Long Time Behavior of a 2D Ginzburg-Landau Model with Fixed Total Magnetic Flux

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Abstract

We prove the long time behavior of a 2D Ginzburg-Landau system in superconductivity with Coulomb gauge and fixed total magnetic flux. This solved a problem left open in Q.Tang [9].

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1 Introduction

We consider the long time behavior of a 2D Ginzburg-Landau model in superconductivity:

\[ \partial_t \psi + i \phi \psi + (i \nabla + A) \psi + \frac{\lambda}{2} (|\psi|^2 - 1) \psi = 0, \quad (1.1) \]
\[ \partial_t A + \nabla \phi + \text{curl}^2 A + \text{Re}\{(i \nabla \psi + \psi A) \overline{\psi}\} = 0, \quad (1.2) \]
in $QT := (0, T) \times \Omega$, with boundary and initial conditions

\[
\nabla \psi \cdot \nu = 0, \quad A \cdot \nu = 0, \quad \int_\Omega \text{curl} \, A \, dx = L, \quad \text{curl} \, A = H(t) \quad \text{on} \quad (0, T) \times \partial\Omega, \tag{3.3}
\]

\[
(\psi, A)(\cdot, 0) = (\psi_0, A_0)(\cdot) \quad \text{in} \quad \Omega. \tag{1.4}
\]

Here, the unknowns $\psi, A,$ and $\phi$ are $\mathbb{C}$-valued, $\mathbb{R}^2$-valued, and $\mathbb{R}$-valued functions, respectively, and they stand for the order parameter, the magnetic potential, and the electric potential, respectively. $\lambda > 0$ is a Ginzburg-Landau constant, $L$ is a given constant and $H(t)$ is the unknown applied magnetic field, and $i := \sqrt{-1}$. $\overline{\psi}$ denotes the complex conjugate of $\psi$, $\text{Re} \psi := (\psi + \overline{\psi})/2$ is the real part of $\psi$, and $|\psi|^2 := \psi \overline{\psi}$ is the density of superconductivity carriers. $T$ is any given positive constant. $\Omega$ is a simply connected and bounded domain with smooth boundary $\partial\Omega$ and $\nu$ is the outward unit normal to $\partial\Omega$.

It is well-known that the Ginzburg-Landau equations are gauge invariant, namely, if $(\psi, A, \phi)$ is a solution of (1.1)-(1.2), then $(\psi e^{i\chi}, A + \nabla \chi, \phi - \partial_t \chi)$ is also a solution for any real-valued smooth function $\chi$. Accordingly, in order to obtain the well-posedness of the problem, we need to impose some gauge condition. From physical point of view, one may usually think of four types of the gauge condition:

1. **Coulomb gauge**: $\text{div} \, A = 0$ in $\Omega$ and $\int_\Omega \phi \, dx = 0$.

2. **Lorentz gauge**: $\phi = -\text{div} \, A$ in $\Omega$.

3. **Lorenz gauge**: $\partial_t \phi = -\text{div} \, A$ in $\Omega$.

4. **Temporal gauge (Weyl gauge)**: $\phi = 0$ in $\Omega$.

For the initial data $(\psi_0, A_0) \in W_0 := \{(\psi_0, A_0) \mid \psi_0 \in L^\infty \cap H^1, A_0 \in H^1\}$, Chen et al. [1, 2], Du [3], and Fan and Ozawa [4] proved the existence and uniqueness of global strong solutions to (1.1)-(1.4) in the case of the Coulomb, Lorenz, and Lorentz as well as temporal gauges.

We denote $\text{curl} \, A := \partial_1 A_2 - \partial_2 A_1$ for vector $A := \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$ and $\text{curl} \, b := \left( \begin{array}{c} \partial_2 b \\ -\partial_1 b \end{array} \right)$ for scalar $b$.

For the initial data $\psi_0, A_0 \in L^2$, under the Coulomb or Lorentz gauge, Tang and Wang (2-D) [5], Fan and Jiang (3-D) [6] proved the global existence of weak solutions. Fan and Ozawa [7] (2-D) and Fan, Gao and Guo [8] (3-D) prove the global existence and uniqueness of weak solutions for $\psi_0, A_0 \in L^d$ with $d = 2, 3$. 
We will assume that the initial data \((\psi_0, A_0)\) satisfy
\[
\|\psi_0\|_{L^\infty} \leq C_0, \psi_0, A_0 \in H^1(\Omega), A_0 \cdot \nu = 0, \int_{\Omega} \text{curl} A_0 \, dx = L. \tag{1.5}
\]

Denote
\[
G := \int_{\Omega} \left( |(i \nabla + A)\psi|^2 + |\text{curl} A|^2 + \lambda \frac{1}{4} (|\psi|^2 - 1)^2 \right) \, dx, \tag{1.6}
\]
then it is well-known that
\[
\frac{1}{2} \frac{d}{dt} G + \int_{\Omega} |\partial_t \psi + i \phi \psi|^2 \, dx + \int_{\Omega} (|\partial_t A|^2 + |\nabla \phi|^2) \, dx = 0. \tag{1.7}
\]

In [9], Tang assumed (1.5) holds true and showed the global existence and uniqueness of strong solutions and proved the following uniform-in-time estimate:
\[
\|\psi\|_{L^\infty(0, \infty; L^\infty)} \leq \max(1, C_0), \|(\psi, A, \phi)\|_{L^\infty(0, \infty; H^1)} \leq C,
\int_0^\infty \int_{\Omega} (|\partial_t \psi|^2 + |\partial_t A|^2 + |\nabla \phi|^2) \, dx \, dt \leq C, \tag{1.8}
\]
and gave some weak results on long-time behavior of the solutions and posed some problems:

Problem 1. Is \(H(t)\) uniform-in-time bounded?

Problem 2. Is \(\lim_{t \to \infty} \|\phi(\cdot, t)\|_{H^1} = 0?\)

The aim of this paper is to solve the above two problems. We will prove

**Theorem 1.1.** Let (1.5) hold true and we choose the Coulomb gauge. Then there exists \(0 < t_0 < \infty\) such that
\[
\|(\psi, A)\|_{L^\infty(t_0, \infty; H^2)} \leq C, \|\partial_t (\psi, A)\|_{L^\infty(t_0, \infty; L^2)} \leq C,
\|\partial_t \psi, A)\|_{L^2(t_0, \infty; H^1)} \leq C, \|\phi\|_{L^\infty(t_0, \infty; H^2)} \leq C,
\|\partial_t \phi\|_{L^2(t_0, \infty; H^1)} \leq C, \lim_{t \to \infty} \|\partial_t (\psi, A)(\cdot, t)\|_{L^2} = 0,
\lim_{t \to \infty} \|\phi(\cdot, t)\|_{H^1} = 0, |H(t)| \leq C \tag{1.9}
\]
with positive constant \(C\) independent of time.

It is important to study the long time behavior of the evolutionary problems because it has a unique solution and may indicate which steady state solutions are preferred in the evolution process.

The physical background of this model is that we are committed to have a mixed state inside the superconductor sample by imposing a fixed total magnetic flux. More precisely, the total magnetic flux is a physically observable quantity which is closely linked with the number of vortices. By adjusting the total magnetic flux, we should have a reasonable estimate of the number of vortices in the sample. In this case, \(H(t)\) has to be adjusted in accordance and is therefore not a given quantity, which is a main difficulty.
Definition 1.1. The quantity \((\psi, A, \phi)\) is called a strong solution if \((\psi, A, \phi) \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2)\) and \(\partial_t \psi, \partial_t A \in L^2(0, T; L^2)\) and the equations (1.1) and (1.2) hold true almost everywhere.

In the following proofs, we will use the following inequality [10]:
\[
\|A\|_{H^2(\Omega)} \lesssim \|A\|_{H^1(\Omega)} + \|\text{div} A\|_{H^1(\Omega)} + \|\text{curl} A\|_{H^1(\Omega)} + \|A \cdot \nu\|_{H^2(\partial \Omega)}.
\] (1.10)

We will also use the following lemma [11]:

Lemma 1.2. Suppose \(y(t)\) and \(h(t)\) are nonnegative functions, \(y'(t)\) is locally integrable on \((0, \infty)\) and \(y(t), h(t)\) satisfy
\[
\frac{dy}{dt} \leq C_1 y^2 + C_2 y + h(t), \quad \forall t \geq t_1, \tag{1.11}
\]
\[
\int_{t_1}^\infty y(\tau)d\tau \leq C_3, \quad \int_{t_1}^\infty h(\tau)d\tau \leq C_4, \quad \forall t \geq t_1, \tag{1.12}
\]
with \(C_1, C_2, C_3\) and \(C_4\) being positive constants independent of \(t\). Then for any \(r > 0\),
\[
y(t + r) \leq \left(\frac{C_3}{r} + C_2 r + C_4\right) e^{C_1 C_3}, \quad \forall t \geq t_1. \tag{1.13}
\]

Moreover,
\[
\lim_{t \to \infty} y(t) = 0. \tag{1.14}
\]

After having proved Th.1.1, using the standard compactness principle of Aubin-Lions, it is easy to show that \((\psi(\cdot, t), A(\cdot, t), H(t))\) \(\omega\)–converges to the stationary solutions \((\psi_\infty, A_\infty, H_\infty)\) of the problem:

\[
(i\nabla + A)^2 \psi + \frac{\lambda}{2}(|\psi|^2 - 1)\psi = 0, \tag{1.15}
\]
\[
\text{curl}^2 A + \text{Re}\{(i\nabla \psi + \psi A)\bar{\psi}\} = 0, \tag{1.16}
\]
\[
\text{div} A = 0 \quad \text{in} \quad \Omega, \tag{1.17}
\]
\[
\nu \cdot \nabla \psi = 0, \quad \nu \cdot A = 0, \quad \text{curl} A = H_\infty, \quad \int_\Omega \text{curl} A dx = L. \tag{1.18}
\]

We omit the details here.

2 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1, we only need to show the a priori estimates.
First, it follows from (1.8) that there exists $t_0 \in (0, \infty)$ such that

$$(\partial_t \psi, \partial_t A)(\cdot, t_0) \in L^2(\Omega).$$  \hspace{1cm} (2.1)

Here it should be noted that the solutions are smooth when $t > t_0$.

Applying $\text{div}$ to (1.2) and using $\text{div} A = 0$, one has

$$-\Delta \phi = \text{div} \text{Re}[(i \nabla \psi + \psi A)\overline{\psi}].$$  \hspace{1cm} (2.2)

On the other hand, let $b := \text{curl} A - H(t)$, then

$$\int_{\partial \Omega} (\text{curl}^2 A \cdot \nu) \xi dS = \int_{\partial \Omega} (\text{curl} b \cdot \nu) \xi dS = \int_{\Omega} \nabla \xi \text{curl} b dx = -\int_{\Omega} \text{curl} (b \nabla \xi) dx$$

$$= -\int_{\partial \Omega} b \nabla \xi \cdot \tau dS = 0$$ \hspace{1cm} (2.3)

due to $b = 0$ on $\partial \Omega$ for any smooth function $\xi$.

This shows that

$$\text{curl}^2 A \cdot \nu = 0 \text{ on } (0, T) \times \partial \Omega.$$ \hspace{1cm} (2.4)

Now, it follows from (1.2), (1.3) and (2.4) that

$$\frac{\partial \phi}{\partial \nu} = 0 \text{ on } (0, T) \times \partial \Omega.$$ \hspace{1cm} (2.5)

Using $\int_{\Omega} \phi dx = 0$, and the Poincaré inequality

$$\|\phi\|_{L^2} \lesssim \|\nabla \phi\|_{L^2},$$ \hspace{1cm} (2.6)

it follows from (1.8) that

$$\int_0^\infty \int_{\Omega} |\phi|^2 dx dt \leq C.$$ \hspace{1cm} (2.7)

Equation (1.1) can be written as

$$\partial_t \psi - \Delta \psi + i\phi \psi + 2iA \cdot \nabla \psi + |A|^2 \psi + \frac{\lambda}{2}(|\psi|^2 - 1) \psi = 0.$$ \hspace{1cm} (2.8)
Applying $\partial_t$ to (2.8), testing by $\partial_t \overline{\psi}$, taking the real parts, and using (1.8), we find that

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\partial_t \psi|^2 dx + \int_{\Omega} |\nabla \partial_t \psi|^2 dx + \int_{\Omega} |A|^2 |\partial_t \psi|^2 dx \\
\lesssim \int_{\Omega} |\partial_t \phi| |\partial_t \psi| dx + \int_{\Omega} |\partial_t A \cdot \nabla \psi + A \cdot \nabla \partial_t \psi| |\partial_t \psi| dx \\
+ \int_{\Omega} |A| |\partial_t A| |\partial_t \psi| dx + \int_{\Omega} |\partial_t \psi|^2 dx \\
\lesssim ||\partial_t \phi||_{L^2} ||\partial_t \psi||_{L^2} + ||\partial_t A||_{L^4} ||\partial_t \psi||_{L^4} ||\nabla \psi||_{L^2} + ||\nabla \partial_t \psi||_{L^2} ||A||_{L^4} ||\partial_t \psi||_{L^4} \\
+ ||\partial_t A||_{L^4} ||\partial_t \psi||_{L^2} + ||\partial_t \psi||_{L^2} \\
\lesssim ||\partial_t \phi||_{L^2} ||\partial_t \psi||_{L^2} + ||\partial_t A||_{L^4} ||\partial_t \psi||_{L^4} + ||\nabla \partial_t \psi||_{L^2} ||\partial_t \psi||_{L^4} \\
+ ||\partial_t A||_{L^4} ||\partial_t \psi||_{L^2} + ||\partial_t \psi||_{L^2}^2. 
$$

(2.9)

We will use the Gagliardo-Nirenberg inequalities

$$
||\partial_t A||_{L^4}^2 \lesssim ||\partial_t A||_{L^2} ||\partial_t A||_{H^1}, 
$$

(2.10)

$$
||\partial_t \psi||_{L^4}^2 \lesssim ||\partial_t \psi||_{L^2} ||\partial_t \psi||_{H^1}, 
$$

(2.11)

and the Maxwell inequality

$$
||A||_{H^1} \lesssim ||\text{curl} \ A||_{L^2}. 
$$

(2.12)

Applying $\partial_t$ to (2.2), one has

$$
||\partial_t \phi||_{L^2} \lesssim ||\nabla \partial_t \phi||_{L^4} \\
\lesssim ||\nabla \partial_t \psi||_{L^2} + ||\partial_t \psi||_{L^4} ||A||_{L^4} + ||\partial_t A||_{L^2} + ||\partial_t \psi||_{L^4} \\
\lesssim ||\nabla \partial_t \psi||_{L^2} + ||\partial_t \psi||_{L^2} + ||\partial_t A||_{L^2}. 
$$

(2.13)

Inserting (2.10)-(2.13) into (2.9), we obtain

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\partial_t \psi|^2 dx + \int_{\Omega} |\nabla \partial_t \psi|^2 dx \\
\lesssim \epsilon ||\nabla \partial_t \psi||_{L^2}^2 + \epsilon ||\text{curl} \partial_t A||_{L^2}^2 + \left(1 + \frac{1}{\epsilon}\right) ||\partial_t (\phi, A)||_{L^2}^2 
$$

(2.14)

for any $0 < \epsilon < 1$.

Applying $\partial_t$ to (1.2), testing by $\partial_t A$, and using (1.8), (2.10)-(2.12) and
\( \text{div} \, A = 0 \), we compute

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\partial_t A|^2 dx + \int_{\Omega} |\text{curl} \, \partial_t A|^2 dx + \int_{\Omega} |\psi|^2 |\partial_t A|^2 dx \\
\lesssim \int_{\Omega} |\nabla \partial_t \psi| |\partial_t A| dx + \int_{\Omega} |\partial_t \psi| |A| |\partial_t A| dx + \int_{\Omega} |i \nabla \psi + \psi A| |\partial_t \psi| |\partial_t A| dx \\
\lesssim \|\nabla \partial_t \psi\|_{L^2} \|\partial_t A\|_{L^2} + \|\partial_t \psi\|_{L^4} \|A\|_{L^4} \|\partial_t A\|_{L^4} + \|i \nabla \psi + \psi A\|_{L^2} \|\partial_t \psi\|_{L^4} \|\partial_t A\|_{L^4} \\
\lesssim \epsilon \|\nabla \partial_t \psi\|^2_{L^2} + \epsilon \|\text{curl} \, \partial_t A\|^2_{L^2} + \left(1 + \frac{1}{\epsilon}\right) \|\partial_t (\psi, A)\|^2_{L^2} \\
\text{(2.15)}
\]

for any \(0 < \epsilon < 1\).

Here we have used

\[
\int_{\Omega} \partial_t A \text{curl}^2 \partial_t A dx \\
= \int_{\Omega} \partial_t A \text{curl} (\text{curl} \, \partial_t A - H'(t)) dx \\
= \int_{\Omega} \text{curl} \, \partial_t A (\text{curl} \, \partial_t A - H'(t)) dx \\
= \int_{\Omega} |\text{curl} \, \partial_t A|^2 dx
\]

and

\[
\int_{\Omega} \text{curl} \, \partial_t A \cdot H'(t) dx = H'(t) \int_{\Omega} \text{curl} \, \partial_t A dx = 0.
\]

Summing up (2.14) and (2.15) and taking \(\epsilon\) small enough, we arrive at

\[
\frac{d}{dt} \int_{\Omega} |\partial_t (\psi, A)|^2 dx + \int_{\Omega} (|\nabla \partial_t \psi|^2 + |\text{curl} \, \partial_t A|^2) dx \lesssim \|\partial_t (\psi, A)\|^2_{L^2}. \quad \text{(2.16)}
\]

Using Lemma 1.2 and integrating (2.16) over \((t_0, \infty)\), we conclude that

\[
\|\partial_t (\psi, A)(\cdot, t)\|_{L^2} \leq C, \quad \int_{t_0}^{\infty} \|\partial_t (\psi, A)\|^2_{H^1} dt \leq C, \quad \text{(2.17)}
\]

\[
\lim_{t \to \infty} \|\partial_t (\psi, A)(\cdot, t)\|_{L^2} = 0. \quad \text{(2.18)}
\]

Using the \(H^2\)-theory of Poisson equation, it follows from (2.8), (1.8) and (2.17) that

\[
\|\psi(\cdot, t)\|_{H^2} \lesssim \|\partial_t \psi\|_{L^2} + \|\phi\|_{L^2} + \|A\|_{L^4} \|\nabla \psi\|_{L^4} + \|A\|^2_{L^4} + \|\psi\|_{L^2} \\
\lesssim 1 + \|\nabla \psi\|_{L^4}^{\frac{1}{2}} \|\psi\|_{H^2}^{\frac{1}{4}}.
\]
which gives
\[ \| \psi(\cdot, t) \|_{H^2} \leq C \text{ for } t \geq t_0. \] (2.19)

It follows from (1.2), (1.8), (2.17), and (2.19) that
\[ \| \text{curl}^2 A(\cdot, t) \|_{L^2} \lesssim \| \partial_t A \|_{L^2} + \| \nabla \phi \|_{L^2} + \| i \nabla \psi + \psi A \|_{L^2} \leq C. \] (2.20)

On the other hand,
\[ \| \text{curl} A - H \|_{H^1} \lesssim \| \nabla (\text{curl} A - H) \|_{L^2} \lesssim \| \text{curl} (\text{curl} A - H) \|_{L^2} = \| \text{curl}^2 A \|_{L^2} \leq C \]
and thus
\[ \| H \|_{L^\infty} \lesssim \| \text{curl} A \|_{L^2} + \| \text{curl} A - H \|_{L^2} \leq C. \] (2.21)

Here we should note that $H$ is constant in space and that (2.21) is uniform in time for $t > t_0$.

Using (1.10), (2.20), and (2.21), we have
\[ \| A(\cdot, t) \|_{H^2} \leq C \text{ for } t \geq t_0. \] (2.22)

Using (2.2), (2.19), and (2.22), we know that
\[ \| \phi(\cdot, t) \|_{H^2} \leq C \text{ for } t \geq t_0. \] (2.23)

Similarly to (2.13), one has
\[ \| \nabla \partial_t \phi \|_{L^2} \lesssim \| \nabla \partial_t \psi \|_{L^2} + \| \partial_t \psi \|_{L^2} + \| \partial_t A \|_{L^2} + \| i \nabla \psi + \psi A \|_{L^4} \| \partial_t \psi \|_{L^4} \lesssim \| \partial_t \psi \|_{H^1} + \| \partial_t A \|_{L^2}, \]
which implies
\[ \| \partial_t \phi \|_{L^2(t_0, \infty; H^1)} \leq C. \] (2.24)

Using (1.8) and (2.24), we conclude that
\[ \lim_{t \to \infty} \| \phi(\cdot, t) \|_{H^1} = 0. \] (2.25)

In fact, we have
\[ \frac{d}{dt} \| \phi \|_{H^1}^2 = 2 \int_{\Omega} (\phi \partial_t \phi + \nabla \phi \nabla \partial_t \phi) dx \leq \| \phi \|_{H^1}^2 + \| \partial_t \phi \|_{H^1}^2 \]
and then using Lemma 1.2 to conclude it.

This completes the proof. \qed
References


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