A)\|_{L^2}^2 \qquad (2.15)
\end{aligned}$$

for any $0 < \epsilon < 1$.

Here we have used

$$\begin{aligned}
&\int_\Omega \partial_t A \mathrm{curl}^2 \partial_t A \mathrm{d}x \\
&= \int_\Omega \partial_t A \mathrm{curl}\,(\mathrm{curl}\,\partial_t A - H'(t))\mathrm{d}x \\
&= \int_\Omega \mathrm{curl}\,\partial_t A(\mathrm{curl}\,\partial_t A - H'(t))\mathrm{d}x \\
&= \int_\Omega |\mathrm{curl}\,\partial_t A|^2 \mathrm{d}x
\end{aligned}$$

and

$$\int_\Omega \mathrm{curl}\,\partial_t A \cdot H'(t)\mathrm{d}x = H'(t)\int_\Omega \mathrm{curl}\,\partial_t A \mathrm{d}x = 0.$$

Summing up (2.14) and (2.15) and taking ϵ small enough, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\partial_t(\psi, A)|^2 \mathrm{d}x + \int_\Omega (|\nabla\partial_t\psi|^2 + |\mathrm{curl}\,\partial_t A|^2)\mathrm{d}x \lesssim \|\partial_t(\psi, A)\|_{L^2}^2. \qquad (2.16)$$

Using Lemma 1.2 and integrating (2.16) over (t_0, ∞), we conclude that

$$\|\partial_t(\psi, A)(\cdot, t)\|_{L^2} \le C, \int_{t_0}^\infty \|\partial_t(\psi, A)\|_{H^1}^2 \mathrm{d}t \le C, \qquad (2.17)$$

$$\lim_{t\to\infty} \|\partial_t(\psi, A)(\cdot, t)\|_{L^2} = 0. \qquad (2.18)$$

Using the H^2-theory of Poisson equation, it follows from (2.8), (1.8) and (2.17) that

$$\begin{aligned}
\|\psi(\cdot, t)\|_{H^2} &\lesssim \|\partial_t\psi\|_{L^2} + \|\phi\|_{L^2} + \|A\|_{L^4}\|\nabla\psi\|_{L^4} + \|A\|_{L^4}^2 + \|\psi\|_{L^2} \\
&\lesssim 1 + \|\nabla\psi\|_{L^4} \\
&\lesssim 1 + \|\nabla\psi\|_{L^2}^{\frac{1}{2}}\|\psi\|_{H^2}^{\frac{1}{2}},
\end{aligned}$$

which gives

$$\|\psi(\cdot,t)\|_{H^2} \le C \quad \text{for} \quad t \ge t_0. \tag{2.19}$$

It follows from (1.2), (1.8), (2.17), and (2.19) that

$$\|\mathrm{curl}\,^2 A(\cdot,t)\|_{L^2} \lesssim \|\partial_t A\|_{L^2} + \|\nabla\phi\|_{L^2} + \|i\nabla\psi + \psi A\|_{L^2} \le C. \tag{2.20}$$

On the other hand,

$$\begin{aligned}\|\mathrm{curl}\, A - H\|_{H^1} &\lesssim \|\nabla(\mathrm{curl}\, A - H)\|_{L^2} \\ &\lesssim \|\mathrm{curl}\,(\mathrm{curl}\, A - H)\|_{L^2} = \|\mathrm{curl}\,^2 A\|_{L^2} \le C\end{aligned}$$

and thus

$$\|H\|_{L^\infty} \lesssim \|\mathrm{curl}\, A\|_{L^2} + \|\mathrm{curl}\, A - H\|_{L^2} \le C. \tag{2.21}$$

Here we should note that H is constant in space and that (2.21) is uniform in time for $t > t_0$.

Using (1.10), (2.20), and (2.21), we have

$$\|A(\cdot,t)\|_{H^2} \le C \quad \text{for} \quad t \ge t_0. \tag{2.22}$$

Using (2.2), (2.19), and (2.22), we know that

$$\|\phi(\cdot,t)\|_{H^2} \le C \quad \text{for} \quad t \ge t_0. \tag{2.23}$$

Similarly to (2.13), one has

$$\begin{aligned}\|\nabla\partial_t\phi\|_{L^2} &\lesssim \|\nabla\partial_t\psi\|_{L^2} + \|\partial_t\psi\|_{L^2} + \|\partial_t A\|_{L^2} + \|i\nabla\psi + \psi A\|_{L^4}\|\partial_t\psi\|_{L^4} \\ &\lesssim \|\partial_t\psi\|_{H^1} + \|\partial_t A\|_{L^2},\end{aligned}$$

which implies

$$\|\partial_t\phi\|_{L^2(t_0,\infty;H^1)} \le C. \tag{2.24}$$

Using (1.8) and (2.24), we conclude that

$$\lim_{t\to\infty} \|\phi(\cdot,t)\|_{H^1} = 0. \tag{2.25}$$

In fact, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\phi\|_{H^1}^2 = 2\int_\Omega (\phi\partial_t\phi + \nabla\phi\nabla\partial_t\phi)\mathrm{d}x \le \|\phi\|_{H^1}^2 + \|\partial_t\phi\|_{H^1}^2$$

and then using Lemma 1.2 to conclude it.

This completes the proof.

□

References

[1] Z.M. Chen, C. Elliott, Q. Tang, Justification of a two-dimensional evolutionary Ginzburg-Landau superconductivity model, *RAIRO Model. Math. Anal. Numer.*, **32** (1)(1998), 25-50. https://doi.org/10.1051/m2an/1998320100251

[2] Z.M. Chen, K.H. Hoffman, J. Liang, On a nonstationary Ginzburg-Landau superconductivity model, *Math. Methods Appl. Sci.,* **16** (12) (1993), 855-875. https://doi.org/10.1002/mma.1670161203

[3] Q. Du, Global existence and uniqueness of solutions of the time dependent Ginzburg-Landau model for superconductivity, *Applicable Anal.,* **53** (1-2) (1994), 1-17. https://doi.org/10.1080/00036819408840240

[4] J. Fan, T. Ozawa, Global strong solutions of the time-dependent Ginzburg-Landau model for superconductivity with a new gauge, *Int. J. Math. Anal.,* **6** (3) (2012), 1679-1684.

[5] Q. Tang, S. Wang, Time dependent Ginzburg-Landau equations of superconductivity, *Physica D*, **88** (3-4) (1995), 139-166. https://doi.org/10.1016/0167-2789(95)00195-a

[6] J. Fan, S. Jiang, Global existence of weak solutions of a time-dependent 3-D Ginzburg-Landau model, *Appl. Math. Letters*, **16** (3) (2003), 435-440. https://doi.org/10.1016/s0893-9659(03)80069-1

[7] J. Fan, T. Ozawa, Uniqueness of weak solutions to the Ginzburg-Landau model for superconductivity, *Zeit. Angew. Math. Phys.*, **63** (3) (2012), 453-459. https://doi.org/10.1007/s00033-011-0164-x

[8] J. Fan, H. Gao, B. Guo, Uniqueness of weak solutions to the 3D Ginzburg-Landau superconductivity model, *Int. Math. Res. Notices*, **2015** (5) (2015), 1239-1246. https://doi.org/10.1093/imrn/rnt253

[9] Q. Tang, On an evolutionary system of Ginzburg-Landau equations with fixed total magnetic flux, *Commun. PDE*, **20** (1-2) (1995), 1-36. https://doi.org/10.1080/03605309508821085

[10] J.P. Bourguignon, H. Brezis, Remarks on the Euler equation, *J. Funct. Anal.*, **15** (1974), 341-363. https://doi.org/10.1016/0022-1236(74)90027-5

[11] W. Shen, S. Zheng, On the coupled Cahn-Hilliard equations, *Comm. PDE*, **18** (3-4) (1993), 701-727. https://doi.org/10.1080/03605309308820946

Received: August 23, 2023; Published: September 8, 2023