A Trace Inequality for Non-Commuting Hyponormal Tuples

Vasile Lauric

Department of Mathematics
Florida A&M University
Tallahassee, FL 32308 USA

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Abstract

We introduce the class of almost right hyponormal and almost jointly hyponormal multi-operators and give a sufficient condition for some “commutator” associated to such multi-operators to belong to the trace class.

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1. Introduction

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$, and $\mathcal{C}_p(\mathcal{H})$ (or $\mathcal{C}_p$) the Schatten-von Neumann $p$-classes and by $\| \cdot \|_p$, $p \geq 1$, their respective norm. Of particular interest in this note will be the trace class $\mathcal{C}_1$ and the Hilbert-Schmidt class $\mathcal{C}_2$. For arbitrary operators $S, T \in \mathcal{L}(\mathcal{H})$, $[S, T] := ST - TS$ will denote their commutator and $\mathcal{C}(T) := [T^*, T]$ will denote the self-commutator of $T$. An operator $T$ is called almost normal if $\mathcal{C}(T)$ is trace class, it is hyponormal if $\mathcal{C}(T)$ is positive semi-definite and it is almost hyponormal if the negative part of $\mathcal{C}(T)$, $\mathcal{C}(T)_-$, is trace class, where the negative part of a self-adjoint operator $A \in \mathcal{L}(\mathcal{H})$ is $A_- := \frac{1}{2}(A - |A|)$. It is obvious that an
almost hyponormal operator is the sum of a positive semi-definite operator and a trace class. The converse is valid too according to the following.

**Proposition 1.1.** [3, Proposition 1.1]. An operator \( T \in \mathcal{L}(\mathcal{H}) \) is almost hyponormal if and only if \([T^*, T]\) is the sum of a positive semi-definite operator and a trace class. The converse is valid too according to the following.

The Berger-Shaw theorem provides a sufficient condition for a hyponormal operator to have a trace class self-commutator. Recall that an operator \( S \in \mathcal{L}(\mathcal{H}) \) has finite rational multiplicity if there is a finite set of vectors \( x_1, \ldots, x_m \in \mathcal{H} \) so that the closed linear span generated by the set
\[
\{ r(S)x_j \mid r \in \text{Rat}(\sigma(S)), \ j = 1, \ldots, m \}
\]
is the entire space \( \mathcal{H} \), where \( \text{Rat}(\sigma(S)) \) is the algebra of rational functions with poles off the spectrum \( \sigma(S) \). The smallest \( m \) that satisfies such condition is called the rational multiplicity of \( S \) and will be denoted by \( m(S) \). When such a smallest \( m \) does not exist, \( m(S) \) will be \( +\infty \). Thus, Berger-Shaw theorem asserts that a hyponormal operator \( T \in \mathcal{L}(\mathcal{H}) \) with finite rational multiplicity \( m(T) \) is almost normal and
\[
\text{tr}([T^*, T]) \leq \frac{1}{\pi} m(T) \text{area}(\sigma(T)),
\]
where \( \text{area}(\sigma(T)) \) is the planar Lebesgue measure of \( \sigma(T) \). Hadwin and Nordgren extended in [2] this theorem to include the particular case when the rational multiplicity is infinite and the area area of the spectrum of the operator is zero. A beautiful extension of this theorem along with an operator theoretic proof was obtained by D. Voiculescu.

**Theorem 1.2.** (\cite[Proposition 3]{6}). If \( T \in \mathcal{L}(\mathcal{H}) \) is an almost hyponormal operator and \( X \in \mathcal{L}(\mathcal{H}) \) is a Hilbert-Schmidt operator such that \( m(T + X) < +\infty \), then \( T \) is almost normal and
\[
\text{tr}([T^*, T]) \leq \frac{1}{\pi} m(T + X) \text{area}(\sigma(T + X)).
\]

For the last few decades there has been lots of interest in extending Berger-Shaw theorem to multi-operators.

**Definition 1.** For a multi-operator \( \mathbf{t} = (T_1, \ldots, T_m) \in \mathcal{L}(\mathcal{H})^m \), we denote the right-commutator, left-commutator, joint-commutator of \( \mathbf{t} \) the following operators in \( \mathcal{L}(\mathbb{C}^m \otimes \mathcal{H}) \) (or in \( \mathcal{L}(\mathcal{H}^m) \))
\[
\mathcal{C}_r(\mathbf{t}) = ([T_j^*, T_k])_{1 \leq j,k \leq m}, \quad \mathcal{C}_l(\mathbf{t}) = ([T_j, T_k^*])_{1 \leq j,k \leq m}, \quad \mathcal{C}(\mathbf{t}) = \frac{1}{2}(\mathcal{C}_r(\mathbf{t}) - \mathcal{C}_l(\mathbf{t})).
\]

Furthermore, since each of \( \mathcal{C}_r(\mathbf{t}), \mathcal{C}_l(\mathbf{t}) \), \( \mathcal{C}(\mathbf{t}) \) is a self-adjoint operator on \( \mathbb{C}^m \otimes \mathcal{H} \), one can introduce the following.

**Definition 2.** A multi-operator \( \mathbf{t} = (T_1, \ldots, T_m) \) is called right hyponormal (almost right hyponormal) if \( \mathcal{C}_r(\mathbf{t}) \) (\( \mathcal{C}_r(\mathbf{t})_- \)) is positive semi-definite (trace
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class), respectively. Similarly, \(t\) is called left cohyponormal (almost left cohyponormal) if \(C_l(t) (C_l(t)\_+)\) is positive semi-definite (trace class), respectively. A multi-operator \(t\) is called jointly hyponormal (almost jointly hyponormal) if \(C(t) (C(t)\_+)\) is positive semi-definite (trace class), respectively.

The reader can easily guess the definition of right cohyponormal, almost right cohyponormal, left hyponormal, almost left hyponormal, jointly cohyponormal, and almost jointly cohyponormal together with some direct connections between them such as, the adjoint \(T^* = (T_1^*, \ldots, T_m^*)\) of a left cohyponormal (almost left cohyponormal) is right hyponormal (almost right hyponormal), respectively.

An extension of the Berger-Shaw theorem that would involve the Lebesgue measure of the Taylor spectrum as well as a finite polynomial multiplicity of a multi-operator \(t = (T_1, \ldots, T_m)\) requires the commutativity of \(T_i\)'s. Furthermore, such a result is not possible as one can easily see that \(t = (T_1, T_2)\), with \(T_i\) being the multiplication by coordinate function \(z_i\) defined on the Hardy space \(H^2(D^2)\) of the bidisc \(D^2\) has a right commutator that is not compact; more precisely \(C_r(t) = (P \otimes I) \oplus (I \otimes P)\), where \(P\) is the orthogonal projection onto the one dimensional subspace generated by constant function 1 in \(H^2(D)\).

The proposition below suggests that instead of the rational multiplicity, one can consider the modulus of Hilbert-Schmidt quasi-triangularity, which we define first.

**Definition 3.** For a multi-operator \(t = (T_1, \ldots, T_m)\), let the modulus of Hilbert-Schmidt quasi-triangularity of \(t\) be

\[
q_2(t) = \liminf_{P \in \mathcal{P}, P \uparrow I} \left( \sum_{j=1}^{m} |(I - P)T_jP|^2 \right)^{\frac{1}{2}},
\]

where \(\mathcal{P}\) denotes the set of all finite rank orthogonal projections on \(\mathcal{H}\) and the \(\lim inf\) is with respect to the natural order on \(\mathcal{P}\) of operators \(P\) converging in the strong operator topology to \(I\). In particular, for a single operator \(T \in L(\mathcal{H})\),

\[
q_2(T) = \liminf_{P \in \mathcal{P}, P \uparrow I} |(I - P)TP|_2.
\]

**Proposition 1.3.** ([6, Proposition 2]). Let \(T \in L(\mathcal{H})\) be an almost hyponormal operator so that \(q_2(T) < +\infty\). Then \(\text{tr}([T^*, T]) \leq (q_2(T))^2\).

**Proposition 1.4.** ([6, Proposition 1]). Let \(T \in L(\mathcal{H})\) be an be an operator with \(m(T) < \infty\). Then \((q_2(T))^2 \leq m(T) \|T\|^2\).

Since a normal operator \(T\) with infinite dimensional null space has \(m(T) = \infty\), and on other hand \(q_2(T) = 0\) according to [5, Cor. 2.6 and Thm. 4.2], the modulus of Hilbert-Schmidt quasi-triangularity seems a more adequate choice than the rational multiplicity, but with the disadvantage that the majorant does not involve the norm \(\|T\|\).

To the end of this section we recall the following proposition with a small modification, which will be used in the next section.
Proposition 1.5. ([4, Lemma 2.7]). Let $C = (C_{jk})_{1 \leq j,k \leq m}$ be an operator in $L(C^m \otimes H)$. The following are equivalent:
(i) $C$ is positive semi-definite;
(ii) there exists a multi-operator $C = (C_1, \ldots, C_m)$ in $L(H)^m$ such that $C_{jk} = C_j^*C_k$, $1 \leq j, k \leq m$.

2. Main result.

Theorem 2.1. If $t = (T_1, \ldots, T_m)$ is a multi-operator that is almost right hyponormal or almost jointly hyponormal such that $q_2(t) < \infty$, then $[T_j^*, T_k]$ is trace class, $1 \leq j, k \leq m$.

Proof. First, let $Tt$ be an almost right hyponormal multi-operator with $q_2(t)$ finite. According to Proposition 1.1, $C_r(t)$ is the sum of a positive semi-definite operator $Q$ and a trace class operator $K$, both defined on $C^m \otimes H$ which can be identified with $\bigoplus_{i=1}^m \mathcal{H}_i$, where $\mathcal{H}_i = H$. If $P_j$ is the orthogonal projection from $C^m \otimes H$ onto the $j$-th component of $\bigoplus_{i=1}^m \mathcal{H}_i$, then $P_j C_r(t) P_j = [T_j^*, T_j]$ is the sum of a positive semi-definite operator and a trace class operator, that is, each $T_j$ is an almost hyponormal operator. It is straightforward to see that $q_2(T_j) \leq q_2(t)$, $1 \leq j \leq m$, and thus each $q_2(T_j)$ is finite. Thus, according to Proposition 1.3, each $[T_j^*, T_j]$, $1 \leq j \leq m$, is trace class and $tr([T_j^*, T_j]) \leq q_2(T_j)^2$. Let $K_{jk}$ be the $(j,k)$ entry of the trace class operator $K$; thus $K_{jk}$ is trace class too. Furthermore $[T_j^*, T_k] - K_{jk} = Q_{jk}$ and $(Q_{jk})_{1 \leq j,k \leq m}$ is positive semi-definite and according to Proposition 1.5, there exists a multi-operator $C = (C_1, \ldots, C_m)$ such that $Q_{jk} = C_j^*C_k$. Since each $Q_{jj} = [T_j^*, T_j] - K_{jj}$, $1 \leq j \leq m$, is trace class, we obtain that each $C_j$ is a Hilbert-Schmidt operator, and thus $[T_j^*, T_k] = Q_{jk} - K_{jk} = C_j^*C_k - K_{jk}$ trace class.

In the case of an almost jointly hyponormal multi-operator, the proof is similar to the above one and is left for the reader. □

A consequence of the above proof is the following.

Corollary 2.1.1. If $t = (T_1, \ldots, T_m)$ is a right hyponormal or a jointly hyponormal multi-operator such that $q_2(t) < \infty$, then $[T_j^*, T_k]$, $1 \leq j, k \leq m$, is trace class and

$|tr([T_j^*, T_k])| \leq q_2(T_j) q_2(T_k)$.

Proof. If $t$ is a right hyponormal multi-operator, then each $K_{jk} = 0$ and each $Q_{jk} = [T_j^*, T_k] = C_j^*C_k$ is trace class since $K_j$’s are all Hilbert-Schmidt operators. Furthermore

$|tr(C_j^*C_k)| \leq |C_j|_2 \cdot |C_k|_2 \leq q_2(T_j) q_2(T_k)$.

If $t$ is jointly hyponormal, the $(j,k)$ entry of the joint commutator of $T$ is $\frac{1}{2}([T_j^*, T_k] - [T_j, T_k^*])$, and in particular the $(j,j)$ entry is $[T_j^*, T_j]$ and a similar argument to the above one can be applied. □
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