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Four Parametric Extension of the Laplace Transform

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Abstract

In this paper, a four parametric extension of the Laplace transform is introduced. The relation between the extended transform and the classical Laplace transform is established. Moreover, the extended transforms of a number of functions like constant, polynomials, exponential, trigonometric and hyperbolic trigonometric functions, are found using fundamental theorems of integral calculus.

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1 Introduction

During the study of probability theory, the Laplace transform was firstly introduced by Pierre Simon Laplace in 1782 and has become a hot topic among the researchers from the 19th century due to extensive applications in many areas of Mathematics, Physics, Chemistry, Economics and Engineering. For $\mathrm{Re}(\alpha) > 0$; The Laplace transform L of the function $\phi(v)$ is defined as

(1) $$L\{\phi(t)\} = \int_0^\infty e^{-st}\phi(t)dt = \Phi(s)$$

(see [5]). For $\phi^{(0)}(t) = \phi(t)$ and $\phi^{(k)}(t) = \frac{d^k}{dt^k}\phi(t),\ k \in N$; the Laplace transforms $\Phi(s)$ of some functions $\phi(t)$ are listed below:

$\phi(t)$	$\Phi(s)$	$\phi(t)$	$\Phi(s)$
c	$\frac{c}{s}, s>0$	$\sinh(\alpha t)-\sin(\alpha t)$	$\frac{2\alpha^3}{s^4-\alpha^4}$
t	$\frac{c}{s}, s>0$	$\cosh(\alpha t)-\cos(\alpha t)$	$\frac{2\alpha^2 s}{s^4-\alpha^4}$
$t^n, n\geq 0$	$\frac{n!}{s^{n+1}}, s>0$	$1-\cos(\alpha t)$	$\frac{\alpha^2}{s(s^2+\alpha^2)}$
$e^{\alpha t}$	$\frac{1}{s-\alpha}, s>\alpha$	$\alpha t-\sin(\alpha t)$	$\frac{\alpha^3}{s^2(s^2+\alpha^2)}$
$te^{\alpha t}$	$\frac{1}{(s-\alpha)^2}$	$\phi(t)+\psi(t)$	$\Phi(t)+\Psi(t)$
$t^n e^{\alpha t}$	$\frac{n!}{(s-\alpha)^{n+1}}, s>\alpha$	$u_\alpha(t)\phi(t-\alpha)$	$e^{-\alpha s}\Phi(s)$
$\sin(\alpha t)$	$\frac{\alpha}{s^2+\alpha^2}, s>0$	$\phi(ct)$	$\frac{1}{c}\Phi(\frac{s}{c}), c>0$
$\cos(\alpha t)$	$\frac{s}{s^2+\alpha^2}, s>0$	$\int_0^t \phi(u)\,du$	$\frac{1}{s}\Phi(s)$
$\sinh(\alpha t)$	$\frac{\alpha}{s^2-\alpha^2}, s>\lvert\alpha\rvert$	$u_\alpha(t)$	$\frac{e^{-\alpha s}}{s}$
$\cosh(\alpha t)$	$\frac{s}{s^2-\alpha^2}, s>\lvert\alpha\rvert$	$t^n\phi(t)$	$(-1)\frac{d^n}{ds^n}\Phi(s)$
$t\sin(\alpha t)$	$\frac{2\alpha s}{(s^2+\alpha^2)^2}, s>0$	$\frac{\phi(t)}{t}$	$\int_s^\infty \Phi(u)\,du$
$t\cos(\alpha t)$	$\frac{s^2-\alpha^2}{(s^2+\alpha^2)^2}, s>0$	$\phi^{(n)}(t)$	$s^n\,\Phi(s)-\sum_{i=1}^{n} s^{n-i}\phi^{(i-1)}(0)$,

A number of properties and applications of the Laplace transform in real life have been discussed involving mass spring damper system, chemical pollution in a reservoir and transfer function of control system etc (see [6]). The solution of population growth and decay problems have been found which arises in the field of Physics, Chemistry, Biology and Social Sciences etc using the Laplace transform [8]. We refer [1], [2], [3], [4] and [7] for the study of the integral transforms.

Along with other basic properties, the following properties of definite integral are used in the later results

(2) $$\int_{c_1}^{c_2}\phi(x)dx=\int_{c_1}^{c_2}\phi(y)dy$$

and

(3) $$\int_{c_1}^{c_2} \phi(x)dx = \int_{c_1}^{c_3} \phi(x)dx + \int_{c_3}^{c_2} \phi(x)dx, c_3 \in \left]c_1, c_2\right[.$$

Definition 1.1 The number of arrangements of k distinct things taken r (distinct things) at a time, $0 < r \le k,$ is denoted by ${}^k p_r$ and is given by

$$ {}^k P_r = \frac{k!}{(k-r)!}$$

and is in fact the number of arrangements a set of k distinct objects, each taken r at a time, without repetition of an object in an arrangement. One may observe that ${}^k P_k = k!$.

2 The Extended Laplace Transform

Definition 2.1 For $(a,b,c,d) \in R^4$, $c > 0$; we define

(4) $$L_{(a,b,c.d)}\{\phi(t)\} = \int_0^\infty e^{-(as+b)t}\phi(ct+d)dt = \Phi_{(a,b,c,d)}(s).$$

Definition 2.2 If $(a,b,c,d) \in R^4$ and $c, d > 0$, for a function $\phi(t)$; we define a new integral transform I as

(5) $$I\left\{\phi(t)\right\} = \int_0^d e^{-\frac{as+b}{c}t}\phi(t)dt.$$

Definition 2.3 If $(a,b,c,d) \in R^4$ and $c > 0$,

(i) for a real number s; we define s' as

(6) $$s' = \frac{as+b}{c}$$

and

(ii) we define e' as

(7) $$e' = \frac{e^{\frac{d}{c}(as+b)}}{c}.$$

3 Main Results

Firstly, we establish the relationship between the classical and the extended Laplace transforms.

Theorem 3.1 If $\Phi(s)$, $\Phi_{(a,b,c,d)}(s)$, I, s' and e' are defined as in (1), (4), (5), (6) and (7) respectively, then $\Phi_{(a,b,c,d)}(s) = e'\left[\Phi(s') - I\left\{\phi(t)\right\}\right]$.

Proof: For $ct + d = x$; (4) becomes

$$L_{(a,b,c,d)}\{\phi(t)\}=\int_d^{\infty}e^{-(as+b)\frac{x-d}{c}}\phi(x)\frac{1}{c}dx=\frac{1}{c}\int_d^{\infty}e^{-(as+b)\frac{t-d}{c}}\phi(t)dt.$$

Using (2), (3), the above relation leads to the relation

$$L_{(a,b,c,d)}\{\phi(t)\}=\frac{e^{\frac{d}{c}(as+b)}}{c}\left[\Phi(\frac{as+b}{c})-I\{\phi(t)\}\right],$$

which, with collaboration of (6) and (7), leads to the required result.
One may find the values of
$I\{\phi(t)\}$ for $\phi(t)=\alpha$, $\phi(t)=e^{\alpha t}$, $\phi(t)=te^{\alpha t}$, $\phi(t)=t^n e^{\alpha t}$, $\phi(t)=e^{\beta t}\cos(\alpha t)$, $\phi(t)=t\cos(\alpha t)$, $\phi(t)=e^{\beta t}\sin(\alpha t)$, $\phi(t)=t\sin(\alpha t)$, $\phi(t)=\sinh(\alpha t)$, $\phi(t)=\cosh(\alpha t)$, etc, easily by use of basic theorems of definite integrals. The results are listed below:

3.1 The Newly Defined Integral Transform of Some Elementary Functions

The integral transform $I\{\phi(t)\}$, defined in (5), of $\phi(t)$, for different values of $\phi(t)$, is listed in the following table:

$\phi(t)$	$I\{\phi(t)\}$
1	$\frac{1}{s'}[1-\frac{1}{ce'}]$
t	$-\frac{d}{cs'e'}-\frac{1}{ce'(s')^2}+\frac{1}{(s')^2}$
t^n	$-\frac{1}{ce'}\sum_{i=0}^{n-1}\left(\frac{1}{s'}\right)^{i+1}d^{n-i}\,{}^nP_i-{}^nP_{n-1}\left(\frac{1}{s'}\right)^{n+1}\left(\frac{1}{ce'}-1\right)$
$e^{\alpha t}$	$\frac{c}{as+b-c\alpha}-\frac{ce^{-\frac{d}{c}(as+b-c\alpha)}}{(as+b-c\alpha)}$
$t^n e^{\alpha t}$	$-e^{-\frac{d}{c}(as+b-c\alpha)}\sum_{i=0}^{n}\left(\frac{c}{as+b-c\alpha}\right)^{i+1}d^{n-i}\,{}^nP_i+n!\left(\frac{c}{as+b-c\alpha}\right)^{n+1}$
$\sin(\alpha t)$	$-\frac{(as+b)\sin(\alpha d)}{[(as+b)^2+c^2\alpha^2]e'}-\frac{c\alpha\cos(\alpha d)}{[(as+b)^2+c^2\alpha^2]e'}+\frac{c^2\alpha}{(as+b)^2+c^2\alpha^2}$
$\cos(\alpha t)$	$-\frac{(as+b)\cos(\alpha d)}{[(as+b)^2+c^2\alpha^2]e'}+\frac{c\alpha\sin(\alpha d)}{[(as+b)^2+c^2\alpha^2]e'}+\frac{c(as+b)}{(as+b)^2+c^2\alpha^2}$
$\sinh(\alpha t)$	$-\frac{(as+b)\sinh(\alpha d)}{[(as+b)^2-c^2\alpha^2]e'}-\frac{c\alpha\cosh(\alpha d)}{[(as+b)^2-c^2\alpha^2]e'}+\frac{c^2\alpha}{(as+b)^2-c^2\alpha^2}$

$\cosh(\alpha t)$	$-\frac{(as+b)\cosh(\alpha d)}{[(as+b)^2-c^2\alpha^2]e'}-\frac{c\alpha\sinh(\alpha d)}{[(as+b)^2-c^2\alpha^2]e'}+\frac{c(as+b)}{(as+b)^2-c^2\alpha^2}$
$t\sin(\alpha t)$	$-\frac{(as+b)d\sin(\alpha d)}{[(as+b)^2+c^2\alpha^2]e'}-\frac{c\alpha d\cos(\alpha d)}{[(as+b)^2+c^2\alpha^2]e'}+\frac{2c^3\alpha(as+b)}{((as+b)^2+c^2\alpha^2)^2}$ $-\frac{2c^2\alpha(as+b)\cos(\alpha d)}{[((as+b)^2+c^2\alpha^2)^2]e'}+\frac{c^3\alpha^2\sin(\alpha d)}{[(as+b)^2+c^2\alpha^2]e'}-\frac{c(as+b)^2\sin(\alpha d)}{[((as+b)^2+c^2\alpha^2)^2]e'}$
$t\cos(\alpha t)$	$-\frac{d\cos(\alpha d)}{cs'e'}-\frac{c\alpha d\sin(\alpha d)}{[(as+b)^2+c^2\alpha^2]e'}-\frac{c^2\alpha^2 d\cos(\alpha d)}{[(as+b)((as+b)^2+c^2\alpha^2)]e'}$ $+\frac{2c^4\alpha^2}{((as+b)^2+c^2\alpha^2)^2}-\frac{2c^3\alpha^2\cos(\alpha d)}{[((as+b)^2+c^2\alpha^2)^2]e'}+\frac{c^5\alpha^3\sin(\alpha d)}{c[(as+b)((as+b)^2+c^2\alpha^2)]e'}$ $-\frac{c^2\alpha(as+b)\sin(\alpha d)}{[((as+b)^2+c^2\alpha^2)^2]e'}+\frac{c\cos(\alpha d)}{[(as+b)^2+c^2\alpha^2]e'}-\frac{\alpha c^2\sin(\alpha d)}{[(as+b)((as+b)^2+c^2\alpha^2)]e'}$ $-\frac{c^2}{(as+b)^2+c^2\alpha^2}$
$\sinh(\alpha t)-\sin(\alpha t)$	$-\frac{\sinh(\alpha d)(as+b)}{[(as+b)^2-c^2\alpha^2]e'}+\frac{c^2\alpha}{(as+b)^2-c^2\alpha^2}-\frac{c\alpha\cos(\alpha d)}{[(as+b)^2-c^2\alpha^2]e'}$ $+\frac{(as+b)\sin(\alpha d)}{[(as+b)^2+c^2\alpha^2]e'}-\frac{c^2\alpha}{(as+b)^2+c^2\alpha^2}+\frac{c\alpha\cos(\alpha d)}{[(as+b)^2+c^2\alpha^2]e'}$
$\cosh(\alpha t)-\cos(\alpha t)$	$-\frac{(as+b)\cosh(\alpha d)}{[(as+b)^2-c^2\alpha^2]e'}-\frac{c\alpha\sinh(\alpha d)}{[(as+b)^2-c^2\alpha^2]e'}+\frac{c(as+b)}{(as+b)^2-c^2\alpha^2}$ $+\frac{(as+b)\cos(\alpha d)}{[(as+b)^2+c^2\alpha^2]e'}-\frac{c\alpha\sin(\alpha d)}{[(as+b)^2+c^2\alpha^2]e'}-\frac{c(as+b)}{(as+b)^2+c^2\alpha^2}$
$1-\cos(\alpha t)$	$\frac{ce'-1}{cs'e'}+\frac{(as+b)\cos(\alpha d)}{[(as+b)^2+c^2\alpha^2]e'}-\frac{c\alpha\sin(\alpha d)}{[(as+b)^2+c^2\alpha^2]e'}-\frac{c(as+b)}{(as+b)^2+c^2\alpha^2}$
$\alpha t-\sin(\alpha t)$	$-\frac{\alpha d}{(as+b)e'}-\frac{c\alpha}{(as+b)^2e'}+\frac{\alpha c^2}{(as+b)^2}+\frac{(as+b)\sin(\alpha d)}{[(as+b)^2+c^2\alpha^2]e'}$ $+\frac{c\alpha\cos(\alpha d)}{[(as+b)^2+c^2\alpha^2]e'}-\frac{c^2\alpha}{(as+b)^2+c^2\alpha^2}$

Using the results mentioned above in (3.1), (3.2), (6) and (7), respectively, we obtain the results listed below:

3.2 Four Parametric Laplace Transform of Some Functions

The four parametric extended Laplace transform $\Phi_{(a,b,c,d)}(s)$, defined in (4), of $\phi(t)$, for different values of $\phi(t)$, is listed in the following table:

$\phi(t)$	$\Phi_{(a,b,c,d)}(s)$
1	$\frac{1}{as+b},\ as+b>0$
t	$\frac{d}{as+b}+\frac{c}{(as+b)^2},as+b>0$
t^n	$-\frac{1}{ce'}\sum_{i=0}^{n-1}\left(\frac{1}{s'}\right)^{i+1}d^{n-i}\,{}^nP_i-{}^nP_{n-1}\left(\frac{1}{s'}\right)^{n+1}\left(\frac{1}{ce'}-1\right)$
$e^{\alpha t}$	$\frac{e^{d\alpha}}{as+b-c\alpha},as+b>c\alpha$
$t^n e^{\alpha t}$	$-e^{-\frac{d}{c}(as+b-c\alpha)}\sum_{i=0}^{n}\left(\frac{c}{as+b-c\alpha}\right)^{i+1}d^{n-i}\,{}^nP_i+n!\left(\frac{c}{as+b-c\alpha}\right)^{n+1},as+b>c\alpha$
$\sin(\alpha t)$	$\frac{(as+b)\sin(\alpha d)}{(as+b)^2+c^2\alpha^2}+\frac{c\alpha\cos(\alpha d)}{(as+b)^2+c^2\alpha^2},as+b>0$
$\cos(\alpha t)$	$\frac{(as+b)\cos(\alpha d)}{(as+b)^2+c^2\alpha^2}-\frac{c\alpha\sin(\alpha d)}{(as+b)^2+c^2\alpha^2},as+b>0$
$\sinh(\alpha t)$	$\frac{(as+b)\sinh(\alpha d)}{(as+b)^2-c^2\alpha^2}+\frac{c\alpha\cosh(\alpha d)}{(as+b)^2-c^2\alpha^2},as+b>\lvert c\alpha\rvert$
$\cosh(\alpha t)$	$\frac{(as+b)\cosh(\alpha d)}{(as+b)^2-c^2\alpha^2}+\frac{c\alpha\sinh(\alpha d)}{(as+b)^2-c^2\alpha^2},as+b>\lvert c\alpha\rvert$
$t\sin(\alpha t)$	$\frac{(as+b)d\sin(\alpha d)}{(as+b)^2+c^2\alpha^2}+\frac{c\alpha d\cos(\alpha d)}{(as+b)^2+c^2\alpha^2}+\frac{2c^2\alpha(as+b)\cos(\alpha d)}{((as+b)^2+c^2\alpha^2)^2}-\frac{c^3\alpha^2\sin(\alpha d)}{(as+b)^2+c^2\alpha^2}+\frac{c(as+b)^2\sin(\alpha d)}{[(as+b)^2+c^2\alpha^2]^2},as+b>0$
$t\cos(\alpha t)$	$\frac{c^2e'((as+b)^2-c^2\alpha^2)}{(as+b)^2+c^2\alpha^2}+\frac{d\cos(\alpha d)}{as+b}+\frac{c\alpha d\sin(\alpha d)}{(as+b)^2+c^2\alpha^2)}+\frac{c\alpha^2d\cos(\alpha d)}{s'((as+b)^2+c^2\alpha^2)}-\frac{2c^4\alpha^2e'}{((as+b)^2+c^2\alpha^2)^2}+\frac{2c^3\alpha^2\cos(\alpha d)}{((as+b)^2+c^2\alpha^2)^2}-\frac{c^3\alpha^3\sin(\alpha d)}{s'((as+b)^2+c^2\alpha^2)}+\frac{c^2\alpha(as+b)\sin(\alpha d)}{((as+b)^2+c^2\alpha^2)^2}-\frac{c\cos(\alpha d)}{(as+b)^2+c^2\alpha^2}+\frac{\alpha c\sin(\alpha d)}{s'((as+b)^2+c^2\alpha^2)}+\frac{c^2e'}{(as+b)^2+c^2\alpha^2},as+b>0$
$\sinh(\alpha t)-\sin(\alpha t)$	$\frac{2c^4\alpha^3e'}{(as+b)^4-c^4\alpha^4}+\frac{(as+b)\sinh(\alpha d)}{(as+b)^2-c^2\alpha^2}-\frac{c^2\alpha e'}{(as+b)^2-c^2\alpha^2}+\frac{c\alpha\cos(\alpha d)}{(as+b)^2-c^2\alpha^2}-\frac{(as+b)\sin(\alpha d)}{(as+b)^2+c^2\alpha^2}+\frac{c^2\alpha e'}{(as+b)^2+c^2\alpha^2}-\frac{c\alpha\cos(\alpha d)}{(as+b)^2+c^2\alpha^2}$

$\cosh(\alpha t)-\cos(\alpha t)$	$\frac{2\alpha^2c^3(as+b)e'}{(as+b)^4-c^4\alpha^4}+\frac{(as+b)\cosh(\alpha d)}{(as+b)^2-c^2\alpha^2}+\frac{c\alpha\sinh(\alpha d)}{(as+b)^2-c^2\alpha^2}-\frac{ce'(as+b)}{(as+b)^2-c^2\alpha^2}-\frac{(as+b)\cos(\alpha d)}{(as+b)^2+c^2\alpha^2}+\frac{c\sin(\alpha d)}{(as+b)^2+c^2\alpha^2}+\frac{ce'(as+b)}{(as+b)^2+c^2\alpha^2}$
$1-\cos(\alpha t)$	$\frac{c^2\alpha^2e'}{s'((as+b)^2+c^2\alpha^2)}-\frac{ce'}{s'}+\frac{1}{as+b}-\frac{(as+b)\cos(\alpha d)}{(as+b)^2+c^2\alpha^2}+\frac{c\alpha\sin(\alpha d)}{(as+b)^2+c^2\alpha^2}+\frac{ce'(as+b)}{(as+b)^2+c^2\alpha^2}$
$\alpha t-\sin(\alpha t)$	$\frac{c^2\alpha^3e'}{(s')^2((as+b)^2+c^2\alpha^2)}+\frac{\alpha d}{as+b}+\frac{\alpha c}{(as+b)^2}-\frac{\alpha e'}{(s')^2}-\frac{(as+b)\sin(\alpha d)}{(as+b)^2+c^2\alpha^2}-\frac{\alpha c\cos(\alpha d)}{(as+b)^2+c^2\alpha^2}+\frac{\alpha c^2e'}{(as+b)^2+c^2\alpha^2}$

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