Four Parametric Extension of
the Laplace Transform

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Abstract

In this paper, a four parametric extension of the Laplace transform is introduced. The relation between the extended transform and the classical Laplace transform is established. Moreover, the extended transforms of a number of functions like constant, polynomials, exponential, trigonometric and hyperbolic trigonometric functions, are found using fundamental theorems of integral calculus.

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1 Introduction

During the study of probability theory, the Laplace transform was firstly introduced by Pierre Simon Laplace in 1782 and has become a hot topic among the researchers from the 19th century due to extensive applications in many areas of Mathematics, Physics, Chemistry, Economics and Engineering. For \( \text{Re}(\alpha) > 0 \); the Laplace transform \( L \) of the function \( \phi(v) \) is defined as

\[
L\{\phi(t)\} = \int_0^\infty e^{-st} \phi(t) dt = \Phi(s)
\]

(see [5]). For \( \phi^{(0)}(t) = \phi(t) \) and \( \phi^{(k)}(t) = \frac{d^k}{dt^k} \phi(t), k \in N \); the Laplace transforms \( \Phi(s) \) of some functions \( \phi(t) \) are listed below:
A number of properties and applications of the Laplace transform in real life have been discussed involving mass spring damper system, chemical pollution in a reservoir and transfer function of control system etc (see [6]). The solution of population growth and decay problems have been found which arises in the field of Physics, Chemistry, Biology and Social Sciences etc using the Laplace transform [8]. We refer [1], [2], [3], [4] and [7] for the study of the integral transforms.

Along with other basic properties, the following properties of definite integral are used in the later results

\[ \int_{c_1}^{c_2} \phi(x) \, dx = \int_{c_1}^{c_2} \phi(y) \, dy \]
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and

\( \int_{c_1}^{c_2} \phi(x)dx = \int_{c_1}^{c_2} \phi(x)dx + \int_{c_2}^{c_3} \phi(x)dx, c_3 \in [c_1, c_2]. \)

**Definition 1.1** The number of arrangements of \( k \) distinct things taken \( r \) (distinct things) at a time, \( 0 < r \leq k \), is denoted by \( \text{k}_{P_r} \) and is given by

\[ \text{k}_{P_r} = \frac{k!}{(k-r)!} \]

and is in fact the number of arrangements a set of \( k \) distinct objects, each taken \( r \) at a time, without repetition of an object in an arrangement. One may observe that \( \text{k}_{P_k} = k! \).

## 2 The Extended Laplace Transform

**Definition 2.1** For \( (a, b, c, d) \in \mathbb{R}^4 \), \( c > 0; \) we define

\( L_{(a,b,c,d)} \{ \phi(t) \} = \int_0^\infty e^{-(as+bt)} \phi(ct+d)dt = \Phi_{(a,b,c,d)}(s). \)

**Definition 2.2** If \( (a, b, c, d) \in \mathbb{R}^4 \) and \( c, d > 0 \), for a function \( \phi(t) \); we define a new integral transform \( I \) as

\( I \{ \phi(t) \} = \int_0^d e^{-\frac{as+bt}{c}} \phi(t)dt. \)

**Definition 2.3** If \( (a, b, c, d) \in \mathbb{R}^4 \) and \( c > 0 \),

(i) for a real number \( s \); we define \( s' \) as

\( s' = \frac{as + b}{c} \)

and

(ii) we define \( e' \) as

\( e' = e^{\frac{d}{c}} \).

## 3 Main Results

Firstly, we establish the relationship between the classical and the extended Laplace transforms.

**Theorem 3.1** If \( \Phi(s), \Phi_{(a,b,c,d)}(s), I, s' \) and \( e' \) are defined as in (1), (4), (5), (6) and (7) respectively, then \( \Phi_{(a,b,c,d)}(s) = e' \left[ \Phi(s') - I \{ \phi(t) \} \right]. \)

**Proof:** For \( ct + d = x; \) (4) becomes
Using (2), (3), the above relation leads to the relation

\[ L_{(a,b,c,d)} \{ \phi(t) \} = \frac{e^{\int (as+b-td) \, dt}}{c} \left[ \Phi \left( \frac{as+b}{c} \right) - I \{ \phi(t) \} \right], \]

which, with collaboration of (6) and (7), leads to the required result.

One may find the values of \( I \{ \phi(t) \} \) for \( \phi(t) = e^{\alpha t}, \phi(t) = t e^{\alpha t}, \phi(t) = e^{\beta t} \cos(\alpha t), \phi(t) = t \cos(\alpha t), \phi(t) = e^{\beta t} \sin(\alpha t), \phi(t) = t \sin(\alpha t), \phi(t) = \sinh(\alpha t), \) etc, easily by use of basic theorems of definite integrals. The results are listed below:

### 3.1 The Newly Defined Integral Transform of Some Elementary Functions

The integral transform \( I \{ \phi(t) \} \), defined in (5), of \( \phi(t) \), for different values of \( \phi(t) \), is listed in the following table:

<table>
<thead>
<tr>
<th>( \phi(t) )</th>
<th>( I { \phi(t) } )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{1}{s} \left[ 1 - \frac{1}{ce} \right] )</td>
</tr>
<tr>
<td>( t )</td>
<td>( -\frac{d}{cs'e} - \frac{1}{ce's'(s')^2} + \frac{1}{s'} )</td>
</tr>
<tr>
<td>( t^n )</td>
<td>( -\frac{1}{ce'} \sum_{i=0}^{n-1} \left( \frac{1}{s'} \right)^{i+1} d^{n-i} P_i - \frac{n!}{s'} \left( \frac{1}{ce'} - 1 \right) )</td>
</tr>
<tr>
<td>( e^{\alpha t} )</td>
<td>( \frac{c}{as+b-ca} - \frac{e^{-\int (as+b-ca) , dt}}{ce'} )</td>
</tr>
<tr>
<td>( t^n e^{\alpha t} )</td>
<td>( -e^{-\int (as+b-ca) , dt} \sum_{i=0}^{n} \left( \frac{c}{as+b-ca} \right)^{i+1} d^{n-i} P_i + n! \left( \frac{c}{as+b-ca} \right)^{n+1} )</td>
</tr>
<tr>
<td>\sin(\alpha t)</td>
<td>( \frac{(as+b)^2 + c^2 \alpha^2}{[(as+b)^2 + c^2 \alpha^2]} e^{t} - \frac{c \alpha \cos(\alpha t)}{[(as+b)^2 + c^2 \alpha^2]} + \frac{c^2 \alpha}{(as+b)^2 + c^2 \alpha^2} )</td>
</tr>
<tr>
<td>\cos(\alpha t)</td>
<td>( \frac{(as+b)^2 + c^2 \alpha^2}{[(as+b)^2 + c^2 \alpha^2]} e^{t} + \frac{c \alpha \sin(\alpha t)}{[(as+b)^2 + c^2 \alpha^2]} + \frac{c^2 \alpha}{(as+b)^2 + c^2 \alpha^2} )</td>
</tr>
<tr>
<td>\sinh(\alpha t)</td>
<td>( \frac{(as+b)^2 - c^2 \alpha^2}{[(as+b)^2 - c^2 \alpha^2]} e^{t} - \frac{c \alpha \cosh(\alpha t)}{[(as+b)^2 - c^2 \alpha^2]} + \frac{c^2 \alpha}{(as+b)^2 - c^2 \alpha^2} )</td>
</tr>
<tr>
<td>\cosh(\alpha t)</td>
<td>( \frac{(as+b)^2 - c^2 \alpha^2}{[(as+b)^2 - c^2 \alpha^2]} e^{t} + \frac{c \alpha \sinh(\alpha t)}{[(as+b)^2 - c^2 \alpha^2]} + \frac{c^2 \alpha}{(as+b)^2 - c^2 \alpha^2} )</td>
</tr>
</tbody>
</table>
Using the results mentioned above in (3.1), (3.2), (6) and (7), respectively, we obtain the results listed below:

3.2 Four Parametric Laplace Transform of Some Functions

The four parametric extended Laplace transform \( \Phi_{(a,b,c,d)}(s) \), defined in (4), of \( \phi(t) \), for different values of \( \phi(t) \), is listed in the following table:
Four parametric extension of the Laplace transform

\[
cosh(\alpha t) - \cos(\alpha t) = \frac{2\alpha^2 c^3 (as + b)e^t + (as + b)\cosh(\alpha d) + \alpha \c\sinh(\alpha d)}{(as + b)^4 - c^4 \alpha^4} + \frac{\alpha \c\sinh(\alpha d) - c^2 \alpha^2}{(as + b)^2 - c^2 \alpha^2} - \frac{ce'(as + b)}{(as + b)^2 + c^2 \alpha^2}
\]

\[
1 - \cos(\alpha t) = \frac{c^2 \alpha^2 e'^t}{s'((as + b)^2 + c^2 \alpha^2)} + \frac{1}{s'} \frac{as + b}{\cos(\alpha d)} - \frac{as + b}{\cos(\alpha d)} + \frac{ce'(as + b)}{(as + b)^2 + c^2 \alpha^2}
\]

\[
\alpha t - \sin(\alpha t) = \frac{c^2 \alpha^2 e'^t}{(s')^2 ((as + b)^2 + c^2 \alpha^2)} + \frac{ad}{as + b} + \frac{\alpha c}{(as + b)^2} - \frac{\alpha e'}{(s')^2 ((as + b)^2 + c^2 \alpha^2)} - \frac{\alpha c e'}{(as + b)^2 + c^2 \alpha^2}
\]

References


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