Not Convex Densities and Everywhere Hölder Continuity, a Limit Case

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Dedicated to the memory of Fiorella Pettini.

Abstract

In this paper we study the everywhere hölder continuity of the minima of a class of vectorial integral functionals.

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1. Introduction

In this paper we study the regularity of the local minima of the following integral functional

\[
J(u, \Omega) = \int_{\Omega} \sum_{\alpha=1}^{n} f_\alpha (x, u^\alpha (x), \nabla u^\alpha (x)) + G (x, u (x), \nabla u (x)) \, dx \quad (1.1)
\]

where \( \Omega \) is a open subset of \( \mathbb{R}^N \) and \( u \in W^{1,p} (\Omega, \mathbb{R}^n) \) with \( n \geq 2, \, N \geq 1 \) and \( 1 < p < N \).

Moreover the following hypotheses hold

**H.1.1:** For every \( \alpha = 1, \ldots, n \) the function \( f_\alpha : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R} \) is a Caratheodory function and the following growth conditions hold

\[
|\xi^\alpha|^p - b_\alpha (x) |s|^{\gamma_\alpha} - a_\alpha (x) \leq f_\alpha (x, s, \xi^\alpha) \leq L_\alpha (|\xi^\alpha|^p + b_\alpha (x) |s|^{\gamma_\alpha} + a_\alpha (x)) \quad (1.2)
\]
for almost every $x \in \Omega$, for every $s \in \mathbb{R}$ and for every $\xi^a \in \mathbb{R}^N$ where $L_\alpha > 1$, $1 < p \leq \gamma_\alpha < p^* = \frac{Np}{N-p}$, $b_\alpha$ and $a_\alpha$ are two not-negative function, $b_\alpha \in L^{\sigma_\alpha}_{loc}(\Omega)$ and $a_\alpha \in L^{\kappa}_{loc}(\Omega)$ with $\sigma_\alpha = \frac{p^*}{p^* - \gamma_\alpha - p}$, $\kappa = \frac{N}{p^* - \gamma_\alpha}$ and $0 < \epsilon < \frac{p}{N}$.

**H.1.2:** $G : \Omega \times \mathbb{R}^n \times \mathbb{R}^{Nn} \to \mathbb{R}$ is a Caratheodory function and the following growth conditions hold

$$|G(x,u,\xi)| \leq C(|\xi|^q + |u|^q + a(x)) \quad (1.3)$$

for almost every $x \in \Omega$, for every $u \in \mathbb{R}^n$ and for every $\xi \in \mathbb{R}^{Nn}$ where $C > 1$, $1 \leq q < \frac{p^2}{N} < p$, $a$ is a not-negative function and $a \in L^\kappa_{loc}(\Omega)$ with $\kappa = \frac{N}{p^* - \epsilon N}$ and $0 < \epsilon < \frac{p}{N}$.

**H.1.3:** The function $G(x,u,\cdot)$ is rank one convex then

$$G(x,u,\lambda \xi^1 + (1-\lambda) \xi^2) \leq \lambda G(x,u,\xi^1) + (1-\lambda) G(x,u,\xi^2)$$

for a. e. $x \in \Omega$, for every $u \in \mathbb{R}^n$, for every $\lambda \in [0,1]$, and for every $\xi^1, \xi^2 \in \mathbb{R}^{Nn}$ with $\text{rank} \{\xi^1 - \xi^2\} \leq 1$.

**H.1.4:** The function $G(x,\cdot,\xi)$ is Hölder continuous and for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^{Nn}$ it follows

$$|G(x,u,\xi) - G(x,v,\xi)| \leq c(x)|\xi|^\delta|u-v| \quad (1.4)$$

for a. e. $x \in \Omega$, for every $u,v \in \mathbb{R}^n$ and for every $\xi \in \mathbb{R}^{Nn}$ with $0 < \delta < \min \left\{q, p-1, \frac{q(p^*-1)}{p^*} \right\}$, $c$ is a not-negative function and $c(x) \in L^\sigma_{loc}(\Omega)$ with $\sigma > \frac{p^* - \epsilon q}{p^* - \epsilon q - \beta q}$ and

$$\frac{\delta}{p} + \frac{1}{p^*} + \frac{1}{\sigma} < \frac{p}{N} \quad (1.5)$$

or

$$\frac{\delta}{p} + \frac{1}{p^*} < \frac{p}{N} \quad (1.6)$$

if $c(x) \in L^\infty_{loc}(\Omega)$.

Our principal result is the following Theorem.

**Theorem 1.** If $u \in W^{1,p}_{loc}(\Omega, \mathbb{R}^n)$, with $N \geq 2$, $n \geq 1$ and $1 < p < N$, is a local minimizer of the functional $J(u,\Omega)$ and the hypotheses H.1.1, H.1.2, H.1.3 and H.1.4 (or H.1.4 (bis)) hold then every component $u^a$ of the vectorial function $u$ are a locally Hölder continuous functions.

Theorem 1 derives from the following two results.

**Theorem 2.** If $u \in W^{1,p}_{loc}(\Omega, \mathbb{R}^n)$ with $1 < p < N$ is a local minimizer of the functional $J(u,\Omega)$ and the hypotheses H.1.1, H.1.2, H.1.3 and H.1.4 (or H.1.4
Everywhere Hölder continuity

(bis)) hold then two positive constants \( C_{C,1}, C_{C,2} \) and a radius \( R_0 > 0 \) exists such that for every \( 0 < \varrho < R < R_0 \) and for every \( k \in \mathbb{R} \) it follows

\[
\int_{A_{k,e}^\alpha} |\nabla u^\alpha|^p \, dx \leq \frac{C_{C,1}}{(R-\varrho)^p} \int_{A_{k,R}^\alpha} (u^\alpha - k)^p \, dx + C_{C,2} \left( 1 + |k|^p R^{-\epsilon_\alpha N} \right) |A_{k,R}^\alpha|^{1-\frac{p}{N} + \epsilon}
\]

(1.7)

and

\[
\int_{B_{k,e}^\alpha} |\nabla u^\alpha|^p \, dx \leq \frac{C_{C,1}}{(R-\varrho)^p} \int_{B_{k,R}^\alpha} (u^\alpha - k)^p \, dx + C_{C,2} \left( 1 + |k|^p R^{-\epsilon_\alpha N} \right) |B_{k,R}^\alpha|^{1-\frac{p}{N} + \epsilon}
\]

(1.8)

where \( A_{k,s}^\alpha = \{ u^\alpha > k \} \cap B_s(x_0) \) and \( B_{k,s}^\alpha = \{ u^\alpha < k \} \cap B_s(x_0) \).

**Theorem 3.** If \( u \in W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^n) \), with \( N \geq 2, n \geq 1 \) and \( 1 < p < N \), is a local minimizer of the functional \( J(u, \Omega) \) and the hypotheses H.1.1, H.1.2, H.1.3 and H.1.4 (or H.1.4 (bis)) hold then every component \( u^\alpha \) of the vector function \( u \) are a locally boundedness functions.

The results stated in the previous Theorem 1 is not trivial, in particular we recall the following fundamental counterexample by De Giorgi [14]; the function

\[ u(x) = x |x|^{-\delta} \]

with \( \delta = \frac{N}{2} \left( 1 - \frac{1}{\sqrt{1+(2N-2)}} \right) \) it is a local minimum of the functional

\[
F(u, B_1) = \int_{B_1} \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^N A_{\alpha,\beta}^{ij}(x) \partial_i u^\alpha \partial_j u^\beta \, dx
\]

(1.9)

with

\[ A_{\alpha,\beta}^{ij}(x) = \delta_{\alpha,\beta} \delta_{i,j} + \left( (N-2) \delta_{\alpha,i} + N \frac{x_\alpha x_i}{|x|^2} \right) \left( (N-2) \delta_{\beta,j} + N \frac{x_\beta x_j}{|x|^2} \right) \]

\( N \geq 3 \) and \( B_1 = \{ x \in \mathbb{R}^N : |x| \leq 1 \} \). It is easily observed that \( u(x) = x |x|^{-\delta} \) it is not a bounded function. Recently, in [8, 9, 25, 31, 32] new classes of vectorial problems, with regular weak solutions, have been introduced. In particular Theorem 1 is 1 is the limiting case of a previous result given by the author in [32]. In recent years a large number of articles have been written dealing with the regularity of weak solutions of vectorial problems under standard growths, general growths and anisotropic growths, refer to [1, 2, 4-12, 15-22, 23, 25-35, 41-43].

Our results can therefore be framed within a vast area of research called everywhere regularity that was born with the fundamental work Uhlenbeck [43].
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and which has developed up to now with results that generally have concerned functionals of the form

\[ \int_{B_R(x_0)} G(|\nabla u(x)|) \, dx \]

The literature in this area is very wide, in the bibliography we report only some articles. Generally results are obtained on the boundedness of the gradient of the minima using particular conditions of regularity and ellipticity using very different techniques, we mention [1,2, 4-7, 15, 17,18, 20, 21,33-35, 41-43] only because they are those consulted by the author during the preparation of the article but we refer to the bibliographies of the aforementioned articles for more bibliographical references. Our results differ from those previously mentioned as it deals with some functionals of the form

\[ \int_{B_R(x_0)} F(x,u,\nabla u) \, dx \]

in this case, as far as the limited knowledge of the author is concerned, there are few results, in particular we refer to [8, 25, 32]. Theorem 1 therefore generalizes the results presented in [8], [25] and it is, in a certain sense, complementary to the results presented by the author in [31].

2. Preliminary results

Before giving the proofs of Theorem 1 and Theorem 2, for completeness we introduce a list of results that we will use during the proof.

2.1. Lemmata.

Lemma 1 (Young Inequality). Let \( \varepsilon > 0, a, b > 0 \) and \( 1 < p, q < +\infty \) with \( \frac{1}{p} + \frac{1}{q} = 1 \) then it follows

\[ ab \leq \varepsilon \frac{a^p}{p} + \frac{b^q}{\varepsilon^p q} \]  

(2.1)

Lemma 2 (Hölder Inequality). Assume \( 1 \leq p, q \leq +\infty \) with \( \frac{1}{p} + \frac{1}{q} = 1 \) then if \( u \in L^p(\Omega) \) and \( v \in L^q(\Omega) \) it follows

\[ \int_{\Omega} |uv| \, dx \leq \left( \int_{\Omega} |u|^p \, dx \right)^{\frac{1}{p}} \left( \int_{\Omega} |v|^q \, dx \right)^{\frac{1}{q}} \]  

(2.2)

Lemma 3. Let \( Z(t) \) be a nonnegative and bounded function on the set \([\varrho,R]\); if for every \( \varrho \leq t < s \leq R \) we get

\[ Z(t) \leq \theta Z(s) + \frac{A}{(s-t)^\lambda} + \frac{B}{(s-t)^\mu} + C \]  

(2.3)
where $A, B, C \geq 0$, $\lambda > \mu > 0$ and $0 \leq \theta < 1$ then it follows

$$Z(\varrho) \leq C(\theta, \lambda) \left( \frac{A}{(R-\varrho)^\lambda} + \frac{B}{(R-\varrho)^\mu} + C \right)$$  \hspace{1cm} (2.4)$$

where $C(\theta, \lambda) > 0$ is a real constant depending only on $\theta$ and $\lambda$.

2.2. Sobolev Spaces.

**Theorem 4.** (Sobolev Inequality) Let $\Omega$ be an open subset of $\mathbb{R}^N$ if $u \in W^{1,p}_0(\Omega)$ with $1 \leq p < N$ there exists a real positive constant $C_{SN}$, depending only on $p$ and $N$, such that

$$\|u\|_{L^p^*(\Omega)} \leq C_{SN} \|\nabla u\|_{L^p(\Omega)}$$  \hspace{1cm} (2.5)$$

where $p^* = \frac{Np}{N-p}$.

**Theorem 5.** (Rellich-Sobolev Immersion Theorem) Let $\Omega$ be a open bounded subset of $\mathbb{R}^N$ with lipschitz boundary then if $u \in W^{1,p}(\Omega)$ with $1 \leq p < N$ there exists a real positive constant $C_{IS}$, depending only on $p$ and $N$, such that

$$\|u\|_{L^{p^*}(\Omega)} \leq C_{IS} \|u\|_{W^{1,p}(\Omega)}$$  \hspace{1cm} (2.6)$$

where $p^* = \frac{Np}{N-p}$.

For completeness we remember that if $\Omega$ is an open subset of $\mathbb{R}^N$ and $u$ is a Lebesgue measurable function then $L^p(\Omega)$ is the set of the class of the Lebesgue measurable function such that $\int |u|^p \, dx < +\infty$ and $W^{1,p}(\Omega)$ is the set of the function $u \in L^p(\Omega)$ such that its weak derivative $\partial_i u \in L^p(\Omega)$. The spaces $L^p(\Omega)$ and $W^{1,p}(\Omega)$ are Banach spaces with the respective norms

$$\|u\|_{L^p(\Omega)} = \left( \int_\Omega |u|^p \, dx \right)^{\frac{1}{p}}$$  \hspace{1cm} (2.7)$$

and

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{i=1}^N \|\partial_i u\|_{L^p(\Omega)}$$  \hspace{1cm} (2.8)$$

We say that the function $u : \Omega \subset \mathbb{R}^N \rightarrow \mathbb{R}^n$ belong in $W^{1,p}(\Omega, \mathbb{R}^n)$ if $u^\alpha \in W^{1,p}(\Omega)$ for every $\alpha = 1, \ldots, n$, where $u^\alpha$ is the $\alpha$ component of the vector-valued function $u$; we end by remembering that $W^{1,p}(\Omega, \mathbb{R}^n)$ is a Banach space with the norm

$$\|u\|_{W^{1,p}(\Omega, \mathbb{R}^n)} = \sum_{\alpha=1}^n \|u^\alpha\|_{W^{1,p}(\Omega)}$$  \hspace{1cm} (2.9)$$
Definition 1. Let \( \Omega \subset \mathbb{R}^N \) be a bounded open set and \( v : \Omega \rightarrow \mathbb{R} \), we say that \( v \in W^{1,p}_\text{loc}(\Omega) \) belong to the De Giorgi class \( \text{DG}^+(\Omega,p,\lambda,\lambda_s,\chi,\varepsilon,R_0,k_0) \) with \( p > 1, \lambda > 0, \lambda_s > 0, \chi > 0, \varepsilon > 0, R_0 > 0 \) and \( k_0 \geq 0 \) if
\[
\int_{A_{k,R}} |\nabla v|^p \, dx \leq \frac{\lambda}{(R-\varepsilon)^p} \int_{A_{k,R}} (v-k)^p \, dx + \lambda_s (\chi^p + k^p R^{-N\varepsilon}) |A_{k,R}|^{1-\frac{p}{N}+\varepsilon}
\]  
(2.10)
for all \( k \geq k_0 \geq 0 \) and for all pair of balls \( B_\varrho(x_0) \subset B_R(x_0) \subset \subset \Omega \) with \( 0 < \varrho < R < R_0 \) and \( A_{k,s} = B_s(x_0) \cap \{v > k\} \) with \( s > 0 \).

Definition 2. Let \( \Omega \subset \mathbb{R}^N \) be a bounded open set and \( v : \Omega \rightarrow \mathbb{R} \), we say that \( v \in W^{1,p}_{\text{loc}}(\Omega) \) belong to the De Giorgi class \( \text{DG}^-(\Omega,p,\lambda,\lambda_s,\chi,\varepsilon,R_0,k_0) \) with \( p > 1, \lambda > 0, \lambda_s > 0, \chi > 0 \) and \( k_0 \geq 0 \) if
\[
\int_{B_{k,R}} |\nabla v|^p \, dx \leq \frac{\lambda}{(R-\varepsilon)^p} \int_{B_{k,R}} (k-v)^p \, dx + \lambda_s (|k|^p + |v|^p R^{-N\varepsilon}) |B_{k,R}|^{1-\frac{p}{N}+\varepsilon}
\]  
(2.11)
for all \( k \leq -k_0 \leq 0 \) and for all pair of balls \( B_\varrho(x_0) \subset B_R(x_0) \subset \subset \Omega \) with \( 0 < \varrho < R < R_0 \) and \( B_{k,s} = B_s(x_0) \cap \{v < k\} \) with \( s > 0 \).

Definition 3. We set \( \text{DG}(\Omega,p,\lambda,\lambda_s,\chi,\varepsilon,R_0,k_0) = \text{DG}^+(\Omega,p,\lambda,\lambda_s,\chi,\varepsilon,R_0,k_0) \cap \text{DG}^-(\Omega,p,\lambda,\lambda_s,\chi,\varepsilon,R_0,k_0) \).

Theorem 6. Let \( v \in \text{DG}(\Omega,p,\lambda,\lambda_s,\chi,\varepsilon,R_0,k_0) \) and \( \tau \in (0,1) \), then there exists a constant \( C > 1 \) depending only upon the data and not-dependent on \( v \) and \( x_0 \in \Omega \) such that for every pair of balls \( B_{\tau v}(x_0) \subset B_v(x_0) \subset \subset \Omega \) with \( 0 < \varrho < R_0 \)
\[
\|v\|_{L^\infty(B_{\tau v}(x_0))} \leq \max \left\{ \lambda_s^{\frac{Np}{N-p}}, \frac{C}{(1-\tau)^{\frac{Np}{N}}} \left[ \frac{1}{|B_{\varrho}(x_0)|} \int_{B_{\varrho}(x_0)} |v|^p \, dx \right]^{\frac{1}{p}} \right\}. 
\]  
(2.12)

For more details on De Giorgi’s classes and for the proof of the Theorem 6 refer to [13, 22, 24, 26-28, 37, 38].

3. Proof of Theorem 2

Let us consider \( y \in \Omega \) then we fix \( R_0 = \frac{1}{4} \min \left\{ \frac{1}{N-r_N}, \text{dist}(\partial \Omega, y) \right\} \), where \( r_N = |B_1(0)| \), and we define \( \Sigma = \{x \in \Omega : |x-y| \leq R_0\} \). We fix \( x_0 \in \Sigma, R_1 = \frac{1}{4} \text{dist}(\partial \Sigma, x_0) \), \( R_0 < \min_{\alpha \in 1,\ldots,n} \{R_1, A_{\alpha}, B_{\alpha}\} \), where
\[
A_{\alpha} = \frac{1}{\left[ 2 \cdot 4^{\gamma_\alpha} L_\alpha C_{N,p,\gamma_\alpha} D^{\frac{p-\gamma_\alpha}{p}} \|b_\alpha\|_{L^{\gamma_\alpha}(\Sigma)} \|u^\alpha\|_{W^{1,p}(\Sigma)}^{\gamma_\alpha} \varpi_N \right]^{\frac{1}{p}}},
\]
\[
B_{\alpha} = \frac{1}{\left[ 2D^{p-\gamma_\alpha} \|b_\alpha\|_{L^{\gamma_\alpha}(\Sigma)} \|u^\alpha\|_{W^{1,p}(\Sigma)}^{\gamma_\alpha-p} \right]^{\frac{1}{p}}},
\]
$C_{N,p,\gamma}$ and $D^{r-\gamma}_p$ are universal positive constants.

We fix $0 < \varrho < t < s \leq R < R_0$, $B_z(x_0) = \{ x : |x - x_0| < z \}$, $k \in \mathbb{R}$ and we choose $\eta \in C^\infty_c(B_s(x_0))$ such that $\eta = 1$ on $B_t(x_0)$, $0 \leq \eta \leq 1$ on $B_s(x_0)$ and $|\nabla \eta| \leq \frac{2}{s-t}$ on $B_s(x_0)$. Let us define

$$\varphi = -\eta^p w$$

where $w \in W^{1,p} (\Sigma, \mathbb{R}^n)$ with

$$w^1 = \max(u^1 - k, 0), w^\alpha = 0, \alpha = 2, \ldots, n$$

Let us observe that $\varphi = 0$ $\mathcal{L}^N$-a.e. in $\Omega \setminus (\{ \eta > 0 \} \cap \{ u^1 > k \})$ thus

$$\nabla u + \nabla \varphi = \nabla u$$

(3.1)

$\mathcal{L}^N$-a.e. in $\Omega \setminus (\{ \eta > 0 \} \cap \{ u^1 > k \})$. Let us define

$$A = \begin{pmatrix} p\eta^{-1} \nabla \eta (k - u^1) \\ \nabla u^2 \\ \vdots \\ \nabla u^n \end{pmatrix}$$

(3.2)

since

$$\nabla w = \begin{pmatrix} \nabla u^1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

(3.3)

$\mathcal{L}^N$-a.e. in $\Omega \setminus (\{ \eta > 0 \} \cap \{ u^1 > k \})$ then we deduce that

$$\nabla u + \nabla \varphi = (1 - \eta^p) \nabla u + \eta^p A$$

(3.4)

$\mathcal{L}^N$-a.e. in $\Omega \setminus (\{ \eta > 0 \} \cap \{ u^1 > k \})$. Since $u$ is a local minimizer of the functional (1.1) then we get

$$J(u, \Sigma) \leq J(u + \varphi, \Sigma)$$

(3.5)

then proceeding as in [32] it follows

$$\frac{1}{2} \int_{E^1_{k,s}} |\nabla u^1|^p + b_1(x)(u^1)^{\gamma_1} \, dx$$

$$\leq 2c(p, \gamma) \int_{E^1_{k,s}} (1 - \eta^p) (|\nabla u_1|^p + b_1(x)(u^1)^{\gamma_1}) \, dx$$

$$+ (2^{2p-1} p^p + 2p-1) \int_{E^1_{k,s}} \frac{(u^1 - k)^p}{(s-t)^p} \, dx + 2L_1 \int_{E^1_{k,s}} a_1(x) \, dx$$

$$+ |k|^p R^{-\varepsilon N} |A_{k,s}|^{1 - \frac{p}{N} + \varepsilon}$$

$$+ \int_{E^1_{k,s}} \eta^p |G(x, u + \varphi, A) - G(x, u + \varphi, \nabla u)| \, dx$$

$$+ \int_{E^1_{k,s}} |G(x, u + \varphi, \nabla u) - G(x, u, \nabla u)| \, dx$$

(3.6)
Now let’s estimate the following term

$$
\int_{E_{k,s}^1} |G(x, u + \varphi, \nabla u) - G(x, u, \nabla u)| \, dx
$$

using hypothesis H.1.4 and Hölder’s inequality we obtain

$$
\int_{E_{k,s}^1} |G(x, u + \varphi, \nabla u) - G(x, u, \nabla u)| \, dx \\
\leq \int_{E_{k,s}^1} c(x) |\nabla u|^\delta |\varphi| \, dx \\
\leq \left[ \int_{E_{k,s}^1} |\nabla u|^p \, dx \right]^{\frac{\delta}{q}} \left[ \int_{E_{k,s}^1} (c(x))^{\frac{q}{p(\frac{q}{p} - \frac{q}{q - \delta})}} |\varphi|^{\frac{q}{q - \delta}} \, dx \right]^{\frac{q - \delta}{q}}
$$

(3.7)

and using Hölder’s inequality again it follows that

$$
\int_{E_{k,s}^1} |G(x, u + \varphi, \nabla u) - G(x, u, \nabla u)| \, dx \\
\leq \left[ \int_{E_{k,s}^1} \left| \frac{p - q}{p} \left( \int_{E_{k,s}^1} |\nabla u|^p \, dx \right) \right|^{\frac{\delta}{q}} \left[ \int_{E_{k,s}^1} \left( \frac{c(x)}{p^{\frac{p - q}{q - \delta}}} \right)^{\frac{q}{p^q - (q - \delta)}} \, dx \right]^{\frac{q}{p^q - (q - \delta)}} \left( \int_{E_{k,s}^1} |\varphi|^{\frac{q}{p^q - (q - \delta)}} \, dx \right) \right]^{\frac{\delta}{p}} \left[ \int_{E_{k,s}^1} \left| \frac{p - q}{p} \left( \int_{E_{k,s}^1} |\nabla u|^p \, dx \right) \right|^{\frac{\delta}{q}} \left[ \int_{E_{k,s}^1} \left( \frac{c(x)}{p^{\frac{p - q}{q - \delta}}} \right)^{\frac{q}{p^q - (q - \delta)}} \, dx \right]^{\frac{q}{p^q - (q - \delta)}} \left( \int_{E_{k,s}^1} |\varphi|^{\frac{q}{p^q - (q - \delta)}} \, dx \right) \right]^{\frac{q - \delta}{q}}
$$

(3.8)

where $\Theta = \frac{(p - q)\delta}{pq} + \left( 1 - \frac{q}{\sigma(p^q - (q - \delta))} \right) \left( \frac{p^q - (q - \delta)}{p^{\frac{p}{q} - (q - \delta)}} \frac{\frac{q}{p^q - (q - \delta)}}{q - \delta} \right) = 1 - \left( \frac{\delta}{p} + \frac{1}{p^q} + \frac{1}{\sigma} \right)$; moreover, with some algebraic calculations we obtain $\Theta = 1 - \left( \frac{\delta}{p} + \frac{1}{p^q} + \frac{1}{\sigma} \right)$. Using Sobolev’s inequality
we have

\[
\int_{E_{k,s}^1} |G(x, u + \varphi, \nabla u) - G(x, u, \nabla u)| \, dx
\]

\[
\leq C_{SN} \left| E_{k,s}^1 \right|^\Theta \left( \int_{E_{k,s}^1} |\nabla u|^p \, dx \right)^{\frac{\delta}{p}} \|c\|_{L^p(\Sigma)} \left( \int_{B_s(x_0)} |\nabla \varphi|^p \, dx \right)^{\frac{\beta}{p}}
\]

\[
\leq C_{SN} \left| E_{k,s}^1 \right|^\Theta \left( \int_{E_{k,s}^1} |\nabla u|^p \, dx \right)^{\frac{\delta}{p}} \|c\|_{L^p(\Sigma)} \left( \int_{E_{k,s}^1} \eta^p |\nabla u|^p + 2^p \left( \frac{u^1}{s-t} \right)^p \, dx \right)^{\frac{\beta}{p}}
\]

(3.9)

Now, applying Young’s inequality, it follows

\[
\int_{E_{k,s}^1} |G(x, u + \varphi, \nabla u) - G(x, u, \nabla u)| \, dx
\]

\[
\leq \frac{1}{\varepsilon^{\rho(p-\beta)}} \left( C_{SN} \left| E_{k,s}^1 \right|^\Theta \left( \int \left| \nabla u \right|^p \, dx \right)^{\frac{\delta}{p}} \|c\|_{L^p(\Sigma)} \right)^{\frac{\rho}{p-\beta}}
\]

\[
+ \varepsilon \int_{E_{k,s}^1} \eta^p |\nabla u|^p + 2^p \left( \frac{u^1}{s-t} \right)^p \, dx
\]

\[
\leq \frac{C_{\Sigma}}{\varepsilon^{\frac{\rho}{p}} \varepsilon^{\frac{\rho(p-\beta)}}} \left| E_{k,s}^1 \right|^\frac{\rho(p-\beta)}{p-\beta} + \varepsilon \int_{E_{k,s}^1} \eta^p |\nabla u|^p + 2^p \left( \frac{u^1}{s-t} \right)^p \, dx
\]

where

\[
C_{\Sigma} = \left( C_{SN} \left( \int_{\Sigma} |\nabla u|^p \, dx \right)^{\frac{\delta}{p}} \|c\|_{L^p(\Sigma)} \right)^{\frac{p}{p-\beta}}
\]

(3.11)

Fixed \( \varepsilon = \frac{1}{4} \), using (3.6) and (3.10), we get

\[
\frac{1}{4} \int_{E_{k,s}^1} |\nabla u^1|^p + b_1(x)(u^1)^{\gamma_1} \, dx
\]

\[
\leq 2c(p, \gamma) \int_{E_{k,s}^1} (1 - \eta^p) \left( |\nabla u|^p + b_1(x)(u^1)^{\gamma_1} \right) \, dx
\]

\[
+ (2^{2p-1}p + 2^p) \int_{E_{k,s}^1} \left( \frac{u^1-k}{s-t} \right)^p \, dx + 2L_1 \int_{E_{k,s}^1} a_1(x) \, dx
\]

\[
+ \left| k \right| \frac{p}{p-\beta} R^{-\varepsilon} \, dx
\]

\[
+ 4^{\frac{\rho}{p-\beta}} C_{\Sigma} \left| E_{k,s}^1 \right|^\frac{\rho(p-\beta)}{p-\beta}
\]

\[
+ \int_{E_{k,s}^1} \eta^p |G(x, u + \varphi, A) - G(x, u + \varphi, \nabla u)| \, dx
\]

(3.12)
To get our energy estimate we need to estimate the following integral
\[
\int_{E_{k,s}^1} \eta^p |G(x, u + \varphi, A) - G(x, u + \varphi, \nabla u)| \, dx
\]
from hypothesis H.1.2 we have
\[
\int_{E_{k,s}^1} \eta^p |G(x, u + \varphi, A) - G(x, u + \varphi, \nabla u)| \, dx \\
\leq C \int_{E_{k,s}^1} \eta^p |A|^q \, dx + C \int_{E_{k,s}^1} \eta^p |\nabla u|^q \, dx \\
+ 2C \int_{E_{k,s}^1} \eta^p |u + \varphi|^q \, dx \\
+ 2C \int_{E_{k,s}^1} \eta^p a(x) \, dx
\] (3.13)

Now we have to estimate the various elements of (3.13), let us start with the following term
\[
\int_{E_{k,s}^1} \eta^p |A|^q \, dx
\]

since
\[
|A| \leq p\eta^{-1} |\nabla \eta| (u^1 - k) + n |\nabla u| 
\] (3.14)
on \(E_{k,s}^1\), then
\[
\int_{E_{k,s}^1} \eta^p |A|^q \, dx \\
\leq \int_{E_{k,s}^1} \eta^p |p\eta^{-1} |\nabla \eta| (u^1 - k) + n |\nabla u||^q \, dx \\
\leq 2^{q-1} \int_{E_{k,s}^1} \eta^{p+q} |\nabla \eta|^q (u^1 - k)^q + \eta^n |\nabla u|^q \, dx \\
\] (3.15)

Using Young’s inequality we can estimate the term
\[
\int_{E_{k,s}^1} \eta^{p-q} |\nabla \eta|^q (u^1 - k)^q \, dx
\]

getting
\[
\int_{E_{k,s}^1} \eta^{p-q} |\nabla \eta|^q (u^1 - k)^q \, dx \leq \frac{q}{p} \int_{E_{k,s}^1} |\nabla \eta|^p (u^1 - k)^p \, dx + \int_{E_{k,s}^1} \eta^p dx
\] (3.16)

furthermore, with the Hölder inequality we can estimate the term
\[
\int_{E_{k,s}^1} \eta^p |\nabla u|^q \, dx
Everywhere Hölder continuity

getting

$$\int_{E_{k,s}^1} \eta^p |\nabla u|^q \, dx \leq \left[ \int_{E_{k,s}^1} \eta^p \, dx \right]^{\frac{p-q}{p}} \cdot \left[ \int_{E_{k,s}^1} \eta^p |\nabla u|^p \, dx \right]^{\frac{2}{p}}$$ \hspace{1cm} (3.17)

then, remembering the properties of $\eta$ and using (3.15), (3.16) and (3.17) we have

$$\int_{E_{k,s}^1} \eta^p |A|^q \, dx \leq |E_{k,s}^1|^{\frac{p-q}{p}} \left[ \int_{E_{k,s}^1} |\nabla u|^p \, dx \right]^{\frac{2}{p}}$$ \hspace{1cm} (3.18)

To estimate the following term

$$\int_{E_{k,s}^1} \eta^p |u + \varphi|^q \, dx$$

of (3.50) we can use the properties of $\eta$ and the Hölder inequality getting

$$\int_{E_{k,s}^1} \eta^p |\nabla u|^q \, dx \leq |E_{k,s}^1|^{\frac{p-q}{p}} \left[ \int_{\Sigma} |\nabla u|^p \, dx \right]^{\frac{2}{p}}$$ \hspace{1cm} (3.19)

We now proceed to estimate the following integral

$$\int_{E_{k,s}^1} \eta^p |u + \varphi|^q \, dx$$
of (3.13), using Hölder’s inequality we have

\[
\int_{E_{k,s}^1} \eta^p |u + \varphi|^q \, dx \leq \left( \int_{E_{k,s}^1} \eta^p \, dx \right)^{\frac{p-q}{p}} \left( \int_{E_{k,s}^1} \eta^p |u + \varphi|^p \, dx \right)^{\frac{q}{p}} \tag{3.20}
\]

\[
\leq 2^{q-1} \left( \int_{E_{k,s}^1} \eta^p \, dx \right) |E_{k,s}^1|^{\frac{p}{(p^* - p)p}} \left( \int_{E_{k,s}^1} \eta^p |u + \varphi|^p \, dx \right)^{\frac{q}{p^*}}
\]

by the Sobolev Inequality and the Immersion theorem it follows

\[
\left( \int_{E_{k,s}^1} \eta^p |u + \varphi|^p \, dx \right)^{\frac{1}{p^*}} \leq C \|u\|_{W^{1,p}(\Sigma, \mathbb{R}^n)} + C_{SN} \left( \int_{E_{k,s}^1} |\nabla \varphi|^p \, dx \right)^{\frac{1}{p}} \tag{3.21}
\]

Now, using (3.12), (3.13), (3.18), (3.19), (3.20) and (3.21) we get

\[
\frac{1}{4} \int_{E_{k,s}^1} |\nabla u|^p + b_1(x) (u^1)^{\gamma_1} \, dx \\
\leq 2c(p, \gamma) \int_{E_{k,s}^1} (1 - \eta^p) \left( |\nabla u|^p + b_1(x) (u^1)^{\gamma_1} \right) \, dx \\
+ (2^{p-1} p^p + 2) \int_{E_{k,s}^1} \frac{u^1 - k}{(s-t)^p} \, dx + 2L_1 \int_{E_{k,s}^1} a_1(x) \, dx \\
+ |k|^p R^{-\epsilon N} |A_{k,s}^1|^{1 - \frac{p}{\gamma_1} + \epsilon} + |E_{k,s}^1| \left[ \int_{\Sigma} |\nabla u|^p \, dx \right]^{\frac{q}{p}} \left[ \int_{\Sigma} |\nabla u|^q \, dx \right] + 2^{q-1} n^q |E_{k,s}^1|^{\frac{p}{p^*}} \cdot \left[ \int_{\Sigma} |\nabla u|^p \, dx \right]^{\frac{q}{p}} \\
+ 4 \int_{E_{k,s}^1} \frac{u^1 - k}{(s-t)^p} \, dx + 2C_1, \Sigma |E_{k,s}^1|^{\Theta} + 2C_2, \Sigma \int_{E_{k,s}^1 - E_{k,t}^1} \frac{(u^1 - k)^p}{(s-t)^p} \, dx \\
+ 2^{q-1} p^{q-1} q \int_{E_{k,s}^1 - E_{k,t}^1} \frac{(u^1 - k)^p}{(s-t)^p} \, dx + 2^{q-1} p^{p} |E_{k,s}^1|
\]

(3.22)

Since \( \frac{p}{p^* - 3} > 1 \), from hypothesis H.1.4, estimating \( \int_{E_{k,s}^1} a_1(x) \, dx < \|a\|_{L^s(\Sigma)} |E_{k,s}^1|^{1 - \frac{1}{s_1}} \)

with the Hölder inequality and observing that \( E_{k,t}^1 = A_{k,t}^1 \) and \( E_{k,s}^1 \subset A_{k,s}^1 \) from
Everywhere Hölder continuity

(3.22) we obtain
\[
\frac{1}{4} \int_{A_{k,t}^1} |\nabla u^1|^p + b_1 (x) (u^1)^{\gamma_1} \, dx 
\leq 2c (p, \gamma) \int_{A_{k,a}^1 \setminus A_{k,t}^1} (|\nabla u^1|^p + b_1 (x) (u^1)^{\gamma_1}) \, dx 
+ D_1 \Sigma \int_{A_{k,s}^1} \frac{(u^1 - k)^p}{(s - t)^p} \, dx + (D_2 \Sigma + |k|^p R^{-\varepsilon N}) \, |A_{k,s}^1|^{1 - \frac{p}{N} + \varepsilon} 
\]

By adding the quantity
\[
2c (p, \gamma) \int_{A_{k,t}^1} |\nabla u^1|^p + b_1 (x) (u^1)^{\gamma_1} \, dx 
\]
to both terms of (3.23) and carrying out simple algebraic calculations, we have
\[
\int_{A_{k,t}^1} |\nabla u^1|^p + b_1 (x) (u^1)^{\gamma_1} \, dx 
\leq \frac{8c (p, \gamma)}{1 + 8c (p, \gamma)} \int_{A_{k,s}^1} (|\nabla u^1|^p + b_1 (x) (u^1)^{\gamma_1}) \, dx 
+ B_1 \int_{A_{k,s}^1} \frac{(u^1 - k)^p}{(s - t)^p} \, dx + B_2 \left( 1 + |k|^p R^{-\varepsilon N} \right) |A_{k,s}^1|^{1 - \frac{p}{N} + \varepsilon} 
\]
where \( B_1 = \frac{4D_1 \Sigma}{1 + 8c (p, \gamma)} \) and \( B_2 = \frac{4(D_2 \Sigma + 1)}{1 + 8c (p, \gamma)} \).

Now, using Lemma 3 we get
\[
\int_{A_{k,e}^1} |\nabla u^1|^p \, dx \leq \frac{C_{C,1}}{(R - \varepsilon e)^p} \int_{A_{k,R}^1} (u^1 - k)^p \, dx + C_{C,2} \left( 1 + |k|^p R^{-\varepsilon N} \right) |A_{k,s}^1|^{1 - \frac{p}{N} + \varepsilon} 
\]

(3.24)

Since \(-u\) is a local minimizer of the following integral functional
\[
\tilde{J} (v, \Omega) = \int_{\Omega} \sum_{\alpha=1}^{n} \tilde{f}_\alpha (x, v^\alpha (x), \nabla v^\alpha (x)) + \tilde{G} (x, v (x), \nabla v (x)) \, dx 
\]
where \( \tilde{f}_\alpha (x, v^\alpha (x), \nabla v^\alpha (x)) = f_\alpha (x, -v^\alpha (x), -\nabla v^\alpha (x)) \) and \( \tilde{G} (x, v (x), \nabla v (x)) = G (x, -v (x), -\nabla v (x)) \) then we get
\[
\int_{B_{k,e}^1} |\nabla u^1|^p \, dx \leq \frac{C_{C,1}}{(R - \varepsilon e)^p} \int_{B_{k,R}^1} (k - u^1)^p \, dx + C_{C,2} \left( 1 + |k|^p R^{-\varepsilon N} \right) \, |B_{k,s}^1|^{1 - \frac{p}{N} + \varepsilon} 
\]

(3.25)

Similarly we can proceed for \( u^\alpha \) with \( \alpha = 2, \ldots, n \).
4. Proof of Theorem 1

Let $u \in W^{1,p}(\Omega;\mathbb{R}^n)$ a minimizer of the functional (1.1) then by Theorem 2 it follows that $u^\alpha \in DG(\Omega, p, \lambda, \chi, \varepsilon, R_0, k_0)$ for $\alpha = 1, \ldots, n$ then by Theorem 6 it follows that $u \in L^\infty_{loc}(\Omega;\mathbb{R}^n)$; moreover if $\Sigma$ is a compact subset of $\Omega$ then $u \in L^\infty(\Sigma;\mathbb{R}^n)$ and

$$|u| \leq M = \sqrt[n]{\sum_{\alpha=1}^{n} (M^\alpha)^2} \quad (4.1)$$

where

$$M^\alpha = \sup_\Sigma \{|u^\alpha|\} \quad (4.2)$$

Since H.1.1 and (1.2) hold then proceeding as in [22] we get

$$\int_{A_{k,\varrho}^{\alpha}} |\nabla u^\alpha|^p \, dx \leq \frac{\tilde{C}_{C,1}}{(R-\varrho)^p} \int_{A_{k,R}^{\alpha}} (u^\alpha - k)^p \, dx + \tilde{C}_{C,2} \left| A_{k,s}^{\alpha} \right|^{1-\frac{p}{N}+\varepsilon} \quad (4.3)$$

and

$$\int_{B_{k,\varrho}^{\alpha}} |\nabla u^\alpha|^p \, dx \leq \frac{\tilde{C}_{C,1}}{(R-\varrho)^p} \int_{B_{k,R}^{\alpha}} (k - u^\alpha)^p \, dx + \tilde{C}_{C,2} \left| B_{k,s}^{\alpha} \right|^{1-\frac{p}{N}+\varepsilon} \quad (4.4)$$

for every $\alpha = 1, \ldots, n$. Theorem 1 follows using (4.3), (4.4) and Proposition 7.1, Lemma 7.2 and Theorem 7.6 of [24].

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**References**


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