On Hermite-Hadamard-Like Type Integral Inequalities for Convex Functions via Riemann-Liouville Fractional Integrals

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Abstract

In this paper an identity is given in order to find several Hermite-Hadamard-like type inequalities for functions whose powers of absolute values of third derivatives are s-convex in the second and in the first sense respectively. Several consequences are then formulated.

Mathematics Subject Classifications: 26D15

Keywords: Hermite-Hadamard inequality, s-convex functions, Holder’s inequality

1. Introduction

The convex analysis has an increasing role in many fields such as numerical analysis, convex programming, statistics and approximation theory and other new branches of mathematics. In particular, the famous Hermite-Hadamard inequality has many applications, and was improved, extended and generalized in many directions by many authors, see for example, [10, 9, 1, 13, 12, 17, 18, 16, 8, 11, 5, 2, 3, 14, 4, 19, 20] and the references therein.

We start by recalling below the classical definition for the convex functions and then for s-convex functions in the first sense and in the second sense respectively, ([5], [9],[6],[7]).
Definition 1. A function \( f : I \subset \mathbb{R} \to \mathbb{R} \) is said to be convex on an interval \( I \) if the inequality
\[
 f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)
\]
holds for all \( x, y \in I \) and \( t \in [0,1] \). The function \( f \) is said to be concave on \( I \) if the inequality (1) takes place in reversed direction.

Definition 2. A function \( f : I \subset \mathbb{R} \to \mathbb{R} \) is said to be s-convex in the first sense on \( I \) if the inequality
\[
 f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)
\]
for each \( x, y \in I \), \( t \in (0,1) \), and \( s \) fixed, \( s \in (0,1] \).

Definition 3. A function \( f : \mathbb{R} \to \mathbb{R} \) is said to be s-convex in the second sense if the inequality
\[
 f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)
\]
for each \( x, y \in \mathbb{R} \), \( t \in (0,1) \), and \( s \) fixed, \( s \in (0,1] \).

The well-known Hermite-Hadamard’s inequality for convex functions, see [15] is the following
\[
 f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}.
\]
Moreover, if the function \( f \) is concave then the inequality (2) hold in reversed direction.

We restate also below the next definition arising from the fractional calculus theory, which can be seen in [17].

Definition 4. Let \( f \in L([a,b]) \). Then \( J_{a+}^\alpha f \) and \( J_{b-}^\alpha f \) represent the left-side and the right-side Riemann-Liouville integrals of the order \( \alpha \), being defined by
\[
 J_{a+}^\alpha f = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad (0 \leq a < x),
\]
\[
 J_{b-}^\alpha f = \frac{1}{\Gamma(\alpha)} \int_x^b (t-v)^{\alpha-1} f(t) dt, \quad (0 < x < b),
\]
respectively and the Gamma function is \( \Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt \). It is obvious that \( J_{a+}^0 f = f(x) \) and \( J_{b-}^0 f = f(x) \).

The aim of this paper is to give several Hermite-Hadamard-like type inequalities for functions whose powers of absolute values of third derivatives are s-convex by using Riemann-Liouville fractional integrals. For this goal an identity is presented as a main tool in the demonstrations of these results.
2. Main results

Having as a starting point a result from [2], the aim of this section is to give several Hermite-Hadamard-like type inequalities for functions whose powers of absolute values of third derivatives are s-convex by using Riemann-Liouville fractional integrals.

Lemma 1. Let \( f : I \subset \mathbb{R} \to \mathbb{R} \) be a three times differentiable mapping on the interior \( I^0, a, b \in I^0 \) with \( a < b \). If \( f'' \in L_1[a, b] \), then for any \( \alpha > 1 \) and for all \( x \in I^0 \) the following inequality takes place:

\[
(\alpha + 1)(b - a)f'(x) + (x - a)f'(a) - (x - b)f'(b) - (\alpha + 1)(\alpha + 2)f(x)(b - a) + \\
\quad + \Gamma(\alpha + 3)[\frac{J^\alpha_x f(a)}{(x-a)^{\alpha-1}} + \frac{J^\alpha_x f(b)}{(b-x)^{\alpha-1}}] = \\
= \int_0^1 t(1 - t^{\alpha+1})(x - a)^4 f''(tx + (1 - t)a) - (x - b)^4 f''(tx + (1 - t)b)\,dt.
\]

Proof. We begin by denoting \( I_1 = \int_0^1 t(1 - t^{\alpha+1})(x - a)^4 f''(tx + (1 - t)a)\,dt \) and \( I_2 = \int_0^1 t(1 - t^{\alpha+1})(x - b)^4 f''(tx + (1 - t)b)\,dt \). Now by integrating by parts three times \( I_1 \) and \( I_2 \) we find,

\[
I_1 = -(x - a)^3 \int_0^1 (1 - (\alpha + 2)t^{\alpha+1})f''(tx + (1 - t)a)\,dt = \\
= (x - a)^2[-(\alpha + 1)f'(x) - f'(a) + (\alpha + 1)(\alpha + 2) \int_0^1 t^\alpha f'(tx + (1 - t)a)\,dt] = \\
= (x - a)[(\alpha + 1)f'(x) + f'(a) - (\alpha + 1)(\alpha + 2)f(x) + \alpha(\alpha + 1)(\alpha + 2) \int_0^1 t^{\alpha-1}f(tx + (1 - t)a)\,dt]
\]

and here by using \( u = tx + (1 - t)a \), we will have,

\[
I_1 = (x - a)[(\alpha + 1)f'(x) + f'(a) - (\alpha + 1)(\alpha + 2)f(x) + \frac{\alpha(\alpha + 1)(\alpha + 2)}{(x-a)^{\alpha}} \int_a^x (u-a)^{\alpha-1}f(u)\,du],
\]

and analogues

\[
I_2 = (x - b)[(\alpha + 1)f'(x) + f'(b) - (\alpha + 1)(\alpha + 2)f(x) - \frac{\alpha(\alpha + 1)(\alpha + 2)}{(b-x)^{\alpha}} \int_x^b (b-v)^{\alpha-1}f(v)\,dv],
\]

where \( v = tx + (1 - t)b \). Substracting \( I_2 \) from \( I_1 \), we get,

\[
I_1 - I_2 = (\alpha + 1)f'(x)(b - a) + (x - a)f'(a) - (x - b)f'(b) - (\alpha + 1)(\alpha + 2)(b - a)f(x) + \\
\quad + \Gamma(\alpha + 3)[\frac{J^\alpha_x f(a)}{(x-a)^{\alpha-1}} + \frac{J^\alpha_x f(b)}{(b-x)^{\alpha-1}}], \]

i.e. the desired inequality is obtained.
Theorem 1. Let $f : I^0 \subset \mathbb{R} \to \mathbb{R}$ be a three times differentiable mapping on $I^0, a, b \in I^0$ with $a < b$, $\alpha > 1$. If $f'' \in L[a, b]$, and $|f''|$ is s-convex in the second sense on $[a, b]$, then for all $x \in I^0$ the following inequality is satisfied:

$$
|(b-a)(\alpha+1)f'(x) + (x-a)f'(a) - (x-b)f'(b) - (\alpha+1)(\alpha+2)(b-a)f(x) + 
+ \Gamma(\alpha+3)[\frac{J_{x}^{a}f(a)}{(x-a)^{\alpha-1}} + \frac{J_{x}^{a}f(b)}{(b-x)^{\alpha-1}}]| \leq 
\leq [(x-a)^{4}+(x-b)^{4}]|f'''(x)| \frac{\alpha+1}{(s+2)(s+\alpha+3)} + (x-a)^{4}|f''(a)|(\frac{1}{(s+2)(s+1)} - 
-B(\alpha+3,s+1)) + (x-b)^{4}|f''(b)|(\frac{1}{(s+2)(s+1)} - B(\alpha+3,s+1)).
$$

Proof. It will be used here Lemma 1, the definition of convex functions for $|f''|$ and the properties of the Gamma and Beta functions. We have,

$$
|(b-a)(\alpha+1)f'(x) + (x-a)f'(a) - (x-b)f'(b) - (\alpha+1)(\alpha+2)(b-a)f(x) + 
+ \Gamma(\alpha+3)[\frac{J_{x}^{a}f(a)}{(x-a)^{\alpha-1}} + \frac{J_{x}^{a}f(b)}{(b-x)^{\alpha-1}}]| \leq 
\leq (x-a)^{4}\int_{0}^{1} t(1-t^{\alpha+1})|f'''(tx+(1-t)a)|dt + (x-b)^{4}\int_{0}^{1} t(1-t^{\alpha+1})|f''(tx+(1-t)b)|dt \leq 
\leq (x-a)^{4}\int_{0}^{1} t(1-t^{\alpha+1})[t^{s}|f'''(x)| + (1-t)^{s}|f''(a)|]dt + 
+x-b)^{4}\int_{0}^{1} t(1-t^{\alpha+1})[t^{s}|f'''(x)| + (1-t)^{s}|f''(b)|]dt \leq 
\leq (x-a)^{4}|f'''(x)| \int_{0}^{1} t^{s+1}(1-t^{\alpha+1})dt + |f''(a)| \int_{0}^{1} t(1-t^{\alpha+1})(1-t)^{s}dt + 
+x-b)^{4}|f'''(x)| \int_{0}^{1} t^{s+1}(1-t^{\alpha+1})dt + |f''(b)| \int_{0}^{1} t(1-t^{\alpha+1})(1-t)^{s}dt = 
= [(x-a)^{4}+(x-b)^{4}] \frac{\alpha+1}{(s+2)(s+\alpha+3)}|f'''(x)| + (x-a)^{4}|f''(a)|(\frac{1}{(s+2)(s+1)} - 
-B(\alpha+3,s+1)) + (x-b)^{4}|f''(b)|(\frac{1}{(s+2)(s+1)} - B(\alpha+3,s+1)),
$$

where $B(x, y)$ is the Beta function, $B(x, y) = \int_{0}^{1} t^{x-1}(1-t)^{y-1}dt$, $x > 0$, $y > 0$ and the Gamma function is given by $\Gamma(x) = \int_{0}^{\infty} e^{-t}t^{x-1}dt$, $x > 0$. 

\qed
Corollary 1. Let \( f : I^0 \subset \mathbb{R} \to \mathbb{R} \) be a three times differentiable mapping on \( I^0, a, b \in I^0 \) with \( a < b, \alpha > 1 \). If \( f''' \in L[a,b] \), and \( |f'''| \) is \( s \)-convex in the second sense on \([a,b]\), then the following inequality is holds:

\[
\begin{align*}
|\alpha + 1| f'(\frac{a+b}{2}) + \frac{f'(a) + f'(b)}{2} - (\alpha + 1)(\alpha + 2)f(\frac{a+b}{2}) + \\
+\Gamma(\alpha + 3)\frac{2^{\alpha-1}}{(b-a)^\alpha}[J_1^\alpha f(a) + J_2^\alpha f(b)]
\leq \frac{(b-a)^3}{16} \left( 2 + \frac{\alpha + 1}{(s+\alpha+3)} |f'''(\frac{a+b}{2})| + \frac{1}{(s+2)(s+1)} - B(\alpha + 3, s+1) \right) |f'''(a)| + |f'''(b)|
\end{align*}
\]

**Proof.** We put in previous theorem \( x = \frac{a+b}{2} \). \( \square \)

Theorem 2. Let \( f : I^0 \subset \mathbb{R} \to \mathbb{R} \) be a three times differentiable mapping on \( I^0, a, b \in I^0 \) and \( a < b, \alpha > 1 \). If \( f''' \in L[a,b] \), and \( |f'''|^q \) is \( s \)-convex in the second sense on \([a,b]\) for a fixed \( q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( p > \alpha \) then for all \( x \in I^0 \) the following inequality holds:

\[
\begin{align*}
&|b-a)(\alpha + 1)f'(x) + (x-a)f'(a) - (x-b)f'(b) - (\alpha + 1)(\alpha + 2)(b-a)f(x) + \\
&+\Gamma(\alpha + 3)\frac{J_x^\alpha f(a)}{(x-a)^{a-1}} + \frac{J_x^\alpha f(b)}{(b-x)^{a-1}}| \leq \\
&\leq \frac{1}{(\alpha + 1)^\frac{1}{p}} \frac{B_2^\alpha(\frac{p+1}{a+1}, p+1)}{(s+1)^\frac{1}{q}} \{(x-a)^4|f'''(x)|^q + |f'''(a)|^q + (x-b)^4|f'''(x)|^q + |f'''(b)|^q\}.
\end{align*}
\]

**Proof.** We use again Lemma 1 and Holder’s integral inequality and we obtain,

\[
\begin{align*}
&|b-a)(\alpha + 1)f'(x) + (x-a)f'(a) - (x-b)f'(b) - (\alpha + 1)(\alpha + 2)(b-a)f(x) + \\
&+\Gamma(\alpha + 3)\frac{J_x^\alpha f(a)}{(x-a)^{a-1}} + \frac{J_x^\alpha f(b)}{(b-x)^{a-1}}| \leq \\
&\leq (x-a)^4 \left( \int_0^1 [t(1-t^{a+1})]^p dt \right)^\frac{1}{p} \left( \int_0^1 |f'''(tx + (1-t)a)|^q dt \right)^\frac{1}{q} + \\
&+(x-b)^4 \left( \int_0^1 [t(1-t^{a+1})]^p dt \right)^\frac{1}{p} \left( \int_0^1 |f'''(tx + (1-t)b)|^q dt \right)^\frac{1}{q}.
\end{align*}
\]

The definition from hypothesis, that \( |f'''|^q \) is \( s \)-convex in the second sense on \([a,b], \) applied in this point and calculus of the integral from here will complete the proof,

\[
\begin{align*}
&|b-a)(\alpha + 1)f'(x) + (x-a)f'(a) - (x-b)f'(b) - (\alpha + 1)(\alpha + 2)(b-a)f(x) + \\
&+\Gamma(\alpha + 3)\frac{J_x^\alpha f(a)}{(x-a)^{a-1}} + \frac{J_x^\alpha f(b)}{(b-x)^{a-1}}| \leq
\end{align*}
\]
Corollary 2. Let \( f : I^0 \subset \mathbb{R} \to \mathbb{R} \) be a three times differentiable mapping on \( I^0, a, b \in I^0 \) with \( a < b \), \( \alpha > 1 \). If \( f''' \in L[a, b] \), and \( |f'''|^q \) is \( s \)-convex in the second sense on \([a, b]\) for a fixed \( q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( p > \alpha \) then we get the following:

\[
|(\alpha + 1)f'(\frac{a + b}{2}) + \frac{f'(a) + f'(b)}{2} - (\alpha + 1)(\alpha + 2)f(\frac{a + b}{2})| + \\
\quad + \Gamma(\alpha + 3)\frac{2^{\alpha-1}}{(b-a)^\alpha} [J_{\alpha+1}^a f(a) + J_{\alpha+1}^b f(b)] \leq \\
\leq \frac{(b-a)^3}{16(\alpha + 1)^3} \frac{B_{\frac{p+1}{\alpha+1}}(p+1)}{(s+1)^{\frac{1}{p}}} \left\{ ||f'''(x)|^q + |f''(a)|^q + |f''(b)|^q \right\}.
\]

Proof. Like in Corollary 1, we take in Theorem 2, \( x = \frac{a+b}{2} \). □

Theorem 3. Let \( f : I^0 \subset \mathbb{R} \to \mathbb{R} \) be a three times differentiable mapping on \( I^0, a, b \in I^0 \) with \( a < b \) and \( \alpha > 1 \). If \( f''' \in L[a, b] \), and \( |f'''|^q \) is \( s \)-convex in the second sense on \([a, b]\), \( q > 1 \) then for all \( x \in I^0 \) the following inequality takes place:

\[
|(b-a)(\alpha + 1)f'(x) + (x-a)f'(a) - (x-b)f'(b) - (\alpha + 1)(\alpha + 2)(b-a)f(x) + \\
\quad + \Gamma(\alpha + 3)[J_{x+1}^a f(a) + J_{x+1}^b f(b)]| \leq \\
\leq \left( \frac{1}{2} - \frac{1}{\alpha + 2} \right)^{1 - \frac{1}{q}} \frac{1}{s+1} \left\{ (x-a)^4 \frac{\alpha + 1}{(s+1)(s + \alpha + 3)} |f'''(x)|^q + \frac{1}{(s+2)(s+1)} - \\
\quad - B(\alpha + 3, s + 1)) |f''(a)|^q \right\} + (x-b)^4 \frac{\alpha + 1}{(s+2)(s + \alpha + 3)} + \\
\quad + |f''(b)|^q \frac{1}{(s+2)(s+1)} - B(\alpha + 3, s + 1)) \right\}^{\frac{1}{q}}.
\]

Proof. Using Lemma 1, the power-mean integral inequality for \( q \) and that \( |f'''|^q \) is \( s \)-convex function we have successively the following inequalities,

\[
|(b-a)(\alpha + 1)f'(x) + (x-a)f'(a) - (x-b)f'(b) - (\alpha + 1)(\alpha + 2)(b-a)f(x) + \\
\quad + \Gamma(\alpha + 3)[J_{x}^a f(a) + J_{x}^b f(b)]| \leq \\
\leq \left( \frac{1}{2} - \frac{1}{\alpha + 2} \right)^{1 - \frac{1}{q}} \frac{1}{s+1} \left\{ (x-a)^4 \frac{\alpha + 1}{(s+1)(s + \alpha + 3)} |f'''(x)|^q + \frac{1}{(s+2)(s+1)} - \\
\quad - B(\alpha + 3, s + 1)) |f''(a)|^q \right\} + (x-b)^4 \frac{\alpha + 1}{(s+2)(s + \alpha + 3)} + \\
\quad + |f''(b)|^q \frac{1}{(s+2)(s+1)} - B(\alpha + 3, s + 1)) \right\}^{\frac{1}{q}}.
\]
Proof. Now we put $x = \frac{a+b}{2}$ in Theorem 3.

\[\leq (x-a)^4 \int_0^1 t(1-t^\alpha) |f'''(tx+(1-t)a)|dt + (x-b)^4 \int_0^1 t(1-t^\alpha) |f'''(tx+(1-t)b)|dt \leq\]
\[\leq (x-a)^4 \left( \int_0^1 t(1-t^\alpha) dt \right)^{1-\frac{1}{q}} \left( \int_0^1 t(1-t^\alpha) |f'''(tx+(1-t)a)|^q dt \right)^{\frac{1}{q}} +\]
\[+(x-b)^4 \left( \int_0^1 t(1-t^\alpha) dt \right)^{1-\frac{1}{q}} \left( \int_0^1 t(1-t^\alpha) |f'''(tx+(1-t)b)|^q dt \right)^{\frac{1}{q}} \leq\]
\[\leq \left( \frac{1}{2} - \frac{1}{\alpha + 2} \right)^{1-\frac{1}{q}} \{ (x-a)^4 \left[ \frac{1}{s+2} - \frac{1}{s+\alpha+3} \right] |f'''(x)|^q + (B(2,s+1) - B(\alpha + 3, s+1)\right.\]
\[|f'''(a)|^q \left[ \frac{1}{s+2} - \frac{1}{s+\alpha+3} \right] |f'''(x)|^q + (B(2,s+1) - B(\alpha + 3, s+1) |f'''(b)|^q \}^{\frac{1}{q}}.\]

Now by taking into account that $B(2,s+1) = \frac{1}{(s+2)(s+1)}$ we find the inequality from conclusion.

\[\square\]

Corollary 3. Let $f : I^0 \subset \mathbb{R} \rightarrow \mathbb{R}$ be a three times differentiable mapping on $I^0, a, b \in I^0$ with $a < b$, $\alpha > 1$. If $f''' \in L[a,b]$, and $|f'''|^q$ is $s$-convex in the second sense on $[a,b]$, $q > 1$ then the next inequality holds:

\[| (\alpha + 1) f'(\frac{a+b}{2}) + \frac{f'(a) + f'(b)}{2} - (\alpha + 1)(\alpha + 2) f(\frac{a+b}{2}) | +\]
\[+ \Gamma(\alpha + 3) \left| \frac{2^{\alpha - 1}}{(b-a)^\alpha} \left[ J^{\alpha}_{a+b} f(a) + J^{\alpha}_{a+b} f(b) \right] \right| \leq\]
\[\leq \frac{(b-a)^3}{16} \left( \frac{1}{2} - \frac{1}{\alpha + 2} \right)^{1-\frac{1}{q}} \left\{ \frac{\alpha + 1}{(s+2)(s+\alpha+3)} |f'''(a+b/2)|^q + \frac{1}{(s+2)(s+1)} - \right.\]
\[-B(\alpha + 3, s+1)) |f'''(a)|^q \right| \left| \frac{\alpha + 1}{(s+2)(s+\alpha+3)} +\right.\]
\[+ |f'''(b)|^q \left| \frac{1}{(s+2)(s+1)} - B(\alpha + 3, s+1) \right| \left.\right\}^{\frac{1}{q}}.\]

Proof. Now we put $x = \frac{a+b}{2}$ in Theorem 3.
Theorem 4. Suppose that \( f : I^0 \subset \mathbb{R} \rightarrow \mathbb{R} \) is a three times differentiable mapping on \( I^0 \), \( a, b \in I^0 \) with \( a < b \) and \( \alpha > 0 \). If \( f''' \in L[a, b] \), and \( |f'''|^q \) is s-convex in the first sense on \([a, b] \), \( q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( p > \alpha \), then we have the following inequality:

\[
|(b-a)(\alpha+1)f'(x) + (x-a)f'(a) - (x-b)f'(b) - (\alpha + 1)(\alpha + 2)(b-a)f(x) + \Gamma(\alpha + 3)[\frac{J_x^\alpha f(a)}{(x-a)^{\alpha - 1}} + \frac{J_x^\alpha f(b)}{(b-x)^{\alpha - 1}}]| \leq \frac{B^{\frac{1}{2}}(p+1,p+1)}{(\alpha + 1)^{\frac{1}{2}}(s+1)^{\frac{1}{2}}} \{(x-a)^4|f'''(x)|^q + s|f'''(a)|^q \} + \frac{B^{\frac{1}{2}}(p+1,p+1)}{(\alpha + 1)^{\frac{1}{2}}(s+1)^{\frac{1}{2}}} \{(x-b)^4|f'''(x)|^q + s|f'''(b)|^q \}.
\]

Proof. The proof will follow the same reason as in Theorem 3, but here it will be applied the definition of s-convexity in the first sense on \([a, b] \) for \( |f'''|^q \).

Corollary 4. Under conditions of previous theorem, if we take \( x = \frac{a+b}{2} \) the following inequality holds:

\[
|\Gamma(\alpha + 3)\frac{2^{\alpha - 1}}{(b-a)^{\alpha}}[J_x^\alpha f(a) + J_x^\alpha f(b)]| \leq \frac{(b-a)^3}{16} \frac{B^{\frac{1}{2}}(p+1,p+1)}{(\alpha + 1)^{\frac{1}{2}}(s+1)^{\frac{1}{2}}} \{(x-a)^4|f'''(a)|^q + s|f'''(a)|^q \} + \frac{(b-a)^3}{16} \frac{B^{\frac{1}{2}}(p+1,p+1)}{(\alpha + 1)^{\frac{1}{2}}(s+1)^{\frac{1}{2}}} \{(x-b)^4|f'''(b)|^q + s|f'''(b)|^q \}.
\]

Theorem 5. Suppose that \( f : I^0 \subset \mathbb{R} \rightarrow \mathbb{R} \) is a three times differentiable mapping on \( I^0 \), \( a, b \in I^0 \) with \( a < b \) and \( \alpha > 0 \). If \( f''' \in L[a, b] \), and \( |f'''|^q \) is s-convex in the first sense on \([a, b] \), \( q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \) then we have the following inequality:

\[
|(b-a)(\alpha+1)f'(x) + (x-a)f'(a) - (x-b)f'(b) - (\alpha + 1)(\alpha + 2)(b-a)f(x) + \Gamma(\alpha + 3)[\frac{J_x^\alpha f(a)}{(x-a)^{\alpha - 1}} + \frac{J_x^\alpha f(b)}{(b-x)^{\alpha - 1}}]| \leq \left( \frac{1}{2} - \frac{1}{\alpha + 2} \right)^{\frac{1}{2}} \{(x-a)^4|f'''(x)|^q + \frac{\alpha + 1}{(s+2)(s+\alpha + 3)} + s\left( \frac{1}{2(s+2)} \right) - \frac{1}{(\alpha + 3)(s+\alpha + 3)} \} \{|f'''(a)|^q \}^{\frac{1}{2}} + \{(x-b)^4|f'''(x)|^q + \frac{\alpha + 1}{(s+2)(s+\alpha + 3)} + s\left( \frac{1}{2(s+2)} \right) - \frac{1}{(\alpha + 3)(s+\alpha + 3)} \} \{|f'''(b)|^q \}^{\frac{1}{2}}.
\]
Proof. In view of s-convexity in the first sense of $|f''|^q$ on $[a, b]$ and the power-mean integral inequality, the first inequality from the proof of Theorem 1 becomes,

$$|(b-a)(\alpha + 1)f'(x) + (x-a)f'(a) - (x-b)f'(b) - (\alpha + 1)(\alpha + 2)(b-a)f(x) + \Gamma(\alpha + 3)\left[ J_x^a f(a) + J_x^b f(b) \right] | \leq \left( \frac{1}{2} - \frac{1}{\alpha + 2} \right) \frac{1}{2} \left\{ (x-a)^4 \left( \int_0^1 t(1-t^{\alpha + 1})[f''(x)]^q + (1-t^s)[f''(a)]^q dt \right)^{\frac{1}{q}} + (x-b)^4 \left( \int_0^1 t(1-t^{\alpha + 1})[f''(x)]^q + (1-t^s)[f''(b)]^q dt \right)^{\frac{1}{q}} \right\} \frac{1}{4},$$

which by calculus leads to desired inequality. \hfill \Box

Corollary 5. Under conditions of previous theorem, if we take $x = \frac{a+b}{2}$ we have,

$$|(\alpha + 1)f'\left( \frac{a + b}{2} \right) + f'(a) + f'(b) - (\alpha + 1)(\alpha + 2)f\left( \frac{a + b}{2} \right) + \Gamma(\alpha + 3)\left[ J_x^{\frac{a+b}{2}} f(a) + J_x^{\frac{a+b}{2}} f(b) \right] | \leq \frac{(b-a)^3}{16} \left( \frac{1}{2} - \frac{1}{\alpha + 2} \right) \frac{1}{2} \left\{ \left| f''\left( \frac{a + b}{2} \right) \right|^q \frac{\alpha + 1}{(s+2)(s+\alpha + 3)} + s(\frac{1}{2(s+2)} - \frac{1}{(\alpha + 3)(s+\alpha + 3)})\left| f''(a) \right|^q \frac{1}{4} + \left| f''\left( \frac{a + b}{2} \right) \right|^q \frac{\alpha + 1}{(s+2)(s+\alpha + 3)} + s(\frac{1}{2(s+2)} - \frac{1}{(\alpha + 3)(s+\alpha + 3)})\left| f''(b) \right|^q \frac{1}{4} \right\}.$$

References


Received: July 23, 2022; Published: August 20, 2022