The Quasi-Boundary Value Method for Identifying the Initial Value Problem on a Columnar Symmetric Domain

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Abstract

In this paper, we consider the inverse problem for identifying the initial value on the heat equation on the columnar symmetric domain. The problem is ill-posed in the sense that the solution (if it exists) does not depend continuously on the data. The quasi-boundary value regularization method is used to deal with this problem, and then the error estimate between the regularization solution and the exact solution is derived under the priori regularization parameter choice rule.

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1 Introduction

In many industrial applications, it is difficult for one to determine the temperature on the surface of a body. The backward heat conduction problem is a model of this situation. In general, the solution which satisfies the heat conduction equation with final data and the boundary conditions do not exist. Even if a solution exists, it will not be continuously dependent on the final data, i.e., any small perturbation in the input data may cause large change to
the solution. In the past, many regularization methods have been proposed for the BHCP, such as the method of fundamental solutions[1], the Tikhonov regularization method[2], the quasi-reversibility method[3], the wavelet and wavelet-Galerkin method[4], the Fourier regularization method[5] and so on. Many articles analyzed a columnar symmetric heat conduction problems which have been obtained some achievements on research. For example, in[6], cheng utilized a wavelet dual least squares method to determine the surface temperature from a fixed location inside a cylinder. In this paper, we consider homogeneous heat equation on a symmetric domain as follows

\[
\begin{cases}
  u_t - \frac{1}{r} u_r - u_{rr} = 0, & 0 < t < T, \quad 0 < r < r_0, \\
  u(r, 0) = \varphi(r), & 0 \leq r \leq r_0, \\
  u(r_0, t) = 0, & 0 \leq t \leq T, \\
  \lim_{r \to 0} u(r, t) \text{ is bounded}, & 0 < t < T, \quad 0 < r < r_0, \\
  u(r, T) = g(r), & 0 \leq r \leq r_0,
\end{cases}
\]

(1.1)

where \( r_0 \) is the radius, \( \varphi(r) \) is the initial value. We use the additional condition \( u(r, T) = g(r) \) to determine the initial value \( \varphi(r) \). The measured data of \( g(r) \) is \( g^\delta(r) \), which satisfies

\[
\| g^\delta(\cdot) - g(\cdot) \|_{L^2[0,r_0;r]} \leq \delta, \tag{1.2}
\]

where the constant \( \delta > 0 \) represents a noise level of input data. Throughout this paper, \( L^2[0,r_0; r] \) denotes the Hilbert space of Lebesgue measurable function \( \varphi \) with weight \( r \) on \( [0,r_0] \). \((\cdot, \cdot)\) and \( \| \cdot, \cdot \| \) denote the inner and norm on \( L^2[0,r_0; r] \), respectively. The norm of the \( \varphi \) is defined as follows:

\[
\| \varphi \| = \left( \int_0^{r_0} r |\varphi(r)|^2 dr \right)^{\frac{1}{2}}. \tag{1.3}
\]

This problem is ill-posed, we use the quasi-boundary value regularization to solve this problem. The quasi-boundary value method, also called nonlocal boundary value method, is a regularization technique by replacing the final condition or boundary condition by a new approximate condition. This method has been used to solve some inverse problems [7].

Using the separation of variables, we obtain the solution of the problem (1.1) as follows

\[
u(r, t) = \sum_{n=1}^{\infty} e^{-\left(\frac{\mu_n}{r_0}\right)^2 t} \varphi_n \omega_n(r), \tag{1.4}
\]

where

\[
\omega_n(r) = \frac{\sqrt{2}}{r_0 J_1(\mu_n)} J_0\left(\frac{\mu_n r}{r_0}\right), \tag{1.5}
\]
the eigenfunction system \( \omega_n(r) \) is orthonormal with weight \( r \) on \([0, r_0]\). It is also a complete system in \( L^2[0, r_0; r] \). \( J_0(x) \) and \( J_1(x) \) denote the zero order and first order Bessel functions, respectively, and \( \{\mu_n\}_{n=1}^{\infty} \) are the sequence of roots of the equation \( J_0(x) = 0 \) which satisfy
\[
0 < \mu_1 < \mu_2 < \cdots < \mu_n < \cdots, \quad \lim_{n \to \infty} \mu_n = \infty. \tag{1.6}
\]
Now let \( \varphi_n = (\varphi(r), \omega_n(r)) \) and \( g_n = (g(r), \omega_n(r)) \). Using \( u(r, T) = g(r) \), we have
\[
g(r) = \sum_{n=1}^{\infty} e^{-\left(\frac{\mu_n}{r_0}\right)^2 T} \varphi_n \omega_n(r), \quad \tag{1.7}
g_n = e^{-\left(\frac{\mu_n}{r_0}\right)^2 T} \varphi_n. \tag{1.8}
\]
Defined operator \( K : \varphi(r) \to g(r) \), then
\[
g(r) = K\varphi(r) = \sum_{n=1}^{\infty} e^{-\left(\frac{\mu_n}{r_0}\right)^2 T} \varphi_n \omega_n(r). \tag{1.9}
\]
The operator \( K \) is a linear self-adjoint compact operator. \( k_n \) are corresponding eigenvalues of \( K \).
\[
k_n = e^{-\left(\frac{\mu_n}{r_0}\right)^2 T}. \tag{1.10}
\]
Using (1.4) and (1.7), equation (1.8) can be rewritten as
\[
(g(r), \omega_n(r)) = (\varphi(r), \omega_n(r)) e^{-\left(\frac{\mu_n}{r_0}\right)^2 T}. \tag{1.11}
\]
So
\[
\varphi(r) = \sum_{n=1}^{\infty} \frac{g_n}{e^{-\left(\frac{\mu_n}{r_0}\right)^2 T}} \omega_n(r). \tag{1.12}
\]
When \( n \to \infty, \mu_n \to \infty \), so \( (e^{-\left(\frac{\mu_n}{r_0}\right)^2 T})^{-1} \to \infty \), thus the problem (1.1) is ill-posed.

We give a priori bound on the initial value, i.e.,
\[
\| \varphi(\cdot) \|_p \leq E, \quad p > 0, \tag{1.13}
\]
where \( E > 0 \) is a constant and \( \| \cdot \|_p \) denotes the norm in Sobolev space which is defined as follows
\[
\| \varphi(\cdot) \|_p := \left( \sum_{n=1}^{\infty} \left( \frac{H_n}{r_0} \right)^p |(\varphi(\cdot), \omega_n(\cdot))|^2 \right)^{\frac{1}{2}}. \tag{1.14}
\]

**Lemma 2.1** \( \{e^{-\left(\frac{\mu_n}{r_0}\right)^2 T}\}_{n=1}^{\infty} \) mentioned from \([\cdot, \cdot], \mu_n \) are the infinite number real roots of the equation \( J_0(r) = 0 \), then
\[
\frac{C_1}{\mu_n} \leq e^{-\left(\frac{\mu_n}{r_0}\right)^2 T} \leq \frac{C_2}{\mu_n}, \tag{1.15}
\]
where $C_1, C_2$ are constants.

**Lemma 2.2** For any positive constant $p > 0$, $0 < \mu < 1$, $s \geq \mu_1 > 0$, we have

$$F(s) = \frac{\mu s^{1-\frac{s}{2}}}{\mu s + C_1} \leq \begin{cases} C_5 \mu^\frac{p}{2}, & 0 < p < 2, \\ C_6 \mu, & p \geq 2, \end{cases}$$

where $C_5 = C_5(p, C_1)$, $C_6 = C_6(p, \mu_1, C_1)$.

### 2 Regularization method and convergence estimate

In this section, through modifying $u(r, T) = g(r)$ as $u(r, T) + \mu u(r, 0) = g^\delta(r)$, we use the quasi-boundary value method to solve the following problem

$$\begin{cases}
    u_t - \frac{1}{r} u_r - u_{rr} = 0, & 0 < t < T, \quad 0 < r < r_0, \\
    u(r, 0) = \phi^{\mu, \delta}(r), & 0 \leq r \leq r_0, \\
    \lim_{r \to 0} u^\mu, \delta(r, t) \text{ is bounded}, & 0 < t < T, \quad 0 < r < r_0, \\
    u^\mu, \delta(r, T) + \mu u^\mu, \delta(r, 0) = g^\delta(r), & 0 \leq r \leq r_0,
\end{cases}$$

where $\mu$ is regularization parameter. By the separation of variables, we obtain the solution of the problem (2.1) as follows

$$u^\mu, \delta(r, t) = \sum_{n=1}^{\infty} e^{-\left(\frac{t \mu n}{r_0}\right)^2 T} \overline{\phi}^{\mu, \delta}(r) \omega_n(r).$$

Using $u^\mu, \delta(r, T) + \mu u^\mu, \delta(r, 0) = g^\delta(r)$, we obtain

$$g^\delta(r) = \sum_{n=1}^{\infty} (\mu + e^{-\left(\frac{t \mu n}{r_0}\right)^2 T}) \overline{\phi}^{\mu, \delta}(r) \omega_n(r).$$

Hence

$$\phi^{\mu, \delta}(r) = \sum_{n=1}^{\infty} \frac{g^\delta_n}{\mu + e^{-\left(\frac{t \mu n}{r_0}\right)^2 T}} \omega_n(r),$$

which is called the quasi-boundary regularized solution of problem (1.1), and $\mu$ is a constant which will be selected appropriately as regularization parameter.

Before giving the main conclusion of this paper, we first introduce two important lemmas.

**Lemma 2.1** \(\{e^{-\left(\frac{t \mu n}{r_0}\right)^2 T}\}_{n=1}^{\infty}\) mentioned from [?, ?], $\mu_n$ are the infinite number real roots of the equation $J_0(r) = 0$, then

$$\frac{C_1}{\mu_n} \leq e^{-\left(\frac{t \mu n}{r_0}\right)^2 T} \leq \frac{C_2}{\mu_n},$$

(2.5)
Lemma 2.2  For any positive constant \( p > 0 \), \( 0 < \mu < 1 \), \( s \geq \mu_1 > 0 \), we have
\[
F(s) = \frac{\mu s^{1-\frac{\mu}{p}}}{\mu s + C_1} \leq \begin{cases} 
C_1 \mu^{\frac{\mu}{p}}, & 0 < p < 2, \\
C_2 \mu, & p \geq 2,
\end{cases}
\] (2.6)

where \( C_1, C_2 \) are constants.

Theorem 2.1. Let \( \varphi^{\mu, \delta}(r) \) be the regularized solution and \( \varphi(r) \) be the exact solution. Let \( g_\delta(r) \) be the measured data at \( t = T \) satisfying (1.2) and priori condition (1.14) holds for \( p > 0 \). Then we obtain
(a) As \( 0 < p < 2 \), If we select \( \mu = (\frac{\delta}{E})^{\frac{2}{p+2}} \), then
\[
\| \varphi^{\mu, \delta}(\cdot) - \varphi(\cdot) \| \leq (1 + C_1) E^{\frac{\mu}{p+2}} \delta^{\frac{p}{p+2}}.
\] (2.7)

(b) As \( p \geq 2 \), if we select \( \mu = (\frac{\delta}{E})^{\frac{1}{2}} \), then
\[
\| \varphi^{\mu, \delta}(\cdot) - \varphi(\cdot) \| \leq (1 + C_2) E^{\frac{1}{2}} \delta^{\frac{1}{2}}.
\] (2.8)

Proof. By utilizing triangle inequality, we obtain
\[
\| \varphi^{\mu, \delta}(\cdot) - \varphi(\cdot) \| \leq \| \varphi^{\mu, \delta}(\cdot) - \varphi^{\mu}(\cdot) \| + \| \varphi^{\mu}(\cdot) - \varphi(\cdot) \|. \tag{2.9}
\]

We first consider \( \| \varphi^{\mu, \delta}(\cdot) - \varphi^{\mu}(\cdot) \| \).
\[
\| \varphi^{\mu, \delta}(\cdot) - \varphi^{\mu}(\cdot) \|^2 = \left\| \sum_{n=1}^{\infty} \left( \frac{g_n}{\mu + e^{-\left(\frac{\mu n}{r_0}\right)^2 T}} - \frac{g_n}{\mu + e^{-\left(\frac{\mu n}{r_0}\right)^2 T}} \right) \omega_n(r) \right\|^2 = \sum_{n=1}^{\infty} \left( \frac{g_n - g_n}{\mu + e^{-\left(\frac{\mu n}{r_0}\right)^2 T}} \right)^2 = \frac{\delta^2}{\mu^2}.
\]
So
\[
\| \varphi^{\mu, \delta}(\cdot) - \varphi^{\mu}(\cdot) \| \leq \frac{\delta}{\mu}. \tag{2.10}
\]

Now we consider \( \| \varphi^{\mu}(\cdot) - \varphi(\cdot) \| \).
\[
\| \varphi^{\mu}(\cdot) - \varphi(\cdot) \| = \left\| \sum_{n=1}^{\infty} \left( \frac{g_n}{\mu + e^{-\left(\frac{\mu n}{r_0}\right)^2 T}} - \frac{g_n}{e^{-\left(\frac{\mu n}{r_0}\right)^2 T}} \right) \omega_n(r) \right\| = \left( \sum_{n=1}^{\infty} \left( \frac{\mu}{\mu + e^{-\left(\frac{\mu n}{r_0}\right)^2 T}} - \frac{g_n}{e^{-\left(\frac{\mu n}{r_0}\right)^2 T}} \right)^2 \right)^{\frac{1}{2}} = \left( \sum_{n=1}^{\infty} \left( \frac{\mu}{\mu + e^{-\left(\frac{\mu n}{r_0}\right)^2 T}} \right)^{\frac{\mu}{2}} \frac{g_n}{e^{-\left(\frac{\mu n}{r_0}\right)^2 T}} \right)^{\frac{1}{2}}.
\]
\[ A_n = \frac{\mu(\mu_n)^{-\frac{p}{2}}}{\mu + e^{-\left(\frac{\mu}{\mu_n}\right)^2}}. \]

Using Lemma 2.2, we obtain
\[ A_n \leq \frac{\mu\mu_n^{1-\frac{p}{2}}}{\mu\mu_n + C_1} \leq \begin{cases} C_1\mu^2, & 0 < p < 2, \\ C_2\mu, & p \geq 2. \end{cases} \]

So
\[ \| \varphi^{\mu,\delta}(\cdot) - \varphi(\cdot) \| \leq \frac{\delta}{\mu} + \begin{cases} C_1\mu^2 E, & 0 < p < 2, \\ C_2\mu E, & p \geq 2. \end{cases} \tag{2.11} \]

Using \( \mu = (\frac{\delta}{E})^{\frac{2}{p+2}} (0 < p < 2) \) and \( \mu = (\frac{\delta}{E})^{\frac{1}{2}} (p \geq 2) \), we obtain
\[ \| \varphi^{\mu,\delta}(\cdot) - \varphi(\cdot) \| \leq \begin{cases} (1 + C_1)E^{\frac{2}{p+2}}\delta^\frac{p}{p+2}, & 0 < p < 2, \\ (1 + C_2)E^{\frac{1}{2}}\delta^\frac{1}{2}, & p \geq 2. \end{cases} \tag{2.12} \]

### 3 Conclusion

We consider an inverse problem to determine an initial value for homogeneous heat equation on a columnar symmetric domain. We construct the quasi-boundary value method to solve this inverse problem and obtain regularization solution. Moreover, we obtain the Hölder type error estimate under a priori parameter choice rule.

### References

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