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Solutions to Periodic BVP for Second Order Impulsive Differential Equation with a p-Laplacian Operator

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Abstract

In this paper, we study the existence of solutions to a periodic boundary value problem of an impulsive differential equation with a p-Laplacian operator. By employing an existing critical point theorem, we find the range of the control parameter in which the p-Laplacian problem admits at least one non-zero weak solution. The main result is also demonstrated with an example.

Mathematics Subject Classifications: 34B15, 34B18, 34B37, 58E30

Keywords critical point theorem; periodic boundary value problem; p-Laplacian operator

1 Introduction and main results

In this paper, we are concerned with the following second order impulsive systems

$$\begin{cases} -(\Phi_p(u'(t)))' + \Phi_p(u(t)) = \lambda f(t, u(t)), \quad t \neq t_j, t \in [0, T] \,, \\ \Delta\Phi_p(u'(t_j)) = I_j(u(t_j)), j = 1, 2, ..., m, \\ u(0) - u(T) = u'(0) - u'(T) = 0 \end{cases} \tag{1.1}$$

where $f : [0,T]\times\mathbb{R}\to\mathbb{R}$, $I_j\in C(\mathbb{R},\mathbb{R})$, $0=t_0<t_1<t_2<\cdots<t_m<t_{m+1}=T$, $\Phi_p(s)$ is a p-Laplacian operator with $\Phi_p(s)=|s|^{p-2}s,\ 1<p<+\infty$, $\Delta\Phi_p(u'(t_j))=\Phi_p(u'(t_j^+))-\Phi_p(u'(t_j^-))$, where $u'(t_j^+)$, $u'(t_j^-)$ denote the right and left limits, respectively, of $u'(t)$ at $t=t_j, j=1,2...,m$. The parameter λ is a positive real number.

In recent years, there is increasing interest in the existence and multiplicity of solutions for the p-Laplacian differential equation problems. Mainly by applying methods of the coincidence degree, fixed point theorem, continuous theorem and variational method, many existence and multiplicity results of solutions for several types of differential equations with a p-Laplacian operator have been obtained. See, for example, [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17] and the references therein. Moreover, in the last few years, more attention has been paid to applying variational methods to deal with the existence of solutions for impulsive differential equation boundary value problems ([18, 19, 20, 21, 22, 23, 24]). However, to the best of our knowledge, the existence of solutions for the impulsive p-Laplacian differential systems depending on a parameter λ has not been widely investigated. In this paper, we use critical point theory and variational methods to find the range of the control parameter λ in which the p-Laplacian problem (1.1) admits at least one non-zero weak solution.

For the sake of convenience, we list the following conditions.

(H1) f is an L^1-Carathéodory function.

(H2) $0\le\int_0^u I_j(s)ds<\dfrac{T|u|^p}{pm},\ u\in\mathbb{R},\ j=1,\ 2,\ \dots,\ m.$

Given three nonnegative constants c_1,c_2,d, with $c_1<\sqrt[p]{T}cd<\sqrt[p]{2T}cd<c_2$, put

$$a(c_2,d)=pc^p\frac{\int_0^T\max_{|u(t)|\le c_2}F(t,u(t))dt-\int_0^T F(t,d)dt}{c_2^p-2T(cd)^p} \tag{1.2}$$

and

$$b(c_1,d)=pc^p\frac{\int_0^T F(t,d)dt-\int_0^T\max_{|u(t)|\le c_1}F(t,u(t))dt}{2T(cd)^p-c_1^p}, \tag{1.3}$$

where c is the constant specified in (2.1).

Our main results are the following theorems and corollaries.

Theorem 1.1. *Assume that (H1), (H2) are satisfied and there exist three nonnegative constants c_1,c_2,d, with $c_1<\sqrt[p]{T}cd<\sqrt[p]{2T}cd<c_2$, such that*

$$a(c_2,d)<b(c_1,d). \tag{1.4}$$

Then, for each $\lambda \in (1/b(c_1, d),\ 1/a(c_2, d))$*, the system (1.1) admits at least one weak solution* $u(t),\ t \in [0, T]$*, such that* $\dfrac{c_1}{c\sqrt[p]{1+Tc^p}} < ||u||_X < \dfrac{c_2}{c}$.

Theorem 1.2. *Assume that (H1), (H2) hold and there exist two positive constants* γ, d*, with* $\sqrt[p]{2T}cd < \gamma$*, such that*

$$\frac{\int_0^T \max_{|u(t)|\le\gamma} F(t,u(t))dt}{\gamma^p} < \frac{\int_0^T F(t,d)dt}{2T(cd)^p}. \tag{1.5}$$

Then, for each $\lambda \in \left(\dfrac{2Td^p}{p\int_0^T F(t,d)dt},\ \dfrac{\gamma^p}{pc^p\int_0^T \max_{|u(t)|\le\gamma} F(t,u(t))dt}\right)$*, the system (1.1) admits at least one nontrivial weak solution* $u(t),\ t \in [0, T]$*, such that* $||u||_X < \dfrac{\gamma}{c}$.

When the nonlinear term of problem (1.1) is with separable variables, we have the following results. To be precise, let $g \in L^1([0, T])$ such that $g(t) \ge 0$ a.e. $t \in [0, T], g \not\equiv 0$, and let $h : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function. Put

$$H(u) = \int_0^{u(t)} h(s)ds \text{ for } u \in X.$$

Corollary 1.3. *Assume that (H2) is satisfied and there exist three nonnegative constants* c_1, c_2, d*, with* $c_1 < \sqrt[p]{T}cd < \sqrt[p]{2T}cd < c_2$*, such that*

$$\frac{H(c_2) - H(d)}{c_2^p - 2T(cd)^p} < \frac{H(d) - H(c_1)}{2T(cd)^p - c_1^p}. \tag{1.6}$$

Then, for each $\lambda \in \left(\dfrac{2T(cd)^p - c_1^p}{pc^p||g||_1(H(d) - H(c_1))},\ \dfrac{(c_2^p - 2T(cd)^p)}{pc^p||g||_1(H(c_2) - H(d))}\right)$*, the following system*

$$\begin{cases} -(\Phi_p(u'(t)))' + \Phi_p(u(t)) = \lambda g(t)h(u(t)), \quad t \ne t_j, t \in [0, T]\ , \\ \Delta\Phi_p(u'(t_j)) = I_j(u(t_j)), j = 1, 2, ..., m, \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \tag{1.7}$$

admits at least one weak solution $u(t),\ t \in [0, T]$*, such that* $\dfrac{c_1}{c\sqrt[p]{1+Tc^p}} < ||u||_X < \dfrac{c_2}{c}$*, where* $||g||_1 = \int_0^T |g(t)|dt$.

Corollary 1.4. *Assume that (H2) is satisfied and there exist two positive constants* γ, d, *with* $\sqrt[p]{2T}cd < \gamma$, *such that*

$$\frac{H(\gamma)}{\gamma^p} < \frac{H(d)}{2T(cd)^p}$$

Then, for each $\lambda \in \left(\frac{2Td^p}{pH(d)||g||_1}, \frac{\gamma^p}{pc^pH(\gamma)||g||_1}\right)$, *the system (1.7) admits at least one weak solution* $u(t)$, *with* $|u(t)| < \gamma$ *for all* $t \in [0,T]$.

The rest of this paper is organized as follows. In Section 2 we present several definitions and main tools. In Section 3, we prove that problem (1.1) possesses at least one non-zero weak solution when λ lies in an exactly determined open interval. Finally, an example is provided to verify our results.

2 Preliminaries

In the following, we first introduce some notations. Take $X := \{u(t) | u(t) \in W^{1,p}([0,T]),\ u(0) = u(T)\}$, in which we consider the norm

$$||u||_X = \left(\int_0^T |u'(t)|^p dt + \int_0^T |u(t)|^p dt\right)^{\frac{1}{p}}.$$

Note that X is a reflexive Banach space.

Definition 2.1. $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ *is an* L^1*-Carathéodory function if:*
(i) $t \mapsto f(t,u)$ *is measurable for every* $u \in \mathbb{R}$;
(ii) $u \mapsto f(t,u)$ *is continuous for almost every* $t \in [0,T]$;
(iii) for every $s > 0$ *there exists a function* $l_s \in L^1([0,T])$ *such that*

$$\sup_{|u|\le s} |f(t,u)| \le l_s(t) \quad \text{for a.e. } t \in [0,T].$$

Definition 2.2. *The function* $u : [0,T] \to \mathbb{R}$ *is called a weak solution of system (1.1) if* $u \in X$ *and*

$$\int_0^T (\Phi_p(u'(t))v'(t) + \Phi_p(u(t))v(t))dt + \sum_{j=1}^m I_j(u(t_j))v(t_j) = \lambda \int_0^T f(t,u(t))v(t)dt$$

for all $v \in X$.

Lemma 2.3. *For any* $u \in X$, *there exists* $c = 2^{\frac{1}{q}} \max\{T^{-\frac{1}{p}},\ T^{\frac{1}{q}}\}$, $\frac{1}{p} + \frac{1}{q} = 1$, *such that*

$$||u||_\infty \le c||u||_X, \tag{2.1}$$

where $||u||_\infty = \max_{t\in[0,T]} |u(t)|$.

Proof. The proof follows easily from the mean value theorem and the Hölder inequality, so we omit it here. ■

Next we define a functional φ_λ as

$$\varphi_\lambda(u) = \Phi(u) - \lambda\Psi(u),\ u \in X, \tag{2.2}$$

where

$$\Phi(u) = \frac{1}{p}||u||_X^p + \sum_{j=1}^{m} \int_0^{u(t_j)} I_j(s)ds \tag{2.3}$$

and

$$\Psi(u) = \int_0^T F(t, u(t))dt, \tag{2.4}$$

with

$$F(t, u) = \int_0^{u(t)} f(t, s)ds\ .$$

Note that φ_λ is Fréchet differentiable at any $u \in X$ and for any $v \in X$, we have

$$\begin{aligned} \varphi_\lambda'(u)(v) \ &= \ \int_0^T (\Phi_p(u'(t))v'(t) + \Phi(u(t))v(t))dt + \sum_{j=1}^{m} I_j(u(t_j))v(t_j) \\ &\quad -\lambda \int_0^T f(t, u(t))v(t)dt\ . \end{aligned} \tag{2.5}$$

Obviously, φ_λ' is continuous and a critical point of φ_λ, by (2.5), gives a weak solution of system (1.1).

For all $r_1, r_2 \in \mathbb{R}$, with $r_1 < r_2$, we define

$$\beta(r_1, r_2) = \inf_{v\in\Phi^{-1}((r_1,r_2))} \frac{\sup\limits_{u\in\Phi^{-1}((r_1,r_2))} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)}\ , \tag{2.6}$$

$$\alpha(r_1, r_2) = \sup_{v\in\Phi^{-1}((r_1,r_2))} \frac{\Psi(v) - \sup\limits_{u\in\Phi^{-1}((-\infty,r_1))} \Psi(u)}{\Phi(v) - r_1}\ . \tag{2.7}$$

Note that for all $r_1, r_2 \in \mathbb{R}$, with $r_1 < r_2$, we have $\beta(r_1, r_2) \geq 0$, $\alpha(r_1, r_2) \geq 0$. To prove our main results, we need the following critical point theorem.

Theorem 2.4. *[25, Theorem 5.1] Let X be a reflexive real Banach space. Let $\Phi : X \to R$ be a sequentially weakly lower semicontinuous, coercive and continuously Gáteaux differentiable functional whose Gáteaux derivative admits a continuous inverse on X^*; let $\Psi : X \to R$ be a continuously Gáteaux differentiable functional whose Gáteaux derivative is compact. Put $\varphi_\lambda = \Phi - \lambda\Psi$ and assume that there are $r_1,\ r_2 \in \mathbb{R}$, with $r_1 < r_2$, such that*

$$\beta(r_1, r_2) < \alpha(r_1, r_2) \tag{2.8}$$

where β and α are given by (2.6) and (2.7). Then, for each $\lambda \in (1/\alpha(r_1, r_2),\ 1/\beta(r_1, r_2))$ there is $u_{0,\lambda} \in \Phi^{-1}((r_1, r_2))$ such that $\varphi_\lambda(u_{0,\lambda}) \le \varphi_\lambda(u)$ for all $u \in \Phi^{-1}((r_1, r_2))$ and $\varphi'_\lambda(u_{0,\lambda}) = 0$.

3 Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1. By (2.3) and (2.4), we have that Φ is a nonnegative Gâteaux differentiable, coercive and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, and Ψ is a continuously Gâteaux differentiable functional whose Gâteaux is compact. Let

$$r_1 = \frac{c_1^p}{pc^p},\ r_2 = \frac{c_2^p}{pc^p},\ u_0(t) = d, \quad \text{for} \quad t \in [0, T]. \tag{3.1}$$

By condition (H2), we have

$$\begin{aligned} \Phi(u_0) &= \frac{1}{p}||u_0||_X^p + \sum_{j=1}^{m} \int_0^{u_0(t_j)} I_j(s)ds \\ &= \frac{Td^p}{p} + \sum_{j=1}^{m} \int_0^{d} I_j(s)ds \\ &\le \frac{2Td^p}{p}, \end{aligned} \tag{3.2}$$

and

$$\Phi(u_0) \ge \frac{Td^p}{p}. \tag{3.3}$$

Combining $c_1 < \sqrt[p]{T}cd < \sqrt[p]{2T}cd < c_2$, (3.1), (3.2) and (3.3), we have

$$r_1 < \Phi(u_0) < r_2.$$

Obviously, we have $\Psi(u_0) = \int_0^T F(t, u_0(t))dt = \int_0^T F(t, d)dt$. From Lemma 2.3, the estimate $\Phi(u) < r_2,\ u \in X$, implies that

$$|u(t)|^p \le c^p||u||_X^p \le pc^p\Phi(u) < pc^p r_2 = c_2^p,\ t \in [0, T],$$

and

$$\int_0^T F(t, u(t))dt \le \int_0^T \max_{|u(t)| \le c_2} F(t, u)dt.$$

Therefore

$$\sup_{u \in \Phi^{-1}((r_1, r_2))} \Psi(u) \le \sup_{u \in \Phi^{-1}((-\infty, r_2))} \Psi(u) \le \int_0^T \max_{|u(t)| \le c_2} F(t, u(t))dt.$$

For $u \in X$ with $\Phi(u) < r_1$, one can similarly obtain

$$\sup_{u\in\Phi^{-1}((-\infty,r_1))} \Psi(u) \le \int_0^T \max_{|u(t)|\le c_1} F(t,u(t))dt.$$

Therefore, we have

$$\begin{aligned}
\beta(r_1,r_2) &\le \frac{\sup\limits_{u\in\Phi^{-1}((-\infty,r_2))} \Psi(u) - \Psi(u_0)}{r_2 - \Phi(u_0)} \\
&\le pc^p \frac{\int_0^T \max\limits_{|u(t)|\le c_2} F(t,u(t))dt - \int_0^T F(t,d)dt}{c_2^p - 2T(cd)^p} \\
&= a(c_2,d).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\alpha(r_1,r_2) &\ge \frac{\Psi(u_0) - \sup\limits_{u\in\Phi^{-1}((-\infty,r_1))} \Psi(u)}{\Phi(u_0) - r_1} \\
&\ge pc^p \frac{\int_0^T F(t,d)dt - \int_0^T \max\limits_{|u(t)|\le c_1} F(t,u(t))dt}{2T(cd)^p - c_1^p} \\
&= b(c_1,d).
\end{aligned}$$

So, by (1.4), we induce

$$\beta(r_1,r_2) < \alpha(r_1,r_2).$$

Therefore, by Theorem 2.4, for each $\lambda \in (1/b(c_1,d),\ 1/a(c_2,d))$, we have that $\Phi - \lambda\Psi$ admits at least one critical point u such that $r_1 < \Phi(u) < r_2$. Combining (2.3), we get

$$\frac{c_1^p}{pc^p} < \frac{1}{p}||u||_X^p + \sum_{j=1}^m \int_0^{u(t_j)} I_j(s)ds \le \frac{1+Tc^p}{p}||u||_X^p,$$

and

$$\frac{1}{p}||u||_X^p \le \frac{1}{p}||u||_X^p + \sum_{j=1}^m \int_0^{u(t_j)} I_j(s)ds < \frac{c_2^p}{pc^p}.$$

So, the system (1.1) admits at least one weak solution $u(t),\ t \in [0,T]$, such that $\dfrac{c_1}{c\sqrt[p]{1+Tc^p}} < ||u||_X < \dfrac{c_2}{c}$. ■

Proof of Theorem 1.2. Let $c_1 = 0$ and $c_2 = \gamma$, then by (1.2) and (1.3) we get

$$\begin{aligned} a(c_2, d) &= pc^p \frac{\int_0^T \max_{|u(t)|\le\gamma} F(t,u(t))dt - \int_0^T F(t,d)dt}{\gamma^p - 2T(cd)^p} \\ &\le pc^p \frac{\int_0^T \max_{|u(t)|\le\gamma} F(t,u(t))dt - \frac{2T(cd)^p}{\gamma^p}\int_0^T \max_{|u(t)|\le\gamma} F(t,u(t))dt}{\gamma^p - 2T(cd)^p} \\ &= pc^p \frac{\int_0^T \max_{|u(t)|\le\gamma} F(t,u(t))dt}{\gamma^p}, \end{aligned}$$

and

$$b(c_1, d) = pc^p \frac{\int_0^T F(t,d)dt}{2T(cd)^p} = p\frac{\int_0^T F(t,d)dt}{2Td^p}.$$

Therefore, owing to (1.5) we have $a(\gamma, d) < b(0, d)$. Moreover, by Theorem 1.1, we have that for each $\lambda \in \left(\frac{2Td^p}{p\int_0^T F(t,d)dt}, \frac{\gamma^p}{pc^p\int_0^T \max_{|u(t)|\le\gamma} F(t,u(t))dt} \right)$, the system (1.1) admits at least one nontrivial weak solution u such that $||u||_X < \frac{\gamma}{c}$. ■

4. An example

In this section, we give an example to illustrate our main results.

Example 4.1. Consider the boundary value problem

$$\begin{cases} -(|u'(t)|^{-\frac{1}{2}}u'(t))' + |u(t)|^{-\frac{1}{2}}u(t) = \lambda(1+\sin t)e^{-u(t)}, t \ne t_1, t \in [0,1], \\ \Delta(|u'(t)|^{-\frac{1}{2}}u'(t)) = \frac{1}{2}|u(t_1)|^{\frac{1}{2}},\ t_1 = \frac{1}{2}, \\ u(0) - u(1) = u'(0) - u'(1) = 0. \end{cases} \tag{4.1}$$

Compared to system (1.7), $T = 1, p = \frac{3}{2}, g(t) = 1 + \sin t$, $h(u) = e^{-u}$, $I_j(u) = \frac{1}{2}|u|^{\frac{1}{2}}$. Clearly, $c = \sqrt[3]{2}$, (H2) is satisfied and $g \in L^1([0,1])$ such that $g(t) \ge 0$ a.e. $t \in [0,1]$, $g \not\equiv 0$, and $h : \mathbb{R} \to \mathbb{R}$ is a nonnegative continuous function. Choose $d = 0.5, \gamma = 2$.

By simple calculations, we obtain

$$\frac{H(\gamma)}{\gamma^p} \approx 0.3061 < \frac{H(d)}{2T(cd)^p} \approx 0.3949,$$

$$\frac{2Td^p}{pH(d)||g||_1} \approx 1.8406, \frac{\gamma^p}{pc^pH(\gamma)||g||_1} \approx 2.3746$$

Applying Corollary 1.4, when $\lambda \in (1.8406, 2.3746)$, system (4.1) has at least one weak solution u such that $|u(t)| < 2$ for all $t \in [0, 1]$.

Competing interests. The authors declare that they have no competing interests.

Authors' contributions. JX and LZ conceived of the studies, and drafted the manuscript. HQ participated in the discussion. All authors read and approved the final manuscript.

References

[1] Bai, L., Dai, B.X., Three solutions for a p-Laplacian boundary value problem with impulsive effects, *Appl. Math. Comp.*, **217** (2011), 9895-9904. https://doi.org/10.1016/j.amc.2011.03.097

[2] Heidarkhani, S., Afrouzi, G.A., Some multiplicity results to the existence of three solutions for a Dirichlet boundary value problem involving the p-Laplacian, *Math. Model. Anal.*, **16** (2011), 390-400. https://doi.org/10.3846/13926292.2011.601770

[3] Lu, S., Existence of periodic solutions of a p-Laplacian neutral functional differential equation, *Nonlinear Anal.*, **70** (2009), 231-243. https://doi.org/10.1016/j.na.2007.11.053

[4] Ma, S.W., Zhang, Y.X., Existence of infinitely many periodic solutions for ordinary p-Laplacian systems, *J. Math. Anal. Appl.*, **351** (2009), 469-479. https://doi.org/10.1016/j.jmaa.2008.10.027

[5] Xu, B., Tang, C.L., Some existence results on periodic solutions of ordinary p-Laplacian systems, *J. Math. Anal. Appl.*, **333** (2007), 1228-1236. https://doi.org/10.1016/j.jmaa.2006.11.051

[6] Tian, Y., Ge, W.G., Periodic solutions of non-autonomous second-order systems with a p-Laplacian, *Nonlinear Analysis*, **66** (2005), 192-203. https://doi.org/10.1016/j.na.2005.11.020

[7] Lin, X.Y., Tang, X.H., Infinitely many homoclinic orbits of second-order p-Laplacian systems, *Taiwanese Journal of Mathematics*, **17** (2013), 1371-1393. https://doi.org/10.11650/tjm.17.2013.2518

[8] Xu, J.F., Wei, Z.L., Ding, Y.Z., Existence of weak solutions for p-Laplacian probem with impulsive effects, *Taiwanese Journal of Mathematics*, **17** (2013), 501-515. https://doi.org/10.11650/tjm.17.2013.2081

[9] Lin, G.H., Zhou, Z., Yu, J.S., Ground state solutions of discrete asymptotically linear Schrodinger equations with bounded and non-periodic potentials, *Journal of Dynamics and Differential Equations*, **32** (2020), 527-555. https://doi.org/10.1007/s10884-019-09743-4

[10] Ding, Y.Z., O'Regan, D., Positive solutions for a second-order p-Laplacian impulsive boundary value problem, *Advances in Difference Equations*, **2012** (2012), 1-12. https://doi.org/10.1186/1687-1847-2012-159

[11] Jiang, W.H., The existence of solutions for impulsive p-Laplacian boundary value problems at resonance on the half-line, *Boundary Value Problems*, **2015** (2015), 1-13. https://doi.org/10.1186/s13661-015-0299-3

[12] Lian, H.R., Wang, D.L., Bai, Z.B., Agarwal, R.P., Periodic and subharmonic solutions for a class of second-order p-Laplacian Hamiltonian systems, *Boundary Value Problems*, **2014** (2014), 1-15. https://doi.org/10.1186/s13661-014-0260-x

[13] Vincenzo, A, Multiple solutions for a fractional p-Laplacian equation with sign-changing potential, *Electron. J. Differential Equations*, **2016** (2016), 1-12.

[14] Lian, H.R., Wang, D.L., O'Regan, D., Agarwal, R.P., Periodic solutions of nonautonomous second-order differential equations with a p-Laplacian, *Analysis*, **37** (2017), 1-11. https://doi.org/10.1515/anly-2014-1279

[15] Sun, J.T., Chu, J.F., Wu, T.F., Existence and multiplicity of nontrivial solutions for some biharmonic equations with p-Laplacian, *J. Differential Equations*, **262** (2017). https://doi.org/10.1016/j.jde.2016.10.001

[16] Li, Z.Y., Kong, F.C., Positive periodic solutions for p-Laplacian neutral differential equations with a singularity, *Boundary Value Problems*, **2017** (2017), 1-17. https://doi.org/10.1186/s13661-017-0790-0

[17] Bonanno, G., Jebelean, P., Serban, C., Three periodic solutions for discontinuous perturbations of the vector p-Laplacian operator, *Proceedings of the Royal Society of Edinburgh: Section A Mathematics*, **147** (2017), 673-681. https://doi.org/10.1017/s0308210516000317

[18] Nieto, J.J., Regan, D.O., Variational approach to impulsive differential equations, *Nonlinear Anal. RWA,* **10** (2009), 680-690. https://doi.org/10.1016/j.nonrwa.2007.10.022

[19] Qin, D., Li, X., Periodic solutions for ordinary differential equations with sublinear impulsive effects. *J. Math. Appl.*, **303** (2005), 288-303. https://doi.org/10.1016/j.jmaa.2004.08.034

[20] Tian, Y., Ge, W., Applications of variational methods to boundary-value problem for impulsive differential equations, *Proc. Edinb. Math. Soc.*, **51** (2008), 509-527. https://doi.org/10.1017/s0013091506001532

[21] Zhang, H., Li, Z., Variational approach to impulsive differential equations with periodic boundary conditions, *Nonlinear Anal. RWA*, **11** (2010), 67-78. https://doi.org/10.1016/j.nonrwa.2008.10.016

[22] Zhang, X., Tang, X., Subharmonic solutions for a class of non-quadratic second order Hamiltonian systems, *Nonlinear Anal. RWA*, **13** (2012), 113-130. https://doi.org/10.1016/j.nonrwa.2011.07.013

[23] Zhang, Z., Yuan, R., Applications of variational methods to Dirichlet boundary value problem with impulses, *Nonlinear Anal. RWA*, **11** (2010), 155-162. https://doi.org/10.1016/j.nonrwa.2008.10.044

[24] Xie, J.L., Luo, Z.G., The existence of solutions of periodic BVP for second order impulsive differential equations, *Journal of Inequalities and Applications*, **2013** (2013), 1-9. https://doi.org/10.1186/1029-242x-2013-406

[25] Bonanno, G., A critical point theorem via the Ekeland variational principle, *Nonlinear Anal.*, **75** (2012), 2992-3007. https://doi.org/10.1016/j.na.2011.12.003

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