Review of Weierstrass Convergence

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Abstract

The Weierstrass convergence theorem is central to the theory of complex functions. It states that uniformly converging limits of analytic functions on complex regions in 2 dimensions are analytic again (see e.g. [1]). In this paper the theorem is extended to $n > 1$ real and complex dimensions, and the complement of these spaces in the space of continuous functions is studied.

Keywords: Clifford Algebras, Cauchy Theory in n dimensions

1 Introduction and Preliminaries

It is commonly held that a theory of analytic functions could possibly be done on complex vector spaces, only, at least that the functions will need to be complex-valued. While both conditions are sufficient to yield the results, they are not necessary:

Definition 1.1. [Analyticity] An algebra $\mathcal{A}$ (over $\mathbb{R}$) is called extension of $\mathbb{R}^n$, if $\mathcal{A}$ and $\mathbb{R}^n$ are isomorphic as normed vector spaces, and if the product $ab$ for $a,b \in \mathcal{A}$ is a continuous bilinear mapping on $\mathcal{A}$. Such an algebra can be equipped with the norm pulled over from $\mathbb{R}^n$, so that the isomorphism $\iota : \mathbb{R}^n \to \mathcal{A}$ becomes a unitary embedding of $\mathbb{R}^n$ into $\mathcal{A}$. Let $U \subset \mathcal{A}$ be an open subset. A continuous mapping $f : U \to \mathcal{A}$ is called differentiable at some $x_0 \in U$ if and only if there is a $c \in \mathcal{A}$, such that $\|f(x) - cx_0\| = o(\|x - x_0\|)$ and $\|f(x) - x_0c\| = o(\|x - x_0\|)$, where $o(h)$ for $h > 0$ denotes a rest term, such that $o(h)/h \to 0$ as $h \to 0$. That element $c \in \mathcal{A}$ is then called derivative of $f$ at $x_0$ and denoted by $\frac{df(x_0)}{dx} := c$. 
For a constant factor $a \in \mathcal{A}$, we then define $\frac{d(af)(x_0)}{dx} := a \frac{df(x_0)}{dx}$.

Further, $f$ is called locally analytic on $U$, if for each closed and bounded ball $B \subset U$ $f$ is the uniformly converging limit of a power series $f(x) = a \sum_{k>0} a_k x^k$ on $B$, where $a, x, a_k \in \mathcal{A}$, $(k \in \mathbb{N})$, and the $a_k$ and $x$ commute with each other.

If these power series extensions extend globally on $U$, $f$ will be called (globally) analytic on $U$.

Because a function $f : U \to \mathcal{A}$ can be extended to the complex set $U + iU$ as $f(x + iy) := f(x) - if(y)$, we may stick to $\mathbb{R}^n$ and extend to the complex domain at the end.

**Remark 1.2.** For $n = 2$, using the field $\mathbb{C}$ as algebra extension of $\mathbb{R}^2$, the above reduces to the space of analytic (or holomorphic) functions that Weierstraß considered in his convergence theorem.

**Definition 1.3.** [Primitive] Let $f : U \to \mathcal{A}$ be a continuous function of an open subset $U \subset \mathcal{A}$ into $\mathcal{A}$. A function $If : U \to \mathcal{A}$ is called a primitive of $f$ if $\frac{d(If)(x)}{dx} = f(x)$ for all $x \in U$, and accordingly the $m$th primitive of $f$ is defined as function $I^m f : U \to \mathcal{A}$, such that $\left(\frac{d^m}{dx^m}\right) I^m f(x) = f(x)$.

We finally need the algebra extension of $\mathbb{R}^n$ to contain a 1-element, i.e. a neutral element of multiplication $1 \in \mathcal{A}$, that is: $1 \cdot a = a \cdot 1 = a$ is to hold for all $a \in \mathcal{A}$.

Let $U \subset \mathbb{R}^n$ be open with $n \geq 2$ and $\mathcal{C}(U)$ be the vector space of all (real or complex valued) continuous functions on $U$. It is common to define its topology as the projective limit of the restrictions to compact $K \subset U$: $\pi_K : \mathcal{C}(U) \ni f \mapsto f|_K \in \mathcal{C}(K)$, where the $\mathcal{C}(K)$ are Banach spaces under their supremum norm $\|\cdot\| : f \mapsto sup_{x \in K} |f(x)|$. (The projective limit is then defined as the coarsest locally convex topology on $\mathcal{C}(U)$, such that all $\pi_K$ are continuous.) This topology makes $\mathcal{C}(U)$ a complete, metrizable and separable vector space (known as F-space, see: [3]). Further, on F-spaces the open mapping theorem holds, which states that continuous and surjective linear mappings of F-spaces are open (see: [3] again).

By the Weierstraß convergence theorem, the vector space $\mathcal{A}(U)$ of analytic functions on some open $U \subset \mathbb{C}$ becomes a closed subspace of $\mathcal{C}(U)$. Consequently, $\mathcal{A}(U)$ must have an algebraic, if not topological complement in $\mathcal{C}(U)$.

And the problem is: what is this complement? At the same time, one would expect similar results for arbitrary dimensions $n \geq 2$.

### 2 Decomposition Theorem

**Theorem 2.1.** $\mathcal{C}(U)$ is the direct sum the space of constant functions $\text{Const}$ and two (isomorphic) subspaces $\mathcal{C}(U) = \mathcal{X}_+(U) \oplus \mathcal{X}_-(U) \oplus \text{Const}$, for which

\[\left(\frac{d^m}{dx^m}\right) I^m f(x) = f(x)\]
the following holds: \( \mathcal{X}_+(U) \) and \( \mathcal{X}_-(U) \) have algebra extensions \( \mathcal{A}_+(U) \) and \( \mathcal{A}_-(U) \) that embed \( \mathcal{X}_+ \) and \( \mathcal{X}_- \) as locally analytic functions.

In the following, I’ll shortly use \( \mathcal{X}(U) := \mathcal{X}_+(U) \oplus \mathcal{X}_-(U) \).

Proof. First, let us prove that it suffices to show the theorem for a closed ball \( B_r(x_0) \subset U \) around \( x_0 \in U \) of some radius \( r > 0 \):

Since \( U \subset \mathbb{R}^n \) is open, \( U \) is the countable union of a family \( B_k := B_{r_k}(x_k) \) (\( k \in \mathbb{N} \)), of such balls. And for this coverage \( (B_k)_{k \in \mathbb{N}} \) there is a subordinate partition of unity \((g_t)_{t \in \mathbb{N}} \) of functions \( g_t \in \mathcal{C}(U) \) with \( g_t \geq 0 \), each \( g_t \) vanishing outside some \( B_k \), and \( \sum_{t \in \mathbb{N}} g_t(x) = 1 \) for all \( x \in U \). Now, in order to prove that \( \mathcal{C}(U) \) is the topological direct sum of two subspaces, we have to prove that its restrictions to compact \( K \subset U \) are for every compact \( K \subset U \). Then, given any compact \( K \subset U \), there are finitely many of the \( g_t \), \( g_1, \ldots, g_m \), say, such that \( g_1(x) + \cdots + g_m(x) = 1 \) for all \( x \in K \), and each of these \( g_t \) has its support contained in one of the \( B_k \), which then (continuously) map \( f \in U \) to \( g_t f \in C_c(B_k) \), which is the space of continuous functions with support in \( B_k \). So, because we have just a finite set of \( m \in \mathbb{N} \) of \( g_t \), if we prove that \( \mathcal{C}(B_k) = \mathcal{X}_+(B_k) \oplus \mathcal{X}_-(B_k) \oplus \text{Const} \) for each \( B_k \), then we proved that globally for \( \mathcal{C}(U) \). Also note that \( \mathcal{X}_+(B_k) \) does not need to have support in \( B_k \): every \( f \in \mathcal{X}_+(B_k) \) can be added \( m - 1 \) other parts of the other \( \mathcal{X}_+(B_k) \). So, we are left to prove the theorem for \( \mathcal{C}(B_r) \), where \( B_r \) is a ball of some radius \( r > 0 \) around some \( x_0 \in \mathbb{R}^n \), and we may assume that ball center to be the origin 0 itself.

Each \( x \) in the interior of \( B(r) \) has a distance \( d > 0 \) from the boundary of \( B(r) \). So, the integral along all \( k^{th} \) components \( I_k(h) f : (x_1, \ldots, x_k + h, \ldots, x_n) \mapsto \int_0^h f(x_1, \ldots, x_k + \lambda, \ldots, x_n) d\lambda \) is well defined for \( h \) in the open interval \((-d/2, +d/2)\), and so are its compositions \( I_k(h) I_l(h) f \), which integrate first \( f \in \mathcal{X}(B(r)) \) along the \( l^{th} \) component from \( x_l \) to \( x_l + h \), followed by an integration along the \( k^{th} \) component from \( x_k \) to \( x_k + h \). Therefore the derivative \( f_{kl}(x) := \lim_{h \to 0} \frac{I_k(h) I_l(h) f(x)}{2h} \) exists for all \( x \) in the interior of \( B(r) \) and is for real-valued \( f \) a uniformly continuous, \( \mathbb{R}^n \)-valued function on \( B(r) \) with values in the plane spanned by the \( k^{th} \) and \( l^{th} \) coordinates, each, and splits into the sum of an \( \mathbb{R}^n \)-valued and an \( i\mathbb{R}^n \)-valued function in case of complex-valued \( f \). For \( k = l \), \( f \) is twice integrated from \( x_k \) to \( x_k + h \) along the \( k^{th} \) coordinate, \( I_k(h) I_k(h) f(x) = O(h^2) \), so \( f_{kk} \equiv 0 \) follows for all \( k = 1, \ldots, n \).

What we get is a linear mapping \( \Lambda : \mathcal{C}(B_r) \ni f \mapsto (f_{kl})_{1 \leq k, l \leq n} \) from \( \mathcal{C}(B_r) \) into a vector space \( \mathcal{M}(B_r) \) of \( n \times n \)-matrices of elements \( f_{kl} \in \mathcal{C}_r=0(B_r) \), \( 1 \leq k, l \leq n \) of \( \mathbb{R}^n \)-valued, continuous functions on \( B_r \) with zero diagonal elements \( f_{11} = \cdots = f_{nn} \equiv 0 \). With \( \| \cdot \| : (f_{kl})_{kl} \mapsto \max_{1 \leq k, l \leq n} \sup_{x \in B_r} |f_{kl}(x)| \) this vector space becomes a Banachspace and \( \Lambda \) a continuous linear mapping from \( \mathcal{C}(B_r) \) into \( \mathcal{M}(B_r) \). The kernel of \( \Lambda \) now is
the 1-dimensional space $\text{Const}$ of constant (vector) functions: First, a constant $f \in C(U)$ (which means that $f$ is constant on each connected subset of $U$) is continuously differentiable, so the partial derivatives commute with the path integrals, these derivatives are all zero, and so all path integrals from some $x \in B_r$ within $B_r$ to some $x + y$ in $B_r$ vanish, so $\Lambda$ does contain the constant functions in its kernel. And because $f_{kl} = 0$ for all $k, l$ implies that the sum of the differentials, once along positive $k$ and then along positive $l$, plus the one along negative $k$ and then positive $l$, vanishes, the preimage $\Lambda^{-1}\{0\}$ consists of all $f$, for which all partial derivatives are defined and vanish. And these are the constant functions. So, the kernel is the space of constant functions. The constant functions are a 1-dimensional closed subspace $\text{Const}$ of $C(U)$, likewise is $C(B_r) / \text{Const}$ a closed subspace of $C(B_r)$. So the embedding, $I : \text{Const} \oplus (C(B_r) / \text{Const}) \to C(B_r)$ defines a continuous linear map from the topological direct sum $\text{Const} \oplus (C(B_r) / \text{Const})$ into $C(U)$, and since it is onto, it is open (by open-mapping theorem). Hence, $\Lambda : C(B_r) / \text{Const} \to \mathcal{M}(B_r)$ is an isomorphism of Banach spaces.

$\mathcal{M}(B_r)$ itself plainly offers to be split into the direct sum of two (closed and open) subspaces $\mathcal{M}(B_r) = \mathcal{M}_+(B_r) \oplus \mathcal{M}_-(B_r)$ through $(f_{kl}) \mapsto (\frac{1}{2}f_{kl}) + (f_{kl})^t + (\frac{1}{2}f_{lk}) - (f_{lk})^t$, where $(f_{kl})^t := (f_{lk})$ is the transpose of $(f_{lk})$: a symmetric subspace $\mathcal{M}_+(B_r)$ and its antisymmetric counterpart $\mathcal{M}_-(B_r)$. Then again, $\mathcal{M}(B_r)$ is by the open mapping theorem the topological direct sum of $\mathcal{M}_+(B_r)$ and $\mathcal{M}_-(B_r)$. And by continuity of $\Lambda$, $\mathcal{X}(B_r)$ then is the topological direct sum of $\mathcal{X}_+(B_r) := \Lambda^{-1}\mathcal{M}_+(B_r)$ and its antisymmetric complement $\mathcal{X}_-(B_r) := \Lambda^{-1}\mathcal{M}_-(B_r)$. That shows the first part of the theorem: $C(U) / \text{Const}$ splits into the direct sum of two subspaces, a symmetric one, $\mathcal{X}_+(U)$, and an anti-symmetric subspace $\mathcal{X}_-(U)$.

Now, every $(f_{kl})_{kl} \in \mathcal{X}_+\mathcal{X}(B_r)$ defines a twice integrable 2-form: because of its symmetry, the path integrals are only dependent on the endpoints, so it integrates to a vector field $(F_1, \ldots, F_n)$ on $B_r$, and that is integrable again into a scalar field $G$, because the partial derivatives commute (in here):

The space $C(B_r, \mathbb{R}^n)$ of all uniformly continuous function $f$ from $B(r) \subset \mathbb{R}^n$ into $\mathbb{R}^n$ is a Banachspace with its supremum norm. So, its complex direct sum $C(B_r, n) := C(B_r, \mathbb{R}^n) \oplus iC(B_r, \mathbb{R}^n)$ also is a Banachspace, on which we may do the same as with $C(B_r)$ before: path integrating from each $x$ in the interior of $B(r)$ along along the $l^{th}$ coordinate to $x_l + h$ for small $h \in \mathbb{R}$, followed by the path integration along the $k^{th}$ coordinate from $x_k$ to $x_k + h$, and calculating its slope for $2h \to 0$. This maps $f \in C(B_r, n)$ to a matrix-valued function $(f_{kl})_{1 \leq k, l \leq n}$, which extends onto $B_r$ as a uniformly continuous function. Again, $f_{11} = \cdots = f_{nn} = 0$, and defining $\mathcal{M}(B_r, n)$ to be that Banach space of matrix valued functions, there are just 2 differences with the space $\mathcal{M}(B_r)$: First, the $f_{ml}$ are complex valued, instead of vector-valued, and secondly, the kernel of the mapping $\iota : C(B_r, n) \to \mathcal{M}(B_r, n)$ now is the n-
dimensional space of constant vectors instead of just the 1-dimensional space of constants. And again \( M(B_r, n) \) splits into the direct topological sum of a subspace of symmetric matrices \( M_+(B_r, n) \) and anti-symmetric matrices \( M_-(B_r, n) \). Therefore again, \( C(B_r, n) \) decomposes modulo vector space of constants into a subspace \( X_+(B_r, n) \) of integrable vector fields and a disjoint subspace \( X_+(B_r, n) \). Then, given \( f \in X_+(B_r) \), its primitive \( I f = (I f_1, \ldots, I f_n) \) exists and is in \( C(B_r, n) \), and because (modulo \( \text{Const} \)) \( \frac{\partial I f}{\partial x_k} = \frac{\partial I f}{\partial x_l} = f + \text{Const} \) for all \( 1 \leq k \neq l \leq n \), \( I f \in X_+(B_r, n) \). So, by induction, the primitives \( I^m f \) exist for all orders \( m \in \mathbb{N} \), and they are members of \( X_+(B_r, n) \) for odd \( m \) or functions in \( X_+(B_r) \), when \( m \) is even.

So far, we ignored to deal with the integrability of \( X_-(B_r) \): It is not true that it cannot and therefore must not be integrable: Poincaré’s lemma itself mandates its integrability: Because every \( f \in X(B_r) \) defines a zero 2-form \( \alpha := \sum_{l<k\leq n} f dx_k \wedge dx_l \equiv 0 \), and zero differential forms are always twice integrable: once, because \( \alpha \equiv 0 \Rightarrow d\alpha \equiv 0 \), and its integral \( \beta \) say, is integrable again, since \( d\beta = \alpha \equiv 0 \). In particular, Poincaré’s lemma implies the existence of integrals of any order for all \( f \in X(B_r) \) (and therefore the local integrability of all orders for all \( f \in X(U) \)).

It is straightforward to use the conjugation \( \chi : M_+(B_r) \rightarrow M_-(B_r) \) to get at the primitives of all \( f \in X_-(B_r) \). (\( \chi \) maps \( f_{kl} \) for \( k > l \) to \( f_{kl} \) and \( f_{kl} \) for \( k < l \) to \( -f_{kl} \), so is an isomorphism on \( M(B_r) \), for which \( \chi^2 \) is the identity.)

On \( f \in X(B_r) := X_+(B_r) \oplus X_-(B_r) \) this conjugation is delivered through the transformation \( \chi : x = (x_1, \ldots, x_n) \mapsto (\alpha_1 x_1, \ldots, \alpha_n x_n) \), where the \( \alpha_k \) are anti-commuting square matrices such that \( \alpha_k^2 = \cdots = \alpha_n^2 = 1 \), 1 being the unit matrix: Because then, \( \chi : X_+(B_r) \ni (f_+ : x \mapsto f_+(x)) \mapsto (\chi f_+ : x \mapsto f(\alpha_1 x_1, \ldots, \alpha_n x_n) \in X) \) maps between symmetric and antisymmetric functions and extends as a conjugation on \( X(B_r) \). (The \( \alpha_k \) can be picked from the \( n^2 - 1 \) generators of the Lie algebra \( su(n) \).) This defines nothing but an extension of \( \mathbb{R}^n \) to the \( n \)-dimensional orthogonal Clifford algebra \( Cl(n) \): this algebra is defined as the algebra (over \( \mathbb{R} \) or \( \mathbb{C} \)) generated by \( n \) orthogonal unit vectors \( e_1, \ldots, e_n \) for which \( e_k^2 := e_k e_k = 1 \), \( 1 \leq k \leq n \) and \( e_i e_j = -e_j e_i \), \( 1 \leq k < l \leq n \) holds. Then \( \mathbb{R}^n \ni (x_1, \ldots, x_n) \mapsto x_1 e_1 + \cdots + x_n e_n := (x_1 e_1, \ldots, x_n e_n) \in Cl(n) \) is an embedding into \( Cl(n) \), the (symmetric) functions \( X_+(B_r) \) map to functions \( x_1 e_1 + \cdots + x_n e_n \mapsto f(x_1 e_1 + \cdots + x_n e_n) \), the antisymmetric ones become functions \( x_1 e_1 + \cdots + x_n e_n \mapsto f(x_1, \ldots, x_n) \), and \( x_1 e_e + \cdots + x_n e_n \mapsto (x_1, \ldots, x_n) \) defines the conjugation \( \chi \) on \( X(B_r) \), mapping \( X_\pm(B_r) \) onto each other. With this conjugation in place, the elements of \( X_-(B_r) \) can be mapped to \( X_-(B_r) \) and integrated to any order.

(Analogously, \( X_+(B_r, n) \) and \( X_-(B_r, n) \) are conjugated.)

For \( x, y \in \mathbb{R}^n \) the product \( xy := (x_1 e_1 + \cdots + x_n e_n)(y_1 e_1 + \cdots + y_n e_n) = \sum_{k,l} x_k y_l e_k e_l \) is a well-defined, non-commuting, but associative continuous bi-

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linear mapping, in which $e_k \wedge e_l := e_k e_l$ is equivalent with the exterior product of the $k^{th}$ and $l^{th}$ vector components. With the inner product $x \cdot y := \sum_k x_k y_l$ that product rewrites to $xy = x \cdot y + \sum_k x_k \wedge y_l$. In particular, within $x^n$ for $m \in \mathbb{N}$, all exterior products cancel out. Because the $e_k$ are inverse to themselves, for $x \neq 0$, it follows that there is a unique $1/x := y \neq 0$ such that $xy = yx = 1$.

Since $Cl(n)$ has a 1-element, the identity mapping $f : \mathbb{R}^n \ni x \mapsto x \in Cl(n)$ becomes differentiable: its derivative is 1, and the derivative of 1 is zero. In turn, the primitive of 1 is the identity, $f(x) = x$, the primitive of $x$ is $(1/2)x^2$, $(1/3)x^3$ is the primitive of $x^2$, and so forth. That allows the definition of analyticity. Likewise, all powers of $x^{-1}$ are differentiable (outside the origin), but not all are having a primitive: the exception is $1/x$. While integration along a circle around the origin for complex $x \in \mathbb{C}$ gives $2\pi i$, that’s not the case in higher dimensions of 3 or more real components. (The reason is of course that the circle in 3 or more dimensions does not dis-connect the circle’s center from its outside, and the result of path integration of $x^{-1}$ along a circle vanishes for $n > 2$.) Because for $f \in \mathcal{X}_+(U)$ the $m^{th}$ primitives $I^m f$ locally exist for every $m \in \mathbb{N}$, the surface integral around a (sufficiently small) ball $B_r(x_0) \subset U$ of radius $r$ around $x_0 \in U$ exists for every $x_0 \in U$, involves $n - 1$ successive path integrations, one around a full circle, which gives zero.

**Remark 2.2** (Volume, surface integrals and the divergence theorem). For $n \in \mathbb{N}$ dimensions, the volume differential is usually written as $d^n x := dx_1 \cdots dx_n$, while - to be picky - it is the (alternating) differential $n$-form $dx_1 \wedge \cdots \wedge dx_n$. (In particular, it means that it always includes the orientation of the coordinate system.) By successive integration of a smooth function $f : U \to \mathbb{C}$ over a bounded volume $V \subset U \subset \mathbb{R}^2$ along $dx_1, \ldots, dx_n$, the differentials drop out, yielding the volume integral $\int_V \frac{f(x)}{x} dx_1 \wedge \cdots \wedge dx_n \in \mathbb{C}$. If the volume $V$ has a smooth boundary $\Gamma(V)$, then $\Gamma(V)$ has an $(n - 1)$-dimensional tangent space $T_x$ for every $x \in \Gamma(V)$ and its differential volume $da_1 \wedge \cdots \wedge da_{n-1}$ defines the differential surface element in $n$ dimensions, which is commonly written as $\tilde{n} d^{n-1} a$, where $\tilde{n}$ is the unit vector of the orthogonal complement of $T_x$ in $\mathbb{R}^n$.

That association of the differential surface element $da_1 \wedge \cdots \wedge da_{n-1}$ with its normal vector $\tilde{n}$ is arbitrary and misleads to assume that $\int_{\Gamma(V)} f da_1 \wedge \cdots \wedge da_n$ was only defined for vector fields $f : U \in \mathbb{C}^n$, which is untrue. For instance, constant functions on $(n - 1)$-dimensional $r$-spheres can be cut into $2^n$ sections of equal absolute size of area, but with alternating parity of the coordinates; so the surface integral of a constant function over $\Gamma(B_r)$ is zero, and therefore the surface integral of a constant function over the closure of any bounded, convex open set $U \subset \mathbb{R}^n$ always vanishes.

Similarly, within $Cl(n)$, the distinction between vectors and scalars becomes insignificant: divergence of a vector field as well as the gradient of a scalar field boil down to the derivative $\frac{d}{dx}$, and for any $f \in \mathcal{X}_+(U)$ any order $m \geq 0$,
and any \( r\)-ball \( B_r \subset U \) we have: \( \int_{B_r} I^n f(x) d^n x = \int_{\Gamma(B_r)} I^{n+1} d^{n-1} a \), where 
\[ d^n x := dx_1 \wedge \cdots \wedge dx_n \quad \text{and} \quad d^{n-1} a := da_1 \wedge \cdots \wedge da_{n-1}. \]
This is the divergence theorem in a more general form.

**Remark 2.3** (Euler’s formula). In two dimensions, within \( \mathbb{R}^2 \), the unit circle around the origin is given by the set of all \( (x_1, x_2) \) with \( x_1 = \cos(\phi), x_2 = \sin(\phi), \, (0 \leq \phi < 2\pi) \), which map under the \( Cl(n) \)-embedding to \( x_1 e_1 + x_2 e_2 = \cos(\phi)e_1 + \sin(\phi)e_2 = \cos(\phi)e_1 - \sin(\phi)(e_1 e_2)e_1 = e^{-\epsilon_1 e_2} e_1 \).
Since \( (e_1 e_2)^2 = -1 \), \( e_1 e_2 = \pm i \), setting \( e_1 e_2 := -i \), and this becomes equivalent to Euler’s formula \( x + iy = e^{i\phi} \) for \( x + iy \in \mathbb{C} \).

There is a subtle sign mismatch between the differential forms in \( \mathbb{R}^n \) and in \( Cl(n) \): While in \( \mathbb{R}^n \) the \( n\)-form \( dx_1 \wedge \cdots \wedge dx_n \) is defined to be real-valued, within \( Cl(n) \), the \( n\)-form rewrites to
\[ (r \cos(\phi_1), r \sin(\phi_1) \cos(\phi_2), \ldots, r \sin(\phi_1) \cdots \sin(\phi_{n-2}) \cos(\phi_{n-1}), r \sin(\phi_1) \cdots \sin(\phi_{n-2}) \sin(\phi_{n-1})) \]
we have to integrate the polar angles \( \phi_{n-1}, \ldots, \phi_2 \) successively from 0 to \( \pi \), and the azimuthal angle \( \phi_1 \) from 0 to \( 2\pi \). With \( f \) and \( I^m f \), after the first \( n-2 \) integrations along the polar angles, what we get are integrable functions \( I^{n-2} f \) and \( I^{n+m-2} f \) in the \( (r, \phi) \)-plane, which are to be integrated along the r-circle from \( (r, 0) \) to \( (r, 0) \). So these vanish, which proves (i) and (ii). And for \( x \mapsto \frac{1}{(x-x_0)^{n-1}} \)
we get \( \frac{S_{n-1}}{2\pi r} \), which in the \((r, \phi_1)\)-plane integrates to \( \frac{S_{n-1}}{2\pi} 2\pi = S_{n-1} \), which proves (iii). Statement (iv) follows for \( m = 1 \) by integrating \( \int_{\gamma} f(\cdot) (\cdot - x_0) d\gamma \) by parts, where \( \gamma \) is a (piecewise smooth) closed path, which gives \(-\int_{\gamma} (I f) d\gamma \). And because the primitive \( I f \) is integrable, this term vanishes. For \( m > 1 \), (iv) follows by induction. Then the product \( f g \) is integrable on \( B(r) \) for any uniformly converging power series \( g = c \sum c_k x^k \) with \( c_k \in \mathbb{C} \) and \( c \in Cl(n) \). So, (v) follows, since

\[
\frac{1}{(x - x_0)^m} = - \frac{1}{x_0^m} \frac{1}{1 - x_0^{-1} x} = - \frac{1}{x_0^m} \sum_{k \geq 0} \binom{m + k - 1}{k} (x_0^{-1} x)^k
\]

converges absolutely for \( \|x\| < \|x_0\| \) and \( x \in B(r) \), and it extends onto \( B(r) \) via analytic continuation.

To get at the analogue of the Cauchy formula for \( n \geq 2 \) dimensions, we need to consider the integral \( \int_{S_{n-1}(r)} \frac{f(y)}{(x - y)^{n-1}} dy \) over the boundary \( S_{n-1}(r) \) of the \( r \)-ball around \( x \in U \): For \( r \to 0 \):

\[
\left| \int_{S_{n-1}(r)} \frac{f(x) - f(y)}{(x - y)^{n-1}} dy \right| \leq S_{n-1} \sup_{y \in S_{n-1}(r)} |f(x) - f(y)|,
\]

where \( S_{n-1} \) denotes the area of the \((n-1)\)-dimensional unit sphere. Because \( f \) is continuous, \( \sup_{y \in S_{n-1}(r)} |f(x) - f(y)| \) converges to 0 as \( r \to 0 \). So, we are left to calculate the value of the surface integral of \( \frac{1}{x^{n-1}} \) over the \((n-1)\)-dimensional unit sphere around the origin, which we did in the above lemma.

Therefore, the Cauchy formula in \( n \) dimensions is

\[
f(x) = \frac{1}{S_{n-1}} \int_{S_{n-1}(r)} \frac{f(y)}{(x - y)^{n-1}} dy,
\]

which holds for any \( r \)-ball \( B_r(x) \) around \( x \), on which \( f \) is uniformly continuous. In particular, \( f \) is analytic in all points \( x \) in the interior of the \( r \)-ball \( B_r(x) \). This proves the claim. □

References


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