Stability of the General Quintic Functional Equation

Sun Sook Jin and Yang-Hi Lee
Department of Mathematics Education
Gongju National University of Education
Gongju 32533, Republic of Korea

This article is distributed under the Creative Commons by-nc-nd Attribution License. Copyright © 2021 Hikari Ltd.

Abstract

The general quintic functional equation is a generalization of many functional equations such as the additive, the general quadratic, the general cubic, and the general quartic functional equation. In this paper, we investigate the stability of the general quintic functional equation.

Mathematics Subject Classification: 39B82, 39B52

Keywords: stability of a functional equation, general quintic functional equation, general quintic mapping

1 Introduction

In this paper, let $V$, $X$, and $Y$ be a real vector space, a real normed space, and a real Banach space, respectively. In 1940, Ulam [15] raised the question about the stability of group homomorphisms, and in the following year Hyers [4] solved this question about the additive functional equation, which gives a partial answer to Ulam’s question. Since then, many mathematicians have generalized Hyers’ result (refer to [2, 3, 5, 7, 10, 13, 14] for more generalized results). In particular, Gavruta [3] generalized the Hyer’s result as follows:

**Theorem 1.1** Let $(G, +)$ be an abelian group and $\varphi : G^2 \to [0, \infty)$ be a function such that

$$\varphi(x, y) := \sum_{k=0}^{\infty} 2^{-k}\varphi(x, y) < \infty$$
for all \(x, y \in G\). If \(f : G \to Y\) is a mapping such that
\[
\|f(x + y) - f(x) - f(y)\| \leq \varphi(x, y)
\]
for all \(x, y \in G\), then there exists a unique additive mapping \(T : G \to Y\) such that
\[
\|f(x) - T(x)\| \leq \frac{1}{2} \varphi(x, x)
\]
for all \(x \in G\).

In 2019, Lee [12] proved the Hyers-Ulam-Rassias stability of the general quintic functional equation
\[
\sum_{i=0}^{6} 6C_i(-1)^{6-i} f(x + (i - 3)y) = 0
\]
as follows:

**Theorem 1.2** (Theorem 2 in [12]) Let \(p \neq 1, 2, 3, 4, 5\) be a fixed nonnegative real number. Suppose that \(f : X \to Y\) is a mapping such that
\[
\|\sum_{i=0}^{6} 6C_i(-1)^{6-i} f(x + (i - 3)y)\| \leq \theta(\|x\|^p + \|y\|^p)
\]
for all \(x, y \in X\). Then there exists a general quintic mapping \(F\) such that
\[
\|f(x) - f(0) - F(x)\| \leq \left\{
\begin{array}{ll}
\frac{K}{45+2p(2p-2)} + \frac{(128+44-2p)K}{45(2p-32)(2p-8)2p} + \frac{5}{2p(2p-16)(2p-4)} \theta \|x\|^p & \text{if } 5 < p, \\
\frac{2K}{45(2p-8)(2p-2)2p} + \frac{5}{2p(2p-16)(2p-4)} \theta \|x\|^p & \text{if } 4 < p < 5, \\
\frac{2K}{45(2p-8)(2p-2)2p} + \frac{5}{2p(2p-16)(2p-4)} \theta \|x\|^p & \text{if } 3 < p < 4, \\
\frac{K}{90(2p-2)} + \frac{(1282p)K}{90(32-2p)(8-2p)2p} + \frac{5}{2p(2p-16)(2p-4)} \theta \|x\|^p & \text{if } 2 < p < 3, \\
\frac{K}{90(2p-2)} + \frac{(128-2p)K}{90(32-2p)(8-2p)2p} + \frac{5}{(16-2p)(4-2p)} \theta \|x\|^p & \text{if } 1 < p < 2, \\
\frac{K}{180(2-2p)} + \frac{(38-2p)K}{180(32-2p)(8-2p)2p} + \frac{5}{(16-2p)(4-2p)} \theta \|x\|^p & \text{if } 0 \leq p < 1
\end{array}\right.
\]
for all \(x \in X\) and \(F(0) = 0\), where \(K = 182 + 38 \cdot 2p + 6 \cdot 3^p\).

More detailed term for the concept of “a general quintic mapping” can be found in Baker’s paper [1] by the term “generalized polynomial mapping of degree at most 5”. Kim etc. [6] have previously studied the stability of a general quadratic functional equation and Lee [8, 9, 11] has studied the stability of a general cubic functional equation, a general cubic functional equation, and a general quartic functional equation.
In this paper, we will prove that the generalization shown in Proposition 1.1 by Găvruta is also applicable to the generalization of the stability of general quintic functional equation. More specifically, we will investigate the stability of the general quintic functional equation for the mapping $f$ such that

$$\| \sum_{i=0}^{6} C_i (-1)^{6-i} f(x + (i-3)y) \| \leq \varphi(x, y),$$

where $\varphi : V^2 \to [0, \infty)$ be a function such that

$$\sum_{n=0}^{\infty} 32^n \varphi \left( \frac{x}{2^n}, \frac{y}{2^n} \right) < \infty \text{ or } \sum_{n=0}^{\infty} 2^{-n} \varphi (2^n x, 2^n y) < \infty$$

for all $x, y \in V$.

### 2 Main results

Throughout this section, for a given mapping $f : V \to Y$, we use the following abbreviations:

$$f_o(x) := \frac{f(x) - f(-x)}{2}, \quad f_e(x) := \frac{f(x) + f(-x)}{2},$$

$$Df(x, y) := \sum_{i=0}^{6} C_i (-1)^{6-i} f(x + (i-3)y),$$

$$\Gamma f(x) := Df_o(2x, 2x) + 6Df_o(3x, x) + 36Df_o(2x, x) + 70Df_o(x, x),$$

$$\Delta f(x) := Df_e(x, x) + 3Df_e(0, x)$$

for all $x, y \in V$. If $\tilde{f}$ is the mapping defined by $\tilde{f}(x) = f(x) - f(0)$, then the mapping $\tilde{f}$ satisfies the properties $D\tilde{f}(x, y) = Df(x, y)$ and $\tilde{f}(0) = 0$. By laborious computation we can get the equalities

$$\Gamma \tilde{f}(x) = \tilde{f}_o(8x) - 42\tilde{f}_o(4x) + 336\tilde{f}_o(2x) - 512\tilde{f}_o(x),$$

$$\Delta \tilde{f}(x) = \tilde{f}_e(4x) - 20\tilde{f}_e(2x) + 64\tilde{f}_e(x)$$

(2)

for all $x \in V$.

**Lemma 2.1** For a given mapping $f : V \to Y$ with $f(0) = 0$, let $J_n f, J'_n f : V \to Y$ be the mappings defined by

$$J_n f(x) := \left( \frac{16^{n+1}}{12} - \frac{4}{3} \right) f_e \left( \frac{x}{2^n} \right) - \left( \frac{16^{n+1}}{3} - \frac{4^{n+2}}{3} \right) f_e \left( \frac{x}{2^{n+1}} \right) + \frac{2^n - 20 \cdot 8^n + 64 \cdot 32^n}{45} f_o \left( \frac{x}{2^n} \right) - \frac{120 \cdot 2^n - 680 \cdot 8^n + 640 \cdot 32^n}{45} f_o \left( \frac{x}{2^{n+1}} \right) + \frac{256 \cdot 2^n - 1280 \cdot 8^n + 1024 \cdot 32^n}{45} f_o \left( \frac{x}{2^{n+2}} \right)$$

and

$$J'_n f(x) := \left( \frac{16^{n+1}}{12} - \frac{4}{3} \right) f_e \left( \frac{x}{2^n} \right) - \left( \frac{16^{n+1}}{3} - \frac{4^{n+2}}{3} \right) f_e \left( \frac{x}{2^{n+1}} \right) + \frac{2^n - 20 \cdot 8^n + 64 \cdot 32^n}{45} f_o \left( \frac{x}{2^n} \right) - \frac{120 \cdot 2^n - 680 \cdot 8^n + 640 \cdot 32^n}{45} f_o \left( \frac{x}{2^{n+1}} \right) + \frac{256 \cdot 2^n - 1280 \cdot 8^n + 1024 \cdot 32^n}{45} f_o \left( \frac{x}{2^{n+2}} \right).$$
and

\[ J_n f(x) := \left( \frac{4}{4^n} - \frac{1}{16^n} \right) f_0(2^n x) - \left( \frac{1}{4^n} - \frac{1}{16^n} \right) f_e(2^{n+1} x) \]

\[ + \left( \frac{16}{32^n} - \frac{320}{8^n} + \frac{1024}{2^n} \right) f_0(2^n x) - \left( \frac{10}{32^n} - \frac{170}{8^n} + \frac{160}{2^n} \right) f_e(2^{n+1} x) \]

\[ + \left( \frac{1}{32^n} - \frac{5}{8^n} + \frac{4}{2^n} \right) f_o(2^{n+2} x) \]

for all \( x \in V \) and all nonnegative integers \( n \). Then

\[ J_n f(x) - J_{n+1} f(x) = \left( \frac{4^{2n+1}}{3} - \frac{4^n}{3} \right) \Delta f\left( \frac{x}{2^{n+2}} \right) \]

\[ + \left( \frac{2^n}{45} - \frac{4}{9} - \frac{64 \cdot 32^n}{45} \right) \Gamma f\left( \frac{x}{2^{n+3}} \right) \]

\[ J_n' f(x) - J_{n+1}' f(x) = \frac{\Delta f(2^n x)}{48 \cdot 4^n} - \frac{\Delta f(2^n x)}{192 \cdot 16^n} \]

\[ - \frac{\Gamma f(2^n x)}{180 \cdot 2^{n+1}} + \frac{\Gamma f(2^n x)}{144 \cdot 8^{n+1}} - \frac{\Gamma f(2^n x)}{720 \cdot 32^{n+1}} \]

for all \( x \in V \) and all nonnegative integers \( n \).

**Proof.** From the equalities (2) and the definitions of \( J_n f \) and \( J_n' f \), we obtain the equalities

\[ J_n f(x) - J_{n+1} f(x) \]

\[ = \left( \frac{16^{n+1}}{12} - \frac{4^n}{3} \right) f_e\left( \frac{x}{2^{n+1}} \right) - \left( \frac{16^{n+1}}{3} - \frac{4^{n+2}}{3} \right) f_e\left( \frac{x}{2^{n+1}} \right) \]

\[ + \frac{2^n - 2 \cdot 8^n + 64 \cdot 32^n}{45} f_0\left( \frac{x}{2^n} \right) - \frac{40 \cdot 2^n - 680 \cdot 8^n + 640 \cdot 32^n}{45} f_o\left( \frac{x}{2^{n+1}} \right) \]

\[ + \frac{256 \cdot 2^n - 1280 \cdot 8^n + 1024 \cdot 32^n}{45} f_o\left( \frac{x}{2^{n+2}} \right) \]

\[ - \left( \frac{16^{n+2}}{12} - \frac{4^{n+1}}{3} \right) f_e\left( \frac{x}{2^{n+2}} \right) + \left( \frac{16^{n+2}}{3} - \frac{4^{n+3}}{3} \right) f_e\left( \frac{x}{2^{n+2}} \right) \]

\[ - \frac{2^{n+1} - 20 \cdot 8^{n+1} + 64 \cdot 32^{n+1}}{45} f_o\left( \frac{x}{2^{n+1}} \right) \]

\[ + \frac{40 \cdot 2^{n+1} - 680 \cdot 8^{n+1} + 640 \cdot 32^{n+1}}{45} f_o\left( \frac{x}{2^{n+2}} \right) \]

\[ - \frac{256 \cdot 2^{n+1} - 1280 \cdot 8^{n+1} + 1024 \cdot 32^{n+1}}{45} f_o\left( \frac{x}{2^{n+3}} \right) \]

\[ = - \frac{4^n}{3} f_e\left( \frac{x}{2^n} \right) - 20 f_e\left( \frac{x}{2^{n+1}} \right) + 64 f_e\left( \frac{x}{2^{n+2}} \right) \]
Lemma 2.2 If \( f : V \to Y \) is a mapping such that
\[
Df(x, y) = 0
\]
for all $x, y \in V$ with $f(0) = 0$, then

$$J_n f(x) = f(x) \quad \text{and} \quad J'_n f(x) = f(x)$$

for all $x \in V$ and all positive integers $n$.

**Proof.** If $f : V \to Y$ is a mapping such that

$$Df(x, y) = 0$$

for all $x, y \in V$ with $f(0) = 0$, then it follows from the definitions of $\Delta f(x)$ and $\Gamma f(x)$ that $\Delta f(x) = 0$ and $\Gamma f(x) = 0$ for all $x \in V$. Therefore, together with the equality $f(x) - J_n f(x) = \sum_{i=0}^{n-1} (J_i f(x) - J_{i+1} f(x))$ and the equality (3), we conclude that

$$J_n f(x) = f(x)$$

for all $x \in V$ and all positive integers $n$. In the same way we can easily show the equality $J'_n f(x) = f(x)$ for all $x \in V$.

From Lemma 2.1 and Lemma 2.2, we can prove the following stability theorem.

**Theorem 2.3** Let $\varphi : V^2 \to [0, \infty)$ be a function such that

$$\sum_{n=0}^{\infty} 32^n \varphi \left( \frac{x}{2^n}, \frac{y}{2^n} \right) < \infty$$  \hspace{1cm} (5)

for all $x, y \in V$. Suppose that $f : V \to Y$ is a mapping such that

$$\|Df(x, y)\| \leq \varphi(x, y)$$  \hspace{1cm} (6)

for all $x, y \in V$. Then there exists a general quintic mapping $F$ such that

$$\|f(x) - f(0) - F(x)\| \leq \sum_{n=0}^{\infty} \left[ \left( \frac{4^{2n+1}}{3} - \frac{4^n}{3} \right) \Phi \left( \frac{x}{2^{n+2}} \right) + \left( \frac{2^n}{45} - \frac{4 \cdot 8^n}{9} + \frac{64 \cdot 32^n}{45} \right) \Phi' \left( \frac{x}{2^{n+3}} \right) \right]$$  \hspace{1cm} (7)

for all $x \in V$ and $F(0) = 0$, where $\varphi_c : V^2 \to [0, \infty)$ and $\Phi, \Phi' : V \to [0, \infty)$ be the functions defined by

$$\varphi_c(x, y) := \frac{\varphi(x, y) + \varphi(-x, -y)}{2}$$
$$\Phi(x) := \varphi_c(x, x) + 3\varphi_c(0, x)$$
$$\Phi'(x) := \varphi_c(2x, 2x) + 6\varphi_c(3x, x) + 36\varphi_c(2x, x) + 70\varphi_c(x, x).$$
Proof. If \( \tilde{f} \) is the mapping defined by \( \tilde{f}(x) = f(x) - f(0) \), then the mapping \( \tilde{f} \) satisfies the properties \( D\tilde{f}(x, y) = Df(x, y) \) and \( \tilde{f}(0) = 0 \). By (2) and the definitions of \( \Gamma f \) and \( \Delta f \), we have

\[

\|\Gamma \tilde{f}(x)\| = \|D\tilde{f}(2x, 2x) + 6D\tilde{f}(3x, x) + 36D\tilde{f}(2x, x) + 70D\tilde{f}(0, x)\| \\
\leq \Phi'(x),
\]

\[

\|\Delta \tilde{f}(x)\| = \|D\tilde{f}_e(x, x) + 3D\tilde{f}_e(0, x)\| \leq \Phi(x)
\]

for all \( x \in V \). Hence, it follows from (3) and (5) that

\[

\|J_n \tilde{f}(x) - J_{n+1} \tilde{f}(x)\| = \left\| \left( \frac{4^{2n+1}}{3} - \frac{4^n}{3} \right) \Delta f \left( \frac{x}{2^{n+2}} \right) + \left( \frac{2^n}{45} - \frac{4 \cdot 8^n}{9} + \frac{64 \cdot 32^n}{45} \right) \Gamma f \left( \frac{x}{2^{n+3}} \right) \right\| \\
\leq \left( \frac{4^{2n+1}}{3} - \frac{4^n}{3} \right) \Phi \left( \frac{x}{2^{n+2}} \right) + \left( \frac{2^n}{45} - \frac{4 \cdot 8^n}{9} + \frac{64 \cdot 32^n}{45} \right) \Phi' \left( \frac{x}{2^{n+3}} \right)
\]

for all \( x \in V \). Together with the equality \( J_n \tilde{f}(x) - J_{n+m} \tilde{f}(x) = \sum_{i=n}^{n+m-1} (J_i \tilde{f}(x) - J_{i+1} \tilde{f}(x)) \) for all \( x \in V \), we obtain that

\[

\|J_n \tilde{f}(x) - J_{n+m} \tilde{f}(x)\| \leq \sum_{i=n}^{n+m-1} \left[ \left( \frac{4^{2i+1}}{3} - \frac{4^i}{3} \right) \Phi \left( \frac{x}{2^{i+2}} \right) + \left( \frac{2^i}{45} - \frac{4 \cdot 8^i}{9} + \frac{64 \cdot 32^i}{45} \right) \Phi' \left( \frac{x}{2^{i+3}} \right) \right]
\]

for all \( x \in V \) and all nonnegative integers \( n, m \). It follows from (5) and (9) that the sequence \( \{J_n \tilde{f}(x)\} \) is a Cauchy sequence for all \( x \in V \). Since \( Y \) is complete, the sequence \( \{J_n \tilde{f}(x)\} \) converges for all \( x \in V \). Hence we can define a mapping \( F : V \rightarrow Y \) by

\[

F(x) := \lim_{n \rightarrow \infty} J_n \tilde{f}(x)
\]

for all \( x \in V \). Notice that \( J_0 \tilde{f}(x) = f(x) - f(0) \) for all \( x \in V \) and \( F(0) = 0 \) follows from \( \tilde{f}(0) = 0 \). Moreover, letting \( n = 0 \) and passing the limit \( n \rightarrow \infty \) in (9) we get the inequality (7). From the definition of \( F \), we easily get

\[

\|DF(x, y)\| = \lim_{n \rightarrow \infty} \|DJ_n f(x, y)\| \\
\leq \lim_{n \rightarrow \infty} \left\| \left( \frac{16^{n+1}}{12} - \frac{4^n}{3} \right) D\tilde{f}_e \left( \frac{x}{2^n}, \frac{y}{2^n} \right) - \left( \frac{16^{n+1}}{3} - \frac{4^{n+2}}{3} \right) Df \left( \frac{x}{2^{n+1}}, \frac{y}{2^{n+1}} \right) \right\| \\
+ \frac{2^n - 20 \cdot 8^n + 64 \cdot 32^n}{45} Df \left( \frac{x}{2^n}, \frac{y}{2^n} \right) \\
- \frac{40 \cdot 2^n - 680 \cdot 8^n + 640 \cdot 32^n}{45} Df \left( \frac{x}{2^{n+1}}, \frac{y}{2^{n+1}} \right)
\]
for all $x, y \in V$. To prove the uniqueness of $F$, let $F' : V \to Y$ be another general quintic mapping satisfying (7) and $F'(0) = 0$. Instead of the condition (7), it is sufficient to show that there is a unique mapping satisfying the simpler condition

$$\|\tilde{f}(x) - F(x)\| \leq \sum_{i=0}^{\infty} \left( 32^{i+1}\Phi\left(\frac{x}{2^{i+2}}\right) + 32^{i+2}\Phi'\left(\frac{x}{2^{i+3}}\right) \right)$$

for all $x \in V$. By Lemma 2.2, the equality $F'(x) = J_n F'(x)$ holds for all positive integers $n$. So we have

$$\|J_n \tilde{f}(x) - F'(x)\|$$

$$= \|J_n \tilde{f}(x) - J_n F'(x)\|$$

$$= \left\| \left( \frac{16^{n+1}}{12} - \frac{4^n}{3} \right) \left( \tilde{f}_e\left(\frac{x}{2^n}\right) - F'_e\left(\frac{x}{2^n}\right) \right) - \left( \frac{16^{n+1}}{3} - \frac{4^{n+2}}{3} \right) \left( \tilde{f}_e\left(\frac{x}{2^{n+1}}\right) - F'_e\left(\frac{x}{2^{n+1}}\right) \right) + \frac{2^n - 20 \cdot 8^n + 64 \cdot 32^n}{45} \left( \tilde{f}_o\left(\frac{x}{2^n}\right) - F'_o\left(\frac{x}{2^n}\right) \right) - \frac{40 \cdot 2^n - 680 \cdot 8^n + 640 \cdot 32^n}{45} \left( \tilde{f}_o\left(\frac{x}{2^{n+1}}\right) - F'_o\left(\frac{x}{2^{n+1}}\right) \right) + \frac{256 \cdot 2^n - 1280 \cdot 8^n + 1024 \cdot 32^n}{45} \left( \tilde{f}_o\left(\frac{x}{2^{n+2}}\right) - F'_o\left(\frac{x}{2^{n+2}}\right) \right) \right\|$$

$$\leq \frac{32^{n+1}}{2} \left\| \tilde{f}_e\left(\frac{x}{2^n}\right) - F'_e\left(\frac{x}{2^n}\right) \right\| + \frac{32^{n+1}}{2} \left\| \tilde{f}_e\left(\frac{x}{2^{n+1}}\right) - F'_e\left(\frac{x}{2^{n+1}}\right) \right\| + \frac{32^{n+1}}{2} \left\| \tilde{f}_o\left(\frac{x}{2^n}\right) - F'_o\left(\frac{x}{2^n}\right) \right\| + \frac{32^{n+1}}{2} \left\| \tilde{f}_o\left(\frac{x}{2^{n+1}}\right) - F'_o\left(\frac{x}{2^{n+1}}\right) \right\| + \frac{32^{n+1}}{2} \left\| \tilde{f}_o\left(\frac{x}{2^{n+2}}\right) - F'_o\left(\frac{x}{2^{n+2}}\right) \right\|$$

$$\leq \sum_{i=0}^{\infty} \left( 32^{i+n+2}\Phi\left(\frac{x}{2^{i+n+2}}\right) + 32^{i+n+3}\Phi'\left(\frac{x}{2^{i+n+3}}\right) \right) + \sum_{i=0}^{\infty} \left( 32^{i+n+2}\Phi\left(\frac{x}{2^{i+n+3}}\right) + 32^{i+n+3}\Phi'\left(\frac{x}{2^{i+n+4}}\right) \right)
Stability of the general quintic functional equation

\[ + \sum_{i=0}^{\infty} \left( 32^{i+n+2} \Phi \left( \frac{x}{2^{i+n+4}} \right) + 32^{i+n+3} \Phi' \left( \frac{x}{2^{i+n+5}} \right) \right) \]
\[ \leq 3 \sum_{i=n+2}^{\infty} 32^i \Phi \left( \frac{x}{2^i} \right) + 3 \sum_{i=n+3}^{\infty} 32^i \Phi' \left( \frac{x}{2^i} \right) \]

for all \( x \in V \) and all positive integer \( n \). Taking the limit in the above inequality as \( n \to \infty \), we obtain the equality \( F'(x) = \lim_{n \to \infty} J_n \tilde{f}(x) \) for all \( x \in V \), which means that \( F(x) = F'(x) \) for all \( x \in V \).

**Theorem 2.4** Let \( \varphi : V^2 \to [0, \infty) \) be a function such that
\[ \sum_{n=0}^{\infty} 2^{-n} \varphi(2^n x, 2^n y) < \infty \] (10)
for all \( x, y \in V \). Suppose that \( f : V \to Y \) is a mapping such that
\[ \|Df(x, y)\| \leq \varphi(x, y) \] (11)
for all \( x, y \in V \). Then there exists a general quintic mapping \( F \) such that
\[ \|f(x) - f(0) - F(x)\| \leq \sum_{n=0}^{\infty} \left[ \left( \frac{1}{48 \cdot 4^n} - \frac{1}{192 \cdot 16^n} \right) \Phi(2^n x) + \left( \frac{1}{180 \cdot 2^{n+1}} - \frac{1}{144 \cdot 8^{n+1}} + \frac{1}{720 \cdot 32^{n+1}} \right) \Phi'(2^n x) \right] \]
(12)
for all \( x \in V \) and \( F(0) = 0 \), where \( \varphi, \Phi, \Phi' \) be the functions defined as Theorem 2.3.

**Proof.** If \( \tilde{f} \) is the mapping defined by \( \tilde{f}(x) = f(x) - f(0) \), then the mapping \( \tilde{f} \) satisfies the properties \( D\tilde{f}(x, y) = Df(x, y) \) and \( \tilde{f}(0) = 0 \). By the definitions of \( \Gamma f \) and \( \Delta \tilde{f} \), it follows from (4), (8) and (11) that
\[ \|J_n'\tilde{f}(x) - J_{n+1}'\tilde{f}(x)\| \leq \left( \frac{1}{48 \cdot 4^n} - \frac{1}{192 \cdot 16^n} \right) \|\Delta f(2^n x)\| \]
\[ + \left( \frac{1}{180 \cdot 2^{n+1}} + \frac{1}{144 \cdot 8^{n+1}} - \frac{1}{720 \cdot 32^{n+1}} \right) \|\Gamma f(2^n x)\| \]
\[ \leq \left( \frac{1}{48 \cdot 4^n} - \frac{1}{192 \cdot 16^n} \right) \Phi(2^n x) \]
\[ + \left( \frac{1}{180 \cdot 2^{n+1}} - \frac{1}{144 \cdot 8^{n+1}} + \frac{1}{720 \cdot 32^{n+1}} \right) \Phi'(2^n x) \]
for all \( x \in V \). Together with the equality \( J_n'\tilde{f}(x) - J_{n+m}'\tilde{f}(x) = \sum_{i=n}^{n+m-1} (J_i'\tilde{f}(x) - J_{i+1}'\tilde{f}(x)) \) for all \( x \in V \), we obtain that
\[ \|J_n'\tilde{f}(x) - J_{n+m}'\tilde{f}(x)\| \leq \sum_{i=n}^{n+m-1} \left[ \left( \frac{1}{48 \cdot 4^i} - \frac{1}{192 \cdot 16^i} \right) \Phi(2^i x) + \left( \frac{1}{180 \cdot 2^{i+1}} - \frac{1}{144 \cdot 8^{i+1}} + \frac{1}{720 \cdot 32^{i+1}} \right) \Phi'(2^i x) \right] \]
(13)
for all \( x \in V \) and all nonnegative integers \( n, m \). It follows from (10) and (13) that the sequence \( \{ J_n f(x) \} \) is a Cauchy sequence for all \( x \in V \). Since \( Y \) is complete, the sequence \( \{ J_n f(x) \} \) converges for all \( x \in V \). Hence we can define a mapping \( F : V \to Y \) by

\[
F(x) := \lim_{n \to \infty} J_n^o \tilde{f}(x)
\]

for all \( x \in V \). Note that \( F(0) = 0 \) follows from \( \tilde{f}(0) = 0 \). Notice that \( J_n^o \tilde{f}(x) = f(x) - f(0) \) for all \( x \in V \). Moreover, letting \( n = 0 \) and passing the limit \( n \to \infty \) in (13) we get the inequality (12). From the definition of \( F \), we easily get

\[
\| DF(x, y) \|
= \lim_{n \to \infty} \| DJ_n^o f(x, y) \|
\leq \lim_{n \to \infty} \left\| \left( \frac{4}{3^n} - \frac{1}{16^n} \right) Df_e(2^n x, 2^n y) - \left( \frac{1}{3^n} - \frac{1}{16^n} \right) Df_e(2^{n+1} x, 2^{n+1} y) \\
+ \left( \frac{16}{32^n} - \frac{320}{8^n} + \frac{1024}{2^n} \right) Df_o(2^n x, 2^n y) \right. \\
- \left( \frac{10}{32^n} - \frac{170}{8^n} + \frac{160}{2^n} \right) Df_o(2^{n+1} x, 2^{n+1} y) \\
+ \left. \frac{1}{32^n} - \frac{5}{8^n} + \frac{4}{2^n} \right) Df_o(2^{n+2} x, 2^{n+2} y) \right\|
\leq \lim_{n \to \infty} \left( \frac{4}{3^n} \varphi_e(2^n x, 2^n y) \right) + \frac{1}{4^n} \varphi_e(2^{n+1} x, 2^{n+1} y) \\
+ \frac{1}{3^n} \varphi_e(2^n x, 2^n y) + \frac{160}{2^n} \varphi_e(2^{n+1} x, 2^{n+1} y) + \frac{4}{2^n} \varphi_e(2^{n+2} x, 2^{n+2} y) \right)
= 0
\]

for all \( x, y \in V \). To prove the uniqueness of \( F \), let \( F' : V \to Y \) be another general quintic mapping satisfying (12) and \( F'(0) = 0 \). Instead of the condition (12), it is sufficient to show that there is a unique mapping satisfying the simpler condition

\[
\| \tilde{f}(x) - F(x) \| \leq \sum_{i=0}^{\infty} \frac{1}{2^i} (\Phi(2^i x) + \Phi'(2^i x))
\]

for all \( x \in V \). By Lemma 2.2, the equality \( F'(x) = J_n^o F'(x) \) holds for all positive integers \( n \). So we have

\[
\| J_n^o \tilde{f}(x) - F'(x) \|
= \| J_n^o \tilde{f}(x) - J_n^o F'(x) \|
\leq \left\| \left( \frac{4}{3^n} - \frac{1}{16^n} \right) \tilde{f}_e(2^n x) - F'_e(2^n x) - \left( \frac{1}{3^n} - \frac{1}{16^n} \right) \tilde{f}_e(2^{n+1} x) - F'_e(2^{n+1} x) \right\|
\]
$$\begin{align*}
&\left( \frac{16}{32n} - \frac{320}{8n} + \frac{1024}{2n} \right) \tilde{f}_o(2^n x - F_o'(2^n x)) \\
&- \left( \frac{10}{32n} - \frac{170}{8n} + \frac{160}{2n} \right) \tilde{f}_o((2^n x) - F_o'(2^n x)) \\
&+ \left( \frac{1}{32n} - \frac{5}{8n} + \frac{4}{2n} \right) \tilde{f}_o(2^n x - F_o'(2^n x)) \\
&\leq \frac{2}{4^n} \| \tilde{f}_o(2^n x) - F_o'(2^n x) \| + \frac{1}{4n+1} \| \tilde{f}_o(2^n x) - F_o'(2^n x) \| \\
&+ \frac{2}{2^n} \| \tilde{f}_o(2^n x - F_o'(2^n x)) + \frac{1}{2n+2} \| \tilde{f}_o(2^n x) - F_o'(2^n x) \| \\
&+ \frac{1}{2n+2} \| \tilde{f}_o(2^n x) - F_o'(2^n x) \| \\
&\leq \sum_{i=0}^{\infty} \frac{3}{2^n} (\Phi(2^{n+i} x) + \Phi'(2^{n+i} x)) + \sum_{i=0}^{\infty} \frac{1}{2^n} (\Phi(2^{n+i+1} x) + \Phi'(2^{n+i+1} x)) \\
&+ \sum_{i=0}^{\infty} \frac{1}{2^{n+i+2}} (\Phi(2^{n+i+1} x) + \Phi'(2^{n+i+1} x)) \\
&\leq \sum_{i=n}^{\infty} \frac{3}{2^{i}} (\Phi(2^{i} x) + \Phi'(2^{i} x)) + \sum_{i=n+1}^{\infty} \frac{1}{2^{i}} (\Phi(2^{i} x) + \Phi'(2^{i} x)) \\
&+ \sum_{i=n+2}^{\infty} \frac{1}{2^{i}} (\Phi(2^{i} x) + \Phi'(2^{i} x)) \\
&\leq \sum_{i=n}^{\infty} \frac{5}{2^{i}} (\Phi(2^{i} x) + \Phi'(2^{i} x))
\end{align*}$$

for all \( x \in V \) and all positive integer \( n \). Taking the limit in the above inequality as \( n \to \infty \), we obtain the equality \( F'(x) = \lim_{n \to \infty} J_n \tilde{f}(x) \) for all \( x \in V \), which means that \( F(x) = F'(x) \) for all \( x \in V \).

References


Received: August 23, 2021; Published: September 7, 2021