International Journal of Mathematical Analysis
Vol. 15, 2021, no. 7, 291 - 302
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ijma.2021.912363

An Example of Everywhere Regularity for Minima of Vectorial Integral Fuctionals of the Calculus of Variation

Tiziano Granucci [1]

ISIS Leonardo da Vinci
Firenze, via del Terzolle, 50100, Italy

Dedicated to the memory of Fiorella Pettini.

Abstract

In this paper we consider a special case of vectorial integral functional and we proof that its minima are local Hölder contiuous functions.

Mathematics Subject Classification: 49N60, 35J50

Keywords: Everywhere regularity, Hölder continuiuty, minima

[1] I would like to thank all my family and friends for the support given to me over the years: Elisa Cirri, Caterina Granucci, Delia Granucci, Irene Granucci, Laura and Fiorenza Granucci, Massimo Masi and Monia Randolfi. Also I would like to thank my professors Luigi Barletti, Giorgio Busoni, Elvira Mascolo, Giorgio Talenti and Vincenso Vespri.

1. Introduction

In this paper we consider the functional

$$J(u)=\int_{\Omega}\sum_{\alpha=1}^{3}|\nabla u^{\alpha}|^{p}+f\left(u^{1},u^{2},u^{3},\left|\nabla u^{1}\right|,\left|\nabla u^{2}\right|,\left|\nabla u^{3}\right|\right)\,dx \tag{1.1}$$

where $\Omega\subset\mathbb{R}^{3}$ is an open bounded and Lebesgue measurable subset with regular boundary and

$$f(\vec{x},\vec{v})=\left(|\vec{x}-\vec{v}|^{2}+1\right)^{\frac{q}{2}}+\left[(|\vec{x}|+1)^{2}+(|\vec{v}|+1)^{2}\right]^{\frac{q}{2}} \tag{1.2}$$

with $(\vec{x},\vec{v})=(x^{1},x^{2},x^{3},v^{1},v^{2},v^{3})$, $|\vec{x}|=\sqrt{\sum_{\alpha=1}^{3}(x^{\alpha})^{2}}$, $|\vec{v}|=\sqrt{\sum_{\alpha=1}^{3}(v^{\alpha})^{2}}$ and $1\leq q<p<3$; we proof that the minima of the functional (1.1) are locally hölder continuous functions.

I think that the study of this functional is interesting because, to the author's knowledge, there are not many examples of regularities results for minima of vector functionals.

There are counter examples, refer to [4], [8] and [11], which tell us that in general the minima of vector integral functionals are not regular, but as proved by the famous work of K. Uhlenbeck, refer to [18],in the vector case there are very restrictive and particular conditions on the form of the functional which allow to obtain the regularity of the minima of the considered functional, refer olso to [1], [6], [7], [9], [13], [16] and [17]. Recently some articles have been presented, refer to [2] and [6], in which the boundedness of the minima of functionals of type

$$I(u)=\int g(x,u,|\nabla u|)\,dx \tag{1.3}$$

and

$$H(u)=\int_{\Omega}\sum_{\alpha=1}^{m}|\nabla u^{\alpha}|^{p}+f(x,\nabla u)\,dx \tag{1.4}$$

is demonstrated using suitable hypotheses on functions g and f, for more details refer to [6] and [2], refer olso to [3], [4] and [5]. In particular in [2] it is proved that the minima of the functional (1.4) are locally continuous hölder functions. The author believes that it is very interesting to study the minima of the function (1.1) to understand if it is possible to find more general conditions on the form of the functional than those presented in [2] and [6] that allow to obtain the regularity of the minima of such functionals. We observe that the functional (1.1) does not fall within the cases considered in [2] and [6]. The main result of our article is the following regularity theorem.

Theorem 1. *If $u \in W^{1,p}(\Omega, \mathbb{R}^3)$ is a minimum of the functional (1.1) and $1 \le q < \frac{p^2}{3} < 3$ then $u^\alpha \in C^{0,\beta}_{loc}(\Omega)$ for $\alpha = 1, 2, 3$.*

The previous regularity theorem follows from the results of [10] using the following Caccioppoli inequalities, refer olso to [12] (Proposition 7.1 and Lemma 7.2).

Theorem 2. *If $u \in W^{1,p}(\Omega, \mathbb{R}^3)$ is a minimum of the functional (1.1) and $1 \le q < \frac{p^2}{3} < 3$ then a radius $R_0 > 0$ and two positive real numbers D_1 and D_2, dependent only on p and q, exist such that for every $x_0 \in \Omega$, $k \in \mathbb{R}$, $\varrho, R \in \mathbb{R}^+$ with $0 < \varrho \le t < s \le R < R_0$ the following Caccioppoli's inequalities hold*

$$\int_{A^\alpha_{k,\varrho}} |\nabla u^\alpha|^p \, dx \le D_1 \int_{A^\alpha_{k,R}} \left[\frac{(u^\alpha - k)}{R - \varrho}\right]^p dx + D_2 \left[\mathcal{L}^3\left(A^\alpha_{k,R}\right)\right]^{1-\frac{p}{3}+\epsilon} \tag{1.5}$$

and

$$\int_{B^\alpha_{k,\varrho}} |\nabla u^\alpha|^p \, dx \le D_1 \int_{B^\alpha_{k,R}} \left[\frac{(k - u^\alpha)}{R - \varrho}\right]^p dx + D_2 \left[\mathcal{L}^3\left(B^\alpha_{k,s}\right)\right]^{1-\frac{p}{3}+\epsilon} \tag{1.6}$$

where $A^\alpha_{k,r} = \{u^\alpha > k\} \cap B_r(x_0)$ and $B^\alpha_{k,r} = \{u^\alpha < k\} \cap B_r(x_0)$ for $\alpha = 1, 2, 3$.

2. Proof of the Caccioppoili's inequalities

Let $u \in W^{1,p}(\Omega, \mathbb{R}^3)$ be a minimum of the functional (1.1), let us consider $R_0 > 0$ and $x_0 \in \Sigma \subset\subset \Omega \subset \mathbb{R}^3$, with Σ compact, $0 < \varrho \le t < s \le R < \min\left\{R_0, 1, \frac{1}{2} dist(x_0, \partial\Sigma)\right\}$and $\eta \in C^\infty_c(B_s(x_0))$ such that $0 \le \eta \le 1$, $\varphi = 1$ in $B_t(x_0)$ and $|\nabla \eta| \le \frac{2}{s-t}$, let us define

$$\varphi = \eta^\gamma \begin{pmatrix} (u^1 - k)_+ \\ 0 \\ 0 \end{pmatrix} \tag{2.1}$$

with $\gamma > 1$, $k \in \mathbb{R}$ and $(u^1 - k)_+ = \max\{u^1 - k, 0\}$, then since $u \in W^{1,p}(\Omega, \mathbb{R}^3)$ is a minimum of the functional (1.1), it follows

$$\begin{aligned}
0 = &\int_{B_s(x_0)} |\nabla u^1|^{p-2} \nabla u^1 \nabla \varphi^1 \, dx \\
&+q \int_{B_s(x_0)} \left[\sum_{\alpha=1}^{3} (u^\alpha - |\nabla u^\alpha|)^2 + 1\right]^{\frac{q-2}{2}} (u^1 - |\nabla u^1|)\, \varphi^1 \, dx \\
&+q \int_{B_s(x_0)} \left[\left(\sqrt{\sum_{\alpha=1}^{3} (u^\alpha)^2} + 1\right)^2 + \left(\sqrt{\sum_{\alpha=1}^{3} (|\nabla u^\alpha|)^2} + 1\right)^2\right]^{\frac{q-2}{2}} \frac{\sqrt{\sum_{\alpha=1}^{3} (u^\alpha)^2} + 1}{\sqrt{\sum_{\alpha=1}^{3} (u^\alpha)^2}} u^1 \varphi^1 \, dx \\
&+q \int_{B_s(x_0)} \left[\sum_{\alpha=1}^{3} (u^\alpha - |\nabla u^\alpha|)^2 + 1\right]^{\frac{q-2}{2}} (u^1 - |\nabla u^1|)\, \frac{\nabla u^1 \cdot \nabla \varphi^1}{|\nabla u^1|} \, dx \\
&+q \int_{B_s(x_0)} \left[\left(\sqrt{\sum_{\alpha=1}^{3} (u^\alpha)^2} + 1\right)^2 + \left(\sqrt{\sum_{\alpha=1}^{3} (|\nabla u^\alpha|)^2} + 1\right)^2\right]^{\frac{q-2}{2}} \frac{\sqrt{\sum_{\alpha=1}^{3} (|\nabla u^\alpha|)^2} + 1}{\sqrt{\sum_{\alpha=1}^{3} (|\nabla u^\alpha|)^2}} \nabla u^1 \cdot \nabla \varphi^1 \, dx
\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}
0 = &\int_{A_{k,s}} \eta^\gamma |\nabla u^1|^{p-2} \nabla u^1 \cdot \nabla u^1 + \gamma |\nabla u^1|^{p-2} \nabla u^1 \nabla \eta\, \eta^{\gamma-1} \left(u^1 - k\right) dx \\
&+q \int_{A_{k,s}} \left[\sum_{\alpha=1}^{3} (u^\alpha - |\nabla u^\alpha|)^2 + 1\right]^{\frac{q-2}{2}} (u^1 - |\nabla u^1|)\, \eta^\gamma \left(u^1 - k\right) dx \\
&+q \int_{A_{k,s}} \left[\left(\sqrt{\sum_{\alpha=1}^{3} (u^\alpha)^2} + 1\right)^2 + \left(\sqrt{\sum_{\alpha=1}^{3} (|\nabla u^\alpha|)^2} + 1\right)^2\right]^{\frac{q-2}{2}} \frac{\sqrt{\sum_{\alpha=1}^{3} (u^\alpha)^2} + 1}{\sqrt{\sum_{\alpha=1}^{3} (u^\alpha)^2}} u^1 \eta^\gamma \left(u^1 - k\right) dx \\
&+q \int_{A_{k,s}} \left[\sum_{\alpha=1}^{3} (u^\alpha - |\nabla u^\alpha|)^2 + 1\right]^{\frac{q-2}{2}} (u^1 - |\nabla u^1|)\, \frac{\nabla u^1 \cdot \left(\eta^\gamma \nabla u^1 + \gamma \nabla \eta\, \eta^{\gamma-1} \left(u^1 - k\right)\right)}{|\nabla u^1|} \, dx \\
&+q \int_{A_{k,s}} \eta^\gamma \left[\left(\sqrt{\sum_{\alpha=1}^{3} (u^\alpha)^2} + 1\right)^2 + \left(\sqrt{\sum_{\alpha=1}^{3} (|\nabla u^\alpha|)^2} + 1\right)^2\right]^{\frac{q-2}{2}} \frac{\sqrt{\sum_{\alpha=1}^{3} (|\nabla u^\alpha|)^2} + 1}{\sqrt{\sum_{\alpha=1}^{3} (|\nabla u^\alpha|)^2}} \left|\nabla u^1\right|^2 dx \\
&+\gamma q \int_{A_{k,s}} \left[\left(\sqrt{\sum_{\alpha=1}^{3} (u^\alpha)^2} + 1\right)^2 + \left(\sqrt{\sum_{\alpha=1}^{3} (|\nabla u^\alpha|)^2} + 1\right)^2\right]^{\frac{q-2}{2}} \frac{\sqrt{\sum_{\alpha=1}^{3} (|\nabla u^\alpha|)^2} + 1}{\sqrt{\sum_{\alpha=1}^{3} (|\nabla u^\alpha|)^2}} \cdot \\
&\cdot (\nabla u^1 \cdot \nabla \eta)\, \eta^{\gamma-1} \left(u^1 - k\right) dx
\end{aligned} \tag{2.3}$$

Let us define

$$Z=\int\limits_{A_{k,s}}\eta^{\gamma}\left[\left(\sqrt{\sum_{\alpha=1}^{3}(u^{\alpha})^{2}}+1\right)^{2}+\left(\sqrt{\sum_{\alpha=1}^{3}(|\nabla u^{\alpha}|)^{2}}+1\right)^{2}\right]^{\frac{q-2}{2}}\frac{\sqrt{\sum\limits_{\alpha=1}^{3}(|\nabla u^{\alpha}|)^{2}}+1}{\sqrt{\sum\limits_{\alpha=1}^{3}(|\nabla u^{\alpha}|)^{2}}}\left|\nabla u^{1}\right|^{2}dx \tag{2.4}$$

then

$$\begin{aligned}
Z &\geq \int\limits_{A_{k,s}}\eta^{\gamma}\left[\left(\sqrt{\sum_{\alpha=1}^{3}(u^{\alpha})^{2}}+1\right)^{2}+\left(\sqrt{\sum_{\alpha=1}^{3}(|\nabla u^{\alpha}|)^{2}}+1\right)^{2}\right]^{\frac{q-2}{2}}\left|\nabla u^{1}\right|^{2}dx\\
&\geq \int\limits_{A_{k,s}}\eta^{\gamma}\left[\left(\sqrt{\sum_{\alpha=1}^{3}(|\nabla u^{\alpha}|)^{2}}+1\right)^{2}\right]^{\frac{q-2}{2}}\left|\nabla u^{1}\right|^{2}dx\\
&\geq \int\limits_{A_{k,s}}\eta^{\gamma}\left(|\nabla u^{1}|+1\right)^{q-2}\left|\nabla u^{1}\right|^{2}dx\\
&\geq \int\limits_{A_{k,s}}\eta^{\gamma}\left|\nabla u^{1}\right|^{q}dx
\end{aligned} \tag{2.5}$$

Usinig (2.3) and (2.5) we get

$$\begin{aligned}
0 &\leq q\int\limits_{A_{k,s}}\eta^{\gamma}\left|\nabla u^{1}\right|^{q}dx+\int\limits_{A_{k,s}}\eta^{\gamma}\left|\nabla u^{1}\right|^{p}dx\leq qZ+\int\limits_{A_{k,s}}\eta^{\gamma}\left|\nabla u^{1}\right|^{p}dx\\
&= -q\int\limits_{A_{k,s}}\left[\sum_{\alpha=1}^{3}\left(u^{\alpha}-|\nabla u^{\alpha}|\right)^{2}+1\right]^{\frac{q-2}{2}}\left(u^{1}-|\nabla u^{1}|\right)\eta^{\gamma}\left(u^{1}-k\right)dx\\
&-q\int\limits_{A_{k,s}}\left[\left(\sqrt{\sum_{\alpha=1}^{3}(u^{\alpha})^{2}}+1\right)^{2}+\left(\sqrt{\sum_{\alpha=1}^{3}(|\nabla u^{\alpha}|)^{2}}+1\right)^{2}\right]^{\frac{q-2}{2}}\frac{\sqrt{\sum\limits_{\alpha=1}^{3}(u^{\alpha})^{2}}+1}{\sqrt{\sum\limits_{\alpha=1}^{3}(u^{\alpha})^{2}}}u^{1}\eta^{\gamma}\left(u^{1}-k\right)dx\\
&-q\int\limits_{A_{k,s}}\left[\sum_{\alpha=1}^{3}\left(u^{\alpha}-|\nabla u^{\alpha}|\right)^{2}+1\right]^{\frac{q-2}{2}}\left(u^{1}-|\nabla u^{1}|\right)\frac{\nabla u^{1}\cdot\left(\eta^{\gamma}\nabla u^{1}+\gamma\nabla\eta\,\eta^{\gamma-1}\left(u^{1}-k\right)\right)}{|\nabla u^{1}|}dx\\
&-\gamma\int\limits_{A_{k,s}}\left|\nabla u^{1}\right|^{p-2}\nabla u^{1}\cdot\nabla\eta\,\eta^{\gamma-1}\left(u^{1}-k\right)dx\\
&-\gamma q\int\limits_{A_{k,s}}\left[\left(\sqrt{\sum_{\alpha=1}^{3}(u^{\alpha})^{2}}+1\right)^{2}+\left(\sqrt{\sum_{\alpha=1}^{3}(|\nabla u^{\alpha}|)^{2}}+1\right)^{2}\right]^{\frac{q-2}{2}}\frac{\sqrt{\sum\limits_{\alpha=1}^{3}(|\nabla u^{\alpha}|)^{2}}+1}{\sqrt{\sum\limits_{\alpha=1}^{3}(|\nabla u^{\alpha}|)^{2}}}\cdot\\
&\cdot\left(\nabla u^{1}\cdot\nabla\eta\right)\eta^{\gamma-1}\left(u^{1}-k\right)dx
\end{aligned} \tag{2.6}$$

then

$$
\begin{aligned}
&0\le q\int\limits_{A_{k,s}}\eta^{\gamma}\left|\nabla u^{1}\right|^{q}\,dx+\int\limits_{A_{k,s}}\eta^{\gamma}\left|\nabla u^{1}\right|^{p}\,dx\\
&\le q\int\limits_{A_{k,s}}\left[\sum_{\alpha=1}^{3}\left(u^{\alpha}-\left|\nabla u^{\alpha}\right|\right)^{2}+1\right]^{\frac{q-2}{2}}\left|u^{1}-\left|\nabla u^{1}\right|\right|\eta^{\gamma}\left(u^{1}-k\right)\,dx\\
&+q\int\limits_{A_{k,s}}\left[\left(\sqrt{\sum_{\alpha=1}^{3}\left(u^{\alpha}\right)^{2}}+1\right)^{2}+\left(\sqrt{\sum_{\alpha=1}^{3}\left(\left|\nabla u^{\alpha}\right|\right)^{2}}+1\right)^{2}\right]^{\frac{q-2}{2}}\frac{\sqrt{\sum_{\alpha=1}^{3}\left(u^{\alpha}\right)^{2}}+1}{\sqrt{\sum_{\alpha=1}^{3}\left(u^{\alpha}\right)^{2}}}\left|u^{1}\right|\eta^{\gamma}\left(u^{1}-k\right)\,dx\\
&+q\int\limits_{A_{k,s}}\left[\sum_{\alpha=1}^{3}\left(u^{\alpha}-\left|\nabla u^{\alpha}\right|\right)^{2}+1\right]^{\frac{q-2}{2}}\left|u^{1}-\left|\nabla u^{1}\right|\right|\left[\left|\nabla u^{1}\right|\cdot\eta^{\gamma}+\gamma\left|\nabla\eta\right|\eta^{\gamma-1}\left(u^{1}-k\right)\right]\,dx\\
&+\gamma\int\limits_{A_{k,s}}\left|\nabla u^{1}\right|^{p-1}\left|\nabla\eta\right|\eta^{\gamma-1}\left(u^{1}-k\right)\,dx\\
&+\gamma q\int\limits_{A_{k,s}}\left[\left(\sqrt{\sum_{\alpha=1}^{3}\left(u^{\alpha}\right)^{2}}+1\right)^{2}+\left(\sqrt{\sum_{\alpha=1}^{3}\left(\left|\nabla u^{\alpha}\right|\right)^{2}}+1\right)^{2}\right]^{\frac{q-2}{2}}\frac{\sqrt{\sum_{\alpha=1}^{3}\left(\left|\nabla u^{\alpha}\right|\right)^{2}}+1}{\sqrt{\sum_{\alpha=1}^{3}\left(\left|\nabla u^{\alpha}\right|\right)^{2}}}\cdot\\
&\cdot\left(\left|\nabla u^{1}\right|\cdot\left|\nabla\eta\right|\right)\eta^{\gamma-1}\left(u^{1}-k\right)\,dx
\end{aligned}
\tag{2.7}
$$

and

$$
\begin{aligned}
&0\le q\int\limits_{A_{k,s}}\eta^{\gamma}\left|\nabla u^{1}\right|^{q}\,dx+\int\limits_{A_{k,s}}\eta^{\gamma}\left|\nabla u^{1}\right|^{p}\,dx\\
&\le q\int\limits_{A_{k,s}}\left[\sum_{\alpha=1}^{3}\left(u^{\alpha}-\left|\nabla u^{\alpha}\right|\right)^{2}+1\right]^{\frac{q-1}{2}}\eta^{\gamma}\left(u^{1}-k\right)\,dx\\
&+q\int\limits_{A_{k,s}}\left[\left(\sqrt{\sum_{\alpha=1}^{3}\left(u^{\alpha}\right)^{2}}+1\right)^{2}+\left(\sqrt{\sum_{\alpha=1}^{3}\left(\left|\nabla u^{\alpha}\right|\right)^{2}}+1\right)^{2}\right]^{\frac{q-1}{2}}\eta^{\gamma}\left(u^{1}-k\right)\,dx\\
&+q\int\limits_{A_{k,s}}\left[\sum_{\alpha=1}^{3}\left(u^{\alpha}-\left|\nabla u^{\alpha}\right|\right)^{2}+1\right]^{\frac{q-1}{2}}\left[\left|\nabla u^{1}\right|\cdot\eta^{\gamma}+\gamma\left|\nabla\eta\right|\eta^{\gamma-1}\left(u^{1}-k\right)\right]\,dx\\
&+\gamma\int\limits_{A_{k,s}}\left|\nabla u^{1}\right|^{p-1}\left|\nabla\eta\right|\eta^{\gamma-1}\left(u^{1}-k\right)\,dx\\
&+\gamma q\int\limits_{A_{k,s}}\left[\left(\sqrt{\sum_{\alpha=1}^{3}\left(u^{\alpha}\right)^{2}}+1\right)^{2}+\left(\sqrt{\sum_{\alpha=1}^{3}\left(\left|\nabla u^{\alpha}\right|\right)^{2}}+1\right)^{2}\right]^{\frac{q-1}{2}}\cdot\left|\nabla\eta\right|\eta^{\gamma-1}\left(u^{1}-k\right)\,dx
\end{aligned}
\tag{2.8}
$$

Now, using Young Inequality, it follows

$$
\begin{aligned}
&\int\limits_{A_{k,s}}\left[\sum_{\alpha=1}^{3}\left(u^{\alpha}-\left|\nabla u^{\alpha}\right|\right)^{2}+1\right]^{\frac{q-1}{2}}\eta^{\gamma}\left(u^{1}-k\right)\,dx\\
&\le\int\limits_{A_{k,s}}\left[\sum_{\alpha=1}^{3}\left(u^{\alpha}-\left|\nabla u^{\alpha}\right|\right)^{2}+1\right]^{\frac{q}{2}}\eta^{\gamma}\,dx+\int\limits_{A_{k,s}}\eta^{\gamma}\left(u^{1}-k\right)^{q}\,dx,
\end{aligned}
\tag{2.9}
$$

$$\int_{A_{k,s}} \left[\left(\sqrt{\sum_{\alpha=1}^{3} (u^\alpha)^2} + 1 \right)^2 + \left(\sqrt{\sum_{\alpha=1}^{3} (|\nabla u^\alpha|)^2} + 1 \right)^2 \right]^{\frac{q-1}{2}} \eta^\gamma \, (u^1 - k) \, dx$$
$$\leq \int_{A_{k,s}} \left[\left(\sqrt{\sum_{\alpha=1}^{3} (u^\alpha)^2} + 1 \right)^2 + \left(\sqrt{\sum_{\alpha=1}^{3} (|\nabla u^\alpha|)^2} + 1 \right)^2 \right]^{\frac{q}{2}} \eta^\gamma \, dx$$
$$+ \int_{A_{k,s}} \eta^\gamma \, (u^1 - k)^q \, dx, \tag{2.10}$$

$$\int_{A_{k,s}} \left[\sum_{\alpha=1}^{3} (u^\alpha - |\nabla u^\alpha|)^2 + 1 \right]^{\frac{q-1}{2}} \left[|\nabla u^1| \cdot \eta^\gamma + \gamma \, |\nabla \eta| \, \eta^{\gamma-1} \, (u^1 - k) \right] dx$$
$$\leq \int_{A_{k,s}} \left[\sum_{\alpha=1}^{3} (u^\alpha - |\nabla u^\alpha|)^2 + 1 \right]^{\frac{q-1}{2}} |\nabla u^1| \cdot \eta^\gamma \, dx$$
$$+\gamma \int_{A_{k,s}} \left[\sum_{\alpha=1}^{3} (u^\alpha - |\nabla u^\alpha|)^2 + 1 \right]^{\frac{q-1}{2}} |\nabla \eta| \, \eta^{\gamma-1} \, (u^1 - k) \, dx$$
$$\leq \int_{A_{k,s}} \left[\sum_{\alpha=1}^{3} (u^\alpha - |\nabla u^\alpha|)^2 + 1 \right]^{\frac{q}{2}} \eta^\gamma \, dx + \int_{A_{k,s}} |\nabla u^1|^q \, \eta^\gamma \, dx$$
$$+\gamma \int_{A_{k,s}} \left[\sum_{\alpha=1}^{3} (u^\alpha - |\nabla u^\alpha|)^2 + 1 \right]^{\frac{q}{2}} \eta^{\gamma-1} \, dx + \gamma \int_{A_{k,s}} \left[|\nabla \eta| \, (u^1 - k) \right]^q \, dx, \tag{2.11}$$

$$\int_{A_{k,s}} \left[\left(\sqrt{\sum_{\alpha=1}^{3} (u^\alpha)^2} + 1 \right)^2 + \left(\sqrt{\sum_{\alpha=1}^{3} (|\nabla u^\alpha|)^2} + 1 \right)^2 \right]^{\frac{q-1}{2}} \cdot |\nabla \eta| \, \eta^{\gamma-1} \, (u^1 - k) \, dx$$
$$\leq \int_{A_{k,s}} \left[\left(\sqrt{\sum_{\alpha=1}^{3} (u^\alpha)^2} + 1 \right)^2 + \left(\sqrt{\sum_{\alpha=1}^{3} (|\nabla u^\alpha|)^2} + 1 \right)^2 \right]^{\frac{q}{2}} \cdot \, \eta^{\gamma-1} \, dx$$
$$+ \int_{A_{k,s}} \left[|\nabla \eta| \, \eta^{\gamma-1} \, (u^1 - k) \right]^q \, dx \tag{2.12}$$

and, sine $\mathrm{supp}\Big(|\nabla u^1|^{p-1} \, |\nabla \eta| \, \eta^{\gamma-1} \, (u^1 - k) \Big) \subset A_{k,s} \backslash A_{k,t}$, we have

$$\begin{aligned} &\int_{A_{k,s}} |\nabla u^1|^{p-1} \, |\nabla \eta| \, \eta^{\gamma-1} \, (u^1 - k) \, dx \\ &= \int_{A_{k,s} A_{k,t}} |\nabla u^1|^{p-1} \, |\nabla \eta| \, \eta^{\gamma-1} \, (u^1 - k) \, dx \\ &\leq \int_{A_{k,s} A_{k,t}} |\nabla u^1|^p \, \eta^{(\gamma-1)\left(\frac{p}{p-1}\right)} \, dx + \int_{A_{k,s} A_{k,t}} \left[|\nabla \eta| \, (u^1 - k) \right]^p \, dx \end{aligned} \tag{2.13}$$

Using the previous relations (2.8), (2.9), (2.10), (2.11), (2.12) and (2.13) we get

$$
\begin{aligned}
&\int_{A_{k,s}} \eta^{\gamma} \left|\nabla u^1\right|^p \, dx \\
&\leq 2q \int_{A_{k,s}} \left[\sum_{\alpha=1}^{3} \left(u^{\alpha} - \left|\nabla u^{\alpha}\right|\right)^2 + 1\right]^{\frac{q}{2}} \eta^{\gamma} \, dx + 2q \int_{A_{k,s}} \eta^{\gamma} \, \left(u^1 - k\right)^q \, dx \\
&+q \int_{A_{k,s}} \left[\left(\sqrt{\sum_{\alpha=1}^{3} \left(u^{\alpha}\right)^2} + 1\right)^2 + \left(\sqrt{\sum_{\alpha=1}^{3} \left(\left|\nabla u^{\alpha}\right|\right)^2} + 1\right)^2\right]^{\frac{q}{2}} \eta^{\gamma} \, dx \\
&+q \int_{A_{k,s}} \left|\nabla u^1\right|^q \eta^{\gamma} \, dx + q\gamma \int_{A_{k,s}} \left[\sum_{\alpha=1}^{3} \left(u^{\alpha} - \left|\nabla u^{\alpha}\right|\right)^2 + 1\right]^{\frac{q}{2}} \eta^{\gamma-1} \, dx + q\gamma \int_{A_{k,s}} \left[\left|\nabla \eta\right| \, \left(u^1 - k\right)\right]^q \, dx \\
&+\gamma \int_{A_{k,s}A_{k,t}} \left|\nabla u^1\right|^p \, \eta^{(\gamma-1)\left(\frac{p}{p-1}\right)} \, dx + \gamma \int_{A_{k,s}A_{k,t}} \left[\left|\nabla \eta\right| \, \left(u^1 - k\right)\right]^p \, dx \\
&+\gamma q \int_{A_{k,s}} \left[\left(\sqrt{\sum_{\alpha=1}^{3} \left(u^{\alpha}\right)^2} + 1\right)^2 + \left(\sqrt{\sum_{\alpha=1}^{3} \left(\left|\nabla u^{\alpha}\right|\right)^2} + 1\right)^2\right]^{\frac{q}{2}} \cdot \, \eta^{\gamma-1} \, dx \\
&+\gamma q \int_{A_{k,s}} \left[\left|\nabla \eta\right| \, \eta^{\gamma-1} \, \left(u^1 - k\right)\right]^q \, dx
\end{aligned}
\tag{2.14}
$$

Choose $\gamma = p$, remembering the properties of η and using Hölder Inequality it follows

$$
\begin{aligned}
&\int_{A_{k,s}} \eta^{\gamma} \left|\nabla u^1\right|^p \, dx \\
&\leq (2+\gamma)\, q \left[\int_{A_{k,s}} \left[\sum_{\alpha=1}^{3} \left(u^{\alpha} - \left|\nabla u^{\alpha}\right|\right)^2 + 1\right]^{\frac{p}{2}} \, dx\right]^{\frac{q}{p}} \left[\mathcal{L}^3\left(A_{k,s}\right)\right]^{1-\frac{q}{p}} + 2q \int_{A_{k,s}} \eta^{\gamma} \, \left(u^1 - k\right)^q \, dx \\
&+q\,(1+\gamma) \left[\int_{A_{k,s}} \left[\left(\sqrt{\sum_{\alpha=1}^{3} \left(u^{\alpha}\right)^2} + 1\right)^2 + \left(\sqrt{\sum_{\alpha=1}^{3} \left(\left|\nabla u^{\alpha}\right|\right)^2} + 1\right)^2\right]^{\frac{p}{2}} \, dx\right]^{\frac{q}{p}} \left[\mathcal{L}^3\left(A_{k,s}\right)\right]^{1-\frac{q}{p}} \\
&+q \left[\int_{A_{k,s}} \left|\nabla u^1\right|^p \eta^p \, dx\right]^{\frac{q}{p}} \left[\mathcal{L}^3\left(A_{k,s}\right)\right]^{1-\frac{q}{p}} + q\gamma \int_{A_{k,s}} \left[\left|\nabla \eta\right| \, \left(u^1 - k\right)\right]^q \, dx \\
&+\gamma \int_{A_{k,s}A_{k,t}} \left|\nabla u^1\right|^p \, \eta^p \, dx + \gamma \int_{A_{k,s}A_{k,t}} \left[\left|\nabla \eta\right| \, \left(u^1 - k\right)\right]^p \, dx + \gamma q \int_{A_{k,s}} \left[\left|\nabla \eta\right| \, \left(u^1 - k\right)\right]^q \, dx
\end{aligned}
\tag{2.15}
$$

Since

$$\begin{aligned}&\int_{A_{k,s}} \left[\sum_{\alpha=1}^{3} (u^\alpha - |\nabla u^\alpha|)^2 + 1\right]^{\frac{p}{2}} dx \\ &\le 6^{\frac{p}{2}} \int_{A_{k,s}} \left[|u|^2 + |\nabla u|^2 + 1\right]^{\frac{p}{2}} dx \\ &\le 6^{2p} \left[\int_{A_{k,s}} |u|^p \, dx + \int_{A_{k,s}} |\nabla u|^p \, dx + \mathcal{L}^3(A_{k,s})\right]\end{aligned} \tag{2.16}$$

and

$$\begin{aligned}&\int_{A_{k,s}} \left[\left(\sqrt{\sum_{\alpha=1}^{3} (u^\alpha)^2 + 1}\right)^2 + \left(\sqrt{\sum_{\alpha=1}^{3} (|\nabla u^\alpha|)^2 + 1}\right)^2\right]^{\frac{p}{2}} dx \\ &\le 4^{\frac{p}{2}} \int_{A_{k,s}} \left[|u|^2 + |\nabla u|^2 + 1\right]^{\frac{p}{2}} dx \\ &\le 6^{2p} \left[\int_{A_{k,s}} |u|^p \, dx + \int_{A_{k,s}} |\nabla u|^p \, dx + \mathcal{L}^3(A_{k,s})\right]\end{aligned} \tag{2.17}$$

then we get

$$\begin{aligned}&\int_{A_{k,s}} \eta^\gamma |\nabla u^1|^p \, dx \\ &\le 6^{2q}(2+p)(2q)\left[\int_{A_{k,s}} |u|^p \, dx + \int_{A_{k,s}} |\nabla u|^p \, dx + \mathcal{L}^3(A_{k,s})\right]^{\frac{q}{p}} \left[\mathcal{L}^3(A_{k,s})\right]^{1-\frac{q}{p}} + 2q \int_{A_{k,s}} \eta^p \left(u^1 - k\right)^q dx \\ &+ q\left[\int_{A_{k,s}} |\nabla u^1|^p \eta^p \, dx\right]^{\frac{q}{p}} \left[\mathcal{L}^3(A_{k,s})\right]^{1-\frac{q}{p}} + qp \int_{A_{k,s}} \left[|\nabla \eta| \; (u^1 - k)\right]^q dx \\ &+ p \int_{A_{k,s} A_{k,t}} |\nabla u^1|^p \, \eta^p \, dx + p \int_{A_{k,s} A_{k,t}} \left[|\nabla \eta| \; (u^1 - k)\right]^p dx + pq \int_{A_{k,s}} \left[|\nabla \eta| \; (u^1 - k)\right]^q dx\end{aligned} \tag{2.18}$$

But

$$\left[\int_{A_{k,s}} \left|\nabla u^1\right|^p \eta^p \, dx\right]^{\frac{q}{p}} \le \left[\int_{A_{k,s}} |u|^p \, dx + \int_{A_{k,s}} |\nabla u|^p \, dx + \mathcal{L}^3(A_{k,s})\right]^{\frac{q}{p}} \tag{2.19}$$

and, by Young Inequality,

$$\int_{A_{k,s}} \eta^p \left(u^1 - k\right)^q dx \le \int_{A_{k,s}} \eta^p \left(u^1 - k\right)^p + \eta^p \, dx \tag{2.20}$$

and

$$\int_{A_{k,s}} \left[|\nabla \eta| \; \left(u^1 - k\right)\right]^q dx \le \int_{A_{k,s}} \left[|\nabla \eta| \; \left(u^1 - k\right)\right]^p + 1 \, dx \tag{2.21}$$

then, using (2.18), (2.19), (2.20) and (2.21), it follows

$$\int_{A_{k,s}} \eta^p \left|\nabla u^1\right|^p dx$$

$$\leq 6^{2q}(2+p)(3q)\left[\int_{A_{k,s}} |u|^p\ dx + \int_{A_{k,s}} |\nabla u|^p\ dx + \mathcal{L}^3(A_{k,s})\right]^{\frac{q}{p}} \left[\mathcal{L}^3(A_{k,s})\right]^{1-\frac{q}{p}} + 2q\int_{A_{k,s}} \eta^p\ (u^1-k)^p + \eta^p\, dx$$

$$+2qp\int_{A_{k,s}} \left[|\nabla\eta|\ (u^1-k)\right]^p + 1\, dx$$

$$+p\int_{A_{k,s}A_{k,t}} \left|\nabla u^1\right|^p \eta^p\ dx + p\int_{A_{k,s}A_{k,t}} \left[|\nabla\eta|\ (u^1-k)\right]^p\ dx \tag{2.22}$$

then, remembering the properties of η, with simple increases we have

$$\begin{aligned}&\int_{A_{k,t}} \left|\nabla u^1\right|^p dx \leq p\int_{A_{k,s}A_{k,t}} \left|\nabla u^1\right|^p\ \ dx + 6pq2^p\int_{A_{k,s}} \left[\frac{(u^1-k)}{s-t}\right]^p dx\\ &+C_\Omega\left[\mathcal{L}^3(A_{k,s})\right]^{1-\frac{q}{p}}\end{aligned} \tag{2.23}$$

where

$$C_\Omega = 6^{2q}(2+p)(4q)\left[\int_\Omega |u|^p\ \ dx + \int_\Omega |\nabla u|^p\ \ dx + \mathcal{L}^3(\Omega)\right]^{\frac{q}{p}} \tag{2.24}$$

From (2.23) it follows

$$\begin{aligned}&\int_{A_{k,t}} \left|\nabla u^1\right|^p dx \leq \tfrac{p}{p+1}\int_{A_{k,s}A_{k,t}} \left|\nabla u^1\right|^p\ \ dx + \tfrac{6pq2^p}{p+1}\int_{A_{k,s}} \left[\frac{(u^1-k)}{s-t}\right]^p dx\\ &+\tfrac{C_\Sigma}{p+1}\left[\mathcal{L}^3(A_{k,s})\right]^{1-\frac{q}{p}}\end{aligned} \tag{2.25}$$

then by applying the lemma 6.1 of [12] we obtain the following Caccioppoli inequality

$$\int_{A_{k,\varrho}} \left|\nabla u^1\right|^p dx \leq D_1\int_{A_{k,R}} \left[\frac{(u^1-k)}{R-\varrho}\right]^p dx + D_2\left[\mathcal{L}^3(A_{k,s})\right]^{1-\frac{q}{p}} \tag{2.26}$$

where D_1 and D_2 are two positive real constants, independent from x_0 but dependent on the initial data q, p, n and Σ. Similarly, it can be shown that the following Caccioppoli inequality holds

$$\int_{B_{k,\varrho}} \left|\nabla u^1\right|^p dx \leq D_1\int_{B_{k,R}} \left[\frac{(k-u^1)}{R-\varrho}\right]^p dx + D_2\left[\mathcal{L}^3(B_{k,s})\right]^{1-\frac{q}{p}} \tag{2.27}$$

where $B_{k,\varrho} = \{u^1 < k\} \cap B_\varrho(x_0)$. Moreover, from the particular form of the functional we deduce that the previous inequalities also hold for functions u^2 and u^3. Since $1 \leq q < \frac{p^2}{3} < 3$ then the theses of the Theorem 2 hold.

References

[1] D. Breit, B. Stroffolini, A. Verde, A general regularity theorem for functionals with φ-growth, *J. Math. Anal. Appl.*, **383** (2011), 226-233. https://doi.org/10.1016/j.jmaa.2011.05.012

[2] G. Cupini, M. Focardi, F. Leonetti, E. Mascolo, On the Holder continuity for a class of vectorial problems, *Advances in Nonlinear Analysis*, **9** (1) (2019), 1008-1025. https://doi.org/10.1515/anona-2020-0039

[3] G. Cupini, F. Leonetti, E. Mascolo, Local boundedness for minimizers of some polyconvex integrals, *Arch. Rational Mech. Anal.*, **224** (1) (2017), 269-289. https://doi.org/10.1007/s00205-017-1074-7

[4] G. Cupini, P. Marcellini, E. Mascolo, Regularity under sharp anisotropic general growth conditions, *Discrete Contin. Dyn. Syst.*, Ser. B, **11** (1), (2009), 66-86. https://doi.org/10.3934/dcdsb.2009.11.67

[5] G. Cupini, P. Marcellini, E. Mascolo, Local boundedness of solutions to some anisotropic elliptic systems, *Contemporary Mathematics*, **595** (2013), 169-186. https://doi.org/10.1090/conm/595/11803

[6] G. Cupini, P. Marcellini, E. Mascolo, Local boundedness of solutions to quasilinear elliptic systems, *Manuscripta Mathematica*, **137** (2012), 287-315. https://doi.org/10.1007/s00229-011-0464-7

[3] E. De Giorgi, Sulla differenziabilita e l'analicita delle estremali degli integrali multipli regolari, *Mem. Accad. Sci. Torino* (Classe di Sci. mat. fis. e nat.), **3** (3) (1957), 25-43.

[4] E. De Giorgi, Un esempio di estremali discontinue per un problema variazionale di tipo ellittico, *Boll. U.M.I.*, **4** (1968), 135-137.

[6] L. Diening, B. Stro olini, A. Verde, Everywhere regularity of functional with '-growth, *Manus. Math.*, **129** (2009), 449-481. https://doi.org/10.1007/s00229-009-0277-0

[7] M. M. Dougherty, D. Phillips, Higher gradient integrability of equilibria for certain rank-one convex integrals, *SIAM J. Math. Anal.*, **28** (1997), 270-273. https://doi.org/10.1137/s0036141095293384

[8] J. Frehse, A discontinuous solution of a mildly nonlinear system, *Math. Z.*, **124** (1973), 229-230. https://doi.org/10.1007/bf01214096

[9] M. Fuchs, Local Lipschitz regularity of vector valued local minimizers of variational integrals with densities depending on the modulus of the gradient, *Math. Nachr.*, **284** (2011), 266-272. https://doi.org/10.1002/mana.200810189

[10] M. Giaquinta, E. Giusti, On the regularity of minima of variational integrals, *Acta Mathematica*, **148** (1983), 285-298.

[11] E. Giusti, M. Miranda, Un esempio di soluzioni discontinue per un problema di minimo relativo ad un integrale regolare del calcolo delle variazioni, *Boll. U.M.I.*, **2** (1968), 1-8.

[12] E. Giusti, *Metodi diretti nel Calcolo delle Variazioni*, U. M. I., Bologna, 1994.

[13] P. Marcellini, Everywhere regularity for a class of elliptic systems without growth conditions, *Ann. Sc. Norm. Super. Pisa*, **23** (1996), 1-25.

[14] J. Moser, A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations, *Comm. Pure Appl. Math.*, **14** (1961), 457-468. https://doi.org/10.1002/cpa.3160130308

[15] J. Nash, Continuity of solution of parabolic and elliptic equations, *Amer. J. of Math.*, **80** (1958), 931-954. https://doi.org/10.2307/2372841

[16] P. Tolksdorf, A new proof of a regularity theorem, *Invent. Math.*, **71** (1) (1983), 43-49. https://doi.org/10.1007/bf01393338

[17] P. Tolksdorf, Regularity for a More General Class of Quasilinear Elliptic Equations, *J. of Differ. Equ.*, **51** (1984), 126-150. https://doi.org/10.1016/0022-0396(84)90105-0

[18] K. Uhlenbeck, Regularity for a class of nonlinear elliptic systems, *Acta Math.*, **138** (1977), 219-240. https://doi.org/10.1007/bf02392316

Received: September 1, 2021; Published: September 23, 2021