Sub-Linear Estimate of Large Velocities
in a Cosmological Setting

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Abstract

Authors in [8] establish global existence and uniqueness of a classical solution to the Vlasov-Poisson system (VPS) in cosmological setting. In this article, we establish asymptotic behavior growth bounds for the large velocity in the gravitational case.

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1 Introduction

The best describing temporal evolution using VPS of a set of non-colliding particles when interactions are attraction between them as:

\[ \partial_t f(t, x, v) + v \cdot \nabla_x f(t, x, v) - \nabla_x P(t, x, \nabla_v f(t, x, v) = 0, \]  
\[ \Delta P(t, x) = 4\pi p(t, x), \]  
\[ p(t, x) = \int_{\mathbb{R}^3} f(t, x, v) \, dv. \]  

\[ f(t, x, 0) = f_0(x, v), \] is the particle distribution function. The attraction
potential and the spatial mass density are $\mathcal{P}(t, \chi)$ and $p(t, \chi)$, respectively. Most researchers used the system (1.1)-(1.3) to express the particles as stars in the galaxy, while here it was used to express galaxies in the universe, and it represents the time evolution of the universe, where the conditions

$$\lim_{\chi \to \infty} \mathcal{P}(t, \chi) = 0, f(t, \chi, v) = 0 \text{ for } |\chi| > R_0 \text{ or } |v| > P_0; (R_0 > 0, P_0 > 0) \quad (1.4)$$

are not convenient because in a restricted area of the universe the galaxies are not concentrated. In a cosmological setting, the authors in [4,8] structured a straightforward background solution and then global existence and classical solutions were obtained.

The following modified version of VPS (see[8] for more details):

$$\partial_t g + \frac{1}{a(t)} \mathbf{v} \cdot \partial_x g - \frac{1}{a(t)} \left( \partial_3 W + \dot{a} \mathbf{v} \right) \cdot \partial_v = 2aH'(a^2 v^2) \cdot \mathbf{v} \cdot \partial_3 W \quad (1.5)$$

$$\Delta W = 4\pi a^2 \sigma, \quad (1.6)$$

$$\sigma(t, \chi) = \int_{R^3} g(t, x, v) dv. \quad (1.7)$$

Where $H \in C^1_c(R)$ satisfying $\int_{R^3} H(|v|^2) dv = 1$, and $a(t) \in C^2(R^+)$, a background solution from above is defined as

$$f_0(t, \chi, v) = H \left( a^2(t) \left| v - \frac{\dot{a}(t)}{a(t)} \chi \right|^2 \right), \quad \rho_0(t, \chi) = \int_{R^3} f_0(t, \chi, v) dv = a^{-3}(t),$$

$$U_0(t, \chi) = \frac{2\pi}{3} a^{-3}(t)|\chi|^2.$$  Therefore, the solution of the system (1.1)-(1.3) is: $f = f_0 + g, \rho = \rho_0 + \sigma, U = U_0 + W$. For $\mathbf{z} = (\chi, v) \in R^6$ we may define

$$\mathcal{P}(t) = \sup \{|v| : \mathbf{z} \in \text{supp } g(t), 0 \leq s \leq t \},$$

which represents the velocity for high-speed particle in the cloud. The idea of this article is to utilize the discussion in [9] at cosmological setting with the following modification.

**Theorem 1.1** Let $t \geq 0$, then

$$\mathcal{P}(t) \leq \mathcal{E}(1 + t)^\frac{2}{3} \ln t^\frac{2}{3}(2 + t).$$

When $A$ is a set of data, which is chosen as

$$A = \left\{ g_0 \in C^\infty_{[0, \infty]}(Y) \big| \int_Y g_0 = 0, g_0(x, v) + H(v^2) \geq 0 \text{ for all } x, v \in R^3 \right\},$$
where \( C_{n,C}^n(S) \) will be defined later.

Note that the VPS has many applications in astronomy and astrophysics [5,7]. In the status of an isolated, estimates of \( P(t) \) (linear bound) were discussed recently in [6], while Sub-linear estimates were obtained in [3]. The initial and boundary value problems were discussed in [1, 2]. In an accelerating cosmological setting was studied in [4]. The main idea of this paper is to obtain the asymptotic growth bound of estimate in cosmological setting, which produces to modified estimate that will follows the framework of [8].

2 Notation and Preliminary Results

In the following of this article, the function spaces are used with regard to unit cube \( Q = [0, 1]^3 \) also \( Y = Q \times \mathbb{R}^3 \), we symbolize:

\[
\mathcal{O}(Q) := \{ h: \mathbb{R}^3 \to \mathbb{R} | h(\chi + \alpha) = h(\chi), \chi \in \mathbb{R}^3, \alpha \in \mathbb{Z}^3 \},
\]

\[
\mathcal{O}(Y) := \{ h: \mathbb{R}^6 \to \mathbb{R} | h(\chi + \alpha, \nu) = h(\chi, \nu), \chi, \nu \in \mathbb{R}^3, \alpha \in \mathbb{Z}^3 \}.
\]

The periodic function on \( Y \) and \( Q \), which have n-th order derivative

\[
C_{n,C}^n(Y) := \{ h \in C_{n}^n(Y) | \exists u \geq 0 \ s.t. \ h(\chi, \nu) = 0 \ for \ |\nu| \geq u \}.
\]

The small perturbations \( \bar{g} \) are considered in the following only where will be chosen from the permissible set \( \mathcal{A} \). We describe \( X(\bar{s},t,\chi,\nu) \) and \( V(\bar{s},t,\chi,\nu) \) be a solution

\[
\dot{X}(\bar{s}) = \frac{1}{a(\bar{s})} V(\bar{s}), \dot{V}(\bar{s}) = \frac{1}{a(\bar{s})} (\nabla \dot{W}(\bar{s},X(\bar{s}))) + \dot{a}(\bar{s})V(\bar{s}),
\]

then

\[
\bar{g}(t,\chi,\nu) = \bar{g}_0((X,V)(0,t,\chi,\nu)) + \dot{H}(V^2(0,t,\chi,\nu)) - H(\alpha^2(t),\nu^2),(\chi,\nu) \in \mathbb{R}^3, 0 \leq \bar{s} \leq t.
\]

Let \( f = f_0 + \bar{g} \) where \( \bar{g} \) is a solution to (1.5)-(1.7) on the maximal existence interval \([0, +\infty)\). In this section, we recall the important lemma which is a crucial in our proofs (see[4] for its proof).

**Lemma 2.1** The kinetic energy knows \( E_{kin}(t) := \int_0^t |u|^2 f(t,z)dz, t \in [0, \infty)\):

(i) \( E_{kin}(t) \leq \beta a^{-3}(t) \),

(ii) \( \|\sigma(\nu)\|_3 \leq \beta a^{-2}(t) \),

\( \beta \) constant does not rely on \( t \).

For simplicity, we write \( f \lesssim g \) instead of \( f \leq \beta g \), where \( \beta > 0 \) is an arbitrary constant that is not fixed from line to line. Moreover, we define

\[
L(t, \delta) = \sup \int_{t-\delta}^t |\partial_x W(\bar{s},X(\bar{s}))|ds, (\chi,\nu) \in \text{supp } \bar{g}(t),
\]

\[
\Delta(t, \bar{L}) = \sup \{ \delta \in (0, t) : L(t, \delta) \leq \bar{L} \}.
\]
3 A Least Estimate

The equation \( \frac{d}{ds} \left( a(s) V(s) \right) = -\partial_x W(s, X(s)) \) (3.1) with Lemma 2.1, the following estimate is verified

\[ \| \partial_x W(s) \|_{\infty} \leq c_1 a(s) P^\frac{4}{3}(s) \] (3.2)

So \( |a(s_1) V(s_1) - a(s_2) V(s_2)| \leq c_1 a(t) P^\frac{4}{3}(t)|s_1 - s_2|, \quad 0 \leq s_1, s_2 \leq t \) (3.3)

Assume \((X^*(s), V^*(s))\) represent a fix characteristic, then it follows from the proof of Theorem 6.1 in [8] that

\( |a(t)V^*(t) - a(t - 6) V^*(t - 6)| \leq \int_{t - \delta}^t |\partial_x W(s, x^*(s))| ds \leq \int_{t - \delta}^t a^{-1}(s) ds + C \int_{t - \delta}^t a^2(s) \int_{x^*(s) + \tilde{\delta} R^3} |\mathbf{g}(s, x, v)| dv \frac{dx}{|x^*(s) - x|^2} ds, \) (3.4)

Where \( \tilde{\delta} = \left[ -\frac{1}{2}, \frac{1}{2} \right]^3 \), the main tool in computing the above integral is dividing the interval of integrating for the second expression to the

\[
D_1 = \{(s, x, v) | \Lambda(s, v) \leq 200p \}, \quad D_2 = \{(s, x, v) | \Lambda(s, v) > 200p \wedge |X^*(s) - x| \leq \mu(s, v) \}, \quad D_3 = \{(s, x, v) | \Lambda(s, v) > 200p \wedge |X^*(s) - x| > \mu(s, v) \}.
\]

Where \( \Lambda(s, v) = \min \{|v|, |V^*(s) - v|\} \) and \( p > 0 \) is a parameter to be selected later, and here we define for \( R > 0 \)

\[ \mu(s, v) = \frac{R}{\left(1 + |v|^2\right)|V^*(s) - v|}. \] (3.5)

Here, we select the expression \( (1 + |v|^2) \) to ensure there are no singularity. Assume \( L_i \) represent the contribution of \( D_i \) to (3.4). Now we obtain the bound on the contribution of \( D_1 \) as follows

\[ L_1 \leq \int_{t - \delta}^t a^2(s) \int_{X^*(s) + \tilde{\delta} R^3} |\mathbf{g}(s, x, v)||dv| \frac{dx}{|x^*(s) - x|^2} ds \leq \tilde{s} a(t)p^\frac{4}{3}. \] (3.6)

The contribution from \( D_2 \) is:

\[ L_2 \leq \int_{t - \delta}^t a^2(s) \int_{R^3} \frac{R}{(1 + |v|^2)|V^*(s) - v|} dv ds \leq R \int_{t - \delta}^t a^2(s) \left(1 + \ln \frac{1 + p^2(s)}{1 + |V^*(s)|^2}\right) ds.
\]

The transform of \( z' = Z(t, s, z) \), are required to find the contribution of \( J_i \), as \( \mathbf{g} \) is
not stable, the function $f = f_0 + g$ is shifted it and satisfies the equation

$$
p_t f + \frac{1}{a(t)} v \cdot \partial_x f - \frac{1}{a(t)} \left( \partial_x W + \dot{a}(t)v \right) \cdot \partial_y f = 0 \quad (3.7)
$$

Which is stable along any characteristic. Now we obtain the estimate

$$
\mathbb{L}_3 \leq \int_{t-6}^t a^2(s) \int_{X(s) + \bar{Q}} \int_{R^3} |f(s, x, v)| \int_{\mathcal{S}_3} (s, z) dv \frac{ds}{|X^*(s) - x|} \ ds
$$

where $\mathbb{L}_3$ is the integral through $f$ that should appreciated. Assume $\bar{a} \leq \min \left( \frac{1}{2\bar{a}(0)}, \frac{1}{4P(t)} \right)$. Since $\bar{a}$ is positive and decreasing, and using Lemma 2.1

$$
\mathbb{L}_3 \int_{t-6}^t a^2(s) \int_{X(s) + \bar{Q}, R^3} \int_{(X^*(s) + 2\bar{Q}) \times R^3} |f(t, z)| \int_{\mathcal{S}_3} (s, z) dv \frac{dx}{|X^*(s) - X(s)|} \ ds \quad (3.9)
$$

We notice that $Z(s, t, X^*(s) + \bar{Q}, R^3) \cap \text{supp} f(t) \subset (X^*(s) + 2\bar{Q}) \times R^3 =: \mathcal{S}(t)$,

if we let $P(t) \delta \leq \frac{1}{4} t$, it leads to $\delta \leq \min \left( \frac{1}{2\bar{a}(0)}, \frac{1}{4P(t)} \right)$.

Define $I(z) =: \{ s \in [t - \delta, t] \mid (s, Z(s, t, z)) \in \mathcal{S}_3 \}$, we obtain

$$
\mathbb{L}_3 \leq \int_{I(t)} f(t, z) \int_{\mathbb{S}(t)} a^2(s) \int_{X^*(s)} \frac{1}{|X^*(s) - X(s)|^2} \ ds \ dz. \quad (3.10)
$$

The inner time integral on $I(z)$ is very important to estimate firstly, so we have for $\mathcal{S}_1, \mathcal{S}_2 \in [t - \delta, t]$

$$
|V(\mathcal{S}_1) - V(\mathcal{S}_2)| \leq \frac{1}{a(\mathcal{S}_1)} |a(\mathcal{S}_1)V(\mathcal{S}_1) - a(\mathcal{S}_2)V(\mathcal{S}_2)| + \frac{|a(\mathcal{S}_1) - a(\mathcal{S}_2)|}{a(\mathcal{S}_1)} |V(\mathcal{S}_2)|
$$

$$
\leq C_1 \frac{\bar{a}(t)}{a(\mathcal{S}_1)} P(t) \frac{4}{3} \delta + \frac{\bar{a}(0)}{a(\mathcal{S}_1)} P(t) \delta \leq C_2 \delta P(t) \frac{4}{3}, \text{ for } \delta \leq \Lambda \left( \frac{1}{2\bar{a}(0)}, \frac{1}{4P(t)}, \frac{p}{5C_2 P(t)^{3/5}} \right),
$$

for $P(t) \geq 1$ and $\delta$ is not too large, $\delta \leq \Lambda \left( \frac{1}{2\bar{a}(0)}, \frac{1}{4P(t)}, \frac{p}{5C_2 P(t)^{3/5}} \right)$.

The estimate $|V(\mathcal{S}_1) - V(\mathcal{S}_2)| \leq \frac{p}{5}$, holds for any characteristic which hit $\text{supp } g$. Let $z \in \mathcal{S}(t)$, then for any $s \in [t - \delta, t]$ and $u \in I(z)$ the following estimate holds

$$
\frac{4}{5} |V(u, t, z)| \leq |V(s, t, z)| \leq \frac{6}{5} |V(u, t, z)| \quad (3.11)
$$

$$
\frac{3}{5} |V(u, t, z) - V^*(u)| \leq |V(s, t, z) - V^*(s)| \leq \frac{7}{5} |V(u, t, z) - V^*(u)| \quad (3.12)
$$

Now the following lemma shows that the inner time integral of (3.10).
Lemma 3.1 Let \( f(s, z) \neq 0 \) and \( \delta \in [0, t] \) verifying \( \delta \leq \Lambda \left( t, \frac{|v^* - v|}{10} \right) \), \( (3.13) \).

Assume \( \exists \xi > 0 \), such that for any \( s \in [t - \delta, t] \), it yields \( \mu(s, V(s)) \geq \xi \mu(t, v) \), \( (3.14) \)

then \( \int_{t-\delta}^{t} a^2(t) \frac{1}{2} \frac{1}{a^2(t)} \frac{1}{|X^*(s) - X(s)|^2} ds \leq \frac{15}{2} \frac{a^3(t)}{\alpha |v^* - v| \mu(t, v)} \), \( (3.15) \).

Proof. Let \( |\psi(s)| = |X^*(s) - X(s)| \), choose a point \( s_0 \in [t - \delta, t] \), where \( |\psi(s_0)| = \min \{|\psi(s)|, |t - \delta| \leq s \leq t\} \). Define \( \psi(s) = \psi(s_0) + (s - s_0) \psi(s_0) \), then for all \( s \in [t - \delta, t] \), \( (s - s_0) \psi(s_0) \psi(s_0) \geq 0 \) and \( |\psi(s)| \geq |\psi(s_0)||s - s_0| \). On the other hand,

\[
\psi(s) = \psi(s_0) + \frac{2}{a^2(s)} |V^*(s) - V(s)| + \frac{1}{a^2(s)} |\partial_x W(s, X^*(s)) - \partial_x W(s, X(s))| 
\leq 2C_2 \frac{P(s)\psi(s)}{a(s)}
\]

by (3.12) and definition of \( S_3 \), so

\[
|\psi(s) - \bar{\psi}(s)| \leq \frac{P}{5} \frac{1}{a(t - \delta)} |s - s_0| \leq \frac{2}{3} |\psi(s_0)| |s - s_0|, s \in [t - \delta, t].
\]

Since

\[
|\psi(s_0)| = \frac{1}{a(s_0)} |V^*(s_0) - V(s_0)| \geq \frac{4}{5} \frac{1}{a(s_0)} |v^* - v|, \quad (3.16)
\]

Consequently \( |\psi(s)| \geq \frac{4}{15} \frac{1}{a(t)} |v^* - v| |s - s_0| \).

Using (3.14) we find indeed the inequality

\[
\frac{1}{a^2(s)} |X^*(s) - X(s)|^2 \leq \min \left( \frac{1}{|v^* - v|}, \frac{1}{a^2(t)} \right) \left( |\psi(s)| \right) = K( |\psi(s)|).
\]

Since \( K \) is non-increasing and (3.16), we find (3.15) represents upper bound.

If \( (s', Z(s', t, z)) \in S_3 \) and some \( s' \in [t - \delta, t] \), so \( |V(s')|, |V^*(s') - V(s')| \geq 200p \). But then

\[
\delta \leq \Lambda(t, P) \leq \Lambda(t, \frac{|v^* - v|}{200}) \leq \Lambda \left( t, \frac{99}{2} |v^* - v| \right)
\]

therefore (3.13) verified. In similar we find \( \delta \leq \Lambda \left( t, \frac{|v|}{199} \right), \delta \leq \min \left\{ \Lambda \left( t, \frac{|v|}{199} \right), \Lambda \left( t, \frac{99}{2} |v^* - v| \right) \right\} \),

we infer from above \( |V(s)| \leq |v| \) and \( |V^*(s) - V(s)| \leq |v^* - v| \) and (3.14) is satisfied. Using Lemma 3.1 in (3.10) and Lemma 2.1 yields.
\[
\mathcal{L}_{31} \leq R^{-1}a^3(t)\int_{(X'(t) + 2\tilde{q}) \times R^3} (1 + |v|^2 f(t, z) dz \leq R^{-1}
\]  \hspace{1cm} (3.17)

Collecting \( \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3 \) and (3.4) we obtain:
\[
|a(t)V'(t) - a(t) - 6)V'(t - 6)| \leq 6 + 6a(t)P^3 + R^{-1} + R \int_{t-\delta}^{t} a^2(t)(1 + \ln \frac{1 + P^2(t)}{1 + |V'(t)|^2} d\tau.
\]  \hspace{1cm} (3.18)

After optimizing \( R \), we obtain the lower bound on \( \Lambda \).

We give a corollary, that determine the upper limit for \( I(t, \tau) \)[see \([8,11]\)]

**Corollary (3.2):** For any \( t \geq \tau \geq 0 \), it gets:

(i) \( P(t) \leq (1 + t)^2 \).

(ii) \( k(t, \tau) \leq \max(\tau^2 \ln^2(2 + t), \tau \ln(2 + t)) \).

(iii) \( \Lambda(t, l) \geq \min(t, t^2 \ln^{-1}(2 + t), t \ln(2 + t)) \).

We can call the lower bound of the third inequality of Corollary 3.2 as \( \Lambda(t, l) \).

**4 Proof of Theorem 1.1**

The sets \( D_1, D_2 \) and \( D_3 \) are defined in the following as in the prior section, just \( \mu(s, v) \) is changed into \( \mu(s, v) = \frac{R}{(1 + |v|^2) V'(s) - v|m(s, v)|} \).

From the lower bound \( \Lambda^- \) in (Corollary 3.2), \( m(s, v) \) is determine as follows:
\[
m(s, v) = \min\left\{ \frac{s}{10}, \Lambda_-(\frac{|v|}{10}), \Lambda_-\left(\frac{|v' - v|}{10}\right) \right\}.
\]  \hspace{1cm} (4.1)

Let \( t > 0, \; \delta > 0, \; (x, v) \) has been chosen
\[
|v'| = P(t) \geq 200,
\]  \hspace{1cm} (4.2)

Now \( \mathcal{L}_{31} \) will be estimated during (Lemma 3.1) with the definition of \( m \)
\[
m(s, V(s)) \leq \Lambda\left(s, \frac{|V'(s) - V(s)|}{10}\right).
\]  \hspace{1cm} (3.13)

Therefore (3.13) verified when \( t = s, \; \delta = m(s, V(s)) \) and definition of \( m \). Also for any \( u \in (s - m(s, V(s)), s) \)
\[
\mu(s, V(u)) \geq \mu(s, V(s)).
\]  \hspace{1cm} (3.14)

From \( m(s, V(s)) \leq \Lambda(s, \frac{|V'(s)|}{10}) \) with definition of \( \mu \) implies \( |V(s)| \lesssim |V(u)| \).

**Lemma 3.1** and the inclusion \( s_3 \subset s_2 \) yield
\[
\int_{s-m(s, V(s))}^{s} a^2(u) du \leq \int_{s-m(s, V(s))}^{s} a^2(u) \frac{1 + |V(u)|^2}{R} du, \; s \in (0, t).
\]  \hspace{1cm} (4.3)

The following lemma that proved in \([8]\) can be used such as.
Lemma 4.1: Assume \( t > 0 \), \( a, b \in L^1((0, t), R_+), d \in \epsilon ((0, t), R_+) \) and \( \sigma \in (0, t) \) then \( d \leq \sigma \) and \( \int_{\sigma = d(\sigma)}^\sigma a(u)du \leq \int_{\sigma = d(\sigma)}^\sigma b(u)du \).

If suppose \( a(u) = 0 \) when \( d(u) = 0 \). Therefore \(\forall \sigma \in (0, t) \) with \( 6 \epsilon [d(\sigma), \sigma] \),
\[
\int_{\sigma = d(\sigma)}^\sigma a(u)du \leq 2 \int_{\sigma = d(\sigma)}^\sigma b(u)du.
\]

By applying Lemma 4.1 and Lemma 2.3, we obtain from (4.3)
\[
\int f \int_{R^3}^{s = m(s,v(s))} \frac{a^3(s)(1 + |V(s)|^2)}{R} d\sigma dz \leq R^{-1} \eta
\]

On \( D_2 \) depending on the formula of \( m \) several cases appear where
\[
\mathcal{L}_2 \leq \int \int \frac{a^2(s)R}{(1 + |v|^2)|V^*(s) - v|m(s, v)} dud\sigma.
\]

In the first situation, if \( m(s, v) \) has the formula \( \Lambda_\sigma \), \( \frac{|V^*(\sigma) - v|}{10} \) then \( |V^*(\sigma) - v| \leq |v| \) following \( |v| \geq |V^*(\sigma)| \). Whereas \( 100 \) \( m(s, v) = |V^*(\sigma) - v|^2ln^{-1}(2 + \sigma) \) give us \( |V^*(\sigma) - v| \leq ln^7(2 + t) \). We have in both cases
\[
\frac{ln(2 + \sigma)}{ln^2(2 + \sigma)} d\sigma \leq \frac{ln(2 + \sigma)}{ln^{-1}(2 + \sigma)}.
\]

And
\[
\frac{ln(2 + \sigma)}{|V^*(\sigma)|^2} d\sigma \leq \frac{ln^7(2 + \sigma)}{|V^*(\sigma)|^2}.
\]

In the second situation, if \( m(s, v) = \Lambda_\sigma \), \( \frac{|v|}{10} \) then \( |v| \leq |V^*(\sigma) - v| \) following \( |V^*(\sigma)| \leq |V^*(\sigma) - v| \). In both cases \( 100 \) \( m(s, v) = |v|^2ln^{-1}(2 + \sigma) \) and
\[
\int \frac{ln(2 + \sigma)}{(1 + |v|^2)|V^*(\sigma) - v||v|^2} d\sigma \leq \frac{ln(2 + \sigma)}{|V^*(\sigma)|^2}.
\]

Using Corollary 3.2 \( \int |v| \leq \min(p(s), |V^*(\sigma) - v|) (1 + |v|^2)|V^*(\sigma) - v||v| d\sigma \leq \frac{ln^7(2 + \sigma)}{|V^*(\sigma)|^2} \).

At least, if \( m(s, v) = \frac{s}{10} \), then
\[
\int |v| \leq P(s) \frac{1}{(1 + |v|^2)|V^*(\sigma) - v||v|} d\sigma \leq \frac{ln(2 + \sigma)}{|V^*(\sigma)|^2}.
\]

From Corollary 3.2 with taking \( P = \frac{ln(2 + \sigma)}{|V^*(\sigma)|^2} \) and collecting (4.6)-(4.10) with (4.5)

(whenever (4.1) and (4.2) hold true). Obtaining
\[
J_2 \leq 6 \sigma^2(t) \frac{ln^2(2 + t)}{|V^*(t)|^2}.
\]

4.1 Final Estimating:

Combining the contributions (3.6), (4.4) with (4.11), yield
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\[ |a(t)V^*(t) - a(t - \delta)V^*(t - \delta)| \lesssim 6(1 + a(t)^4P^3 + R^{-1} + R^{-2}(t) \frac{\ln(2 + t)}{|V^*(t)|}) \]

We choose a proper $R$ with (4.2) to get $P(t) - P(t - \delta) \lesssim 6|P(t)|^{-\frac{1}{2}} \ln(2 + t)$. Consequently, with the same processing of Corollary 3.2, we have $P(t) \leq C_2(1 + t)^{\frac{2}{3}}\ln^2(2 + t)$, because $P(t - \delta) \leq P(t)$ then Theorem 1.1 is proven.

References


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