Bounded Subsets of the Zygmund $F$-Algebra on the Upper Half Plane

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Abstract

In [7] the author introduced the Zygmund $F$-algebra $N \log^\alpha N(D)$ ($\alpha > 0$) of holomorphic functions $f$ on the upper half plane $D = \{ z \in \mathbb{C} | \text{Im} z > 0 \}$ that satisfy

$$\sup_{y > 0} \int_{\mathbb{R}} \varphi_\alpha (\log(1 + |f(x + iy)|)) \, dx < +\infty,$$

where $\varphi_\alpha (t) = t \log^\alpha (c_\alpha + t)$ for $t \geq 0$ and $c_\alpha = \max (e, e^\alpha)$. In this paper we shall characterize bounded subsets of $N \log^\alpha N(D)$ ($\alpha > 0$).

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1 Introduction

Let $D = \{ z \in \mathbb{C} | \text{Im} z > 0 \}$. The Nevanlinna class on the upper half plane, $N_0(D)$, is the set of all holomorphic functions $f$ on $D$ satisfying

$$\sup_{y > 0} \int_{\mathbb{R}} \log(1 + |f(x + iy)|) \, dx < +\infty.$$

We denote the Smirnov class on $D$ by $N_*(D)$, which consists of all holomorphic functions $f$ on $D$ satisfying $\log(1 + |f(z)|) \leq P[\phi](z)$ ($z \in D$) for some
\( \phi \in L^1(\mathbb{R}), \phi \geq 0, \) where the right side denotes the Poisson integral of \( \phi \) on \( D. \) These classes were introduced by Mochizuki [8]. It is well-known that each function \( f \) in \( N_0(D) \) has the nontangential limit \( f^*(x) = \lim_{y \to 0^+} f(x + iy) \) (a.e. \( x \in \mathbb{R} \)). It is known that, if \( f \in N_0(D), \) \( f \) belongs to \( N^*(D) \) if and only if

\[
\lim_{y \to 0^+} \int_{\mathbb{R}} \log(1 + |f(x + iy)|) \, dx = \int_{\mathbb{R}} \log(1 + |f^*(x)|) \, dx
\]

(see [8]). We define a metric on \( N_*(D) \) by

\[
d_{N_*(D)}(f, g) = \int_{\mathbb{R}} \log(1 + |f^*(x) - g^*(x)|) \, dx \quad (f, g \in N_*(D)).
\]

With this metric \( N_*(D) \) becomes an \( F \)-algebra ([8]), that is, a topological vector space whose topology is given by a complete, translation invariant metric in which multiplication is continuous.

In [5], the Privalov class on \( D \) was introduced; i.e., we denote by \( N^p(D) \) \((p > 1)\) the set of all holomorphic functions \( f \) on \( D \) such that

\[
\sup_{y > 0} \int_{\mathbb{R}} (\log(1 + |f(x + iy)|))^p \, dx < +\infty.
\]

Each \( f \in N^p(D) \) has the nontangential limit \( f^*(x) \) for a.e. \( x \in \mathbb{R}. \) Define a metric

\[
d_{N^p(D)}(f, g) = \left\{ \int_{\mathbb{R}} (\log(1 + |f^*(x) - g^*(x)|))^p \, dx \right\}^{\frac{1}{p}}
\]

for \( f, g \in N^p(D). \) The class is also an \( F \)-algebra with respect to the metric \( d_{N^p(D)}(\cdot, \cdot) \) [5].

The class \( M^p(D) \) \((0 < p < \infty)\) is defined as the set of all holomorphic functions \( f \) on \( D \) for which

\[
\int_{\mathbb{R}} (\log(1 + Mf(x)))^p \, dx < +\infty,
\]

where \( Mf(x) = \sup_{y > 0} |f(x + iy)| \) is a vertical maximal function. \( M^1(D) \) was considered by Ganzhula in [4]. As for \( p > 0, \) Efimov and Subbotin investigated this class [3]. It is known that the following relations hold:

\[
\bigcup_{0 < p \leq 1} H^p(D) \subset M^1(D) \subset N_*(D) \subset N_0(D),
\]

where \( H^p(D) \) denotes the Hardy space on \( D. \) Let \( \alpha_p = \min(1, p). \) The class \( M^p(D) \) with the topology given by the metric defined by

\[
d_{M^p(D)}(f, g) = \left\{ \int_{\mathbb{R}} (\log(1 + M(f - g)(x)))^p \, dx \right\}^{\frac{\alpha_p}{p}} \quad (f, g \in M^p(D))
\]
forms an $F$-algebra (see [3, 4]).

A subset $L$ of a linear topological space $A$ is said to be bounded if for any neighborhood $U$ of zero in $A$ there exists a real number $\lambda$, $0 < \lambda < 1$, such that $\lambda L = \{ \lambda f : f \in L \} \subset U$. In [6], the author considered some characterizations of boundedness in $N_*(D)$ and $N^p(D)$ ($p > 1$). As for $M^p(D)$ with $p = 1$, Ganzhula described the properties of boundedness [4] and Efimov characterized bounded subsets of $M^p(D)$ in the case $0 < p < \infty$ [1, 2].

Motivated by these works, we will investigate some characterizations of boundedness in the Zygmund $F$-algebra on the upper half plane $D$. We let $N_{\log^\alpha} N(D), \alpha > 0$, consist of all holomorphic functions $f$ on $D$ such that

$$\sup_{y > 0} \int \varphi_\alpha (\log(1 + |f(x + iy)|)) \, dx < +\infty,$$

where $\varphi_\alpha(t) = t\{\log(c_\alpha + t)\}^\alpha$ for $t \geq 0$ and $c_\alpha = \max(e, e^\alpha)$. This class was introduced by the author in [7]. We note that, in monograph [9], Zygmund considered this class on the unit disk. It is known that, if $f \in N_{\log^\alpha} N(D)$, the nontangential limit $f^*(x)$ exists a.e. for $x \in \mathbb{R}$. Moreover we can define, for $f, g \in N_{\log^\alpha} N(D)$, a metric

$$d_{N_{\log^\alpha} N(D)}(f, g) = \int \varphi_\alpha (\log(1 + |f^*(x) - g^*(x)|)) \, dx.$$

With this metric $N_{\log^\alpha} N(D)$ is also an $F$-algebra [7].

In this paper, we give a characterization of bounded subsets of $N_{\log^\alpha} N(D)$ ($\alpha > 0$) with respect to the metric.

## 2 Main result

In this section we shall characterize bounded subsets of $N_{\log^\alpha} N(D)$ ($\alpha > 0$).

**Theorem 2.1.** Let $\alpha > 0$. $L \subset N_{\log^\alpha} N(D)$ is bounded if and only if

(i) there exists a $K < \infty$ such that

$$\int \varphi_\alpha (\log(1 + |f^*(x)|)) \, dx < K$$

for all $f \in L$;

(ii) for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_{E} \varphi_\alpha (\log(1 + |f^*(x)|)) \, dx < \varepsilon, \text{ for all } f \in L,$$

for any measurable set $E \subset \mathbb{R}$ with the Lebesgue measure $|E| < \delta$.  

Proof. We follow [6, Theorem 1].

Necessity. Let $L$ be a bounded subset of $N \log^\alpha N(D)$, $\alpha > 0$. We use the notation $|f|_\alpha = d_{N \log^\alpha N(D)}(f, 0)$.

(i) For any $\eta > 0$ there is a number $\lambda = \lambda(\eta)$, $0 < \lambda < 1$, such that

$$d_{N \log^\alpha N(D)}(\lambda f, 0) = \int_\mathbb{R} \varphi_\alpha (\log(1 + \lambda |f^*(x)|)) \, dx < \eta$$

(1)

for all $f \in L$. By using the inequalities $(1 + x)^\lambda \leq 1 + \lambda x$ $(0 < \lambda < 1, \, x \geq 0)$ and $|cf|_\alpha \leq \max(1, \, |c|^{\alpha+1}) |f|_\alpha$ $(c \in \mathbb{C})$, we have, from (1),

$$\int_\mathbb{R} \varphi_\alpha (\log(1 + |f^*(x)|)) \, dx \leq \int_\mathbb{R} \varphi_\alpha \left( \log(1 + \lambda |f^*(x)|) \right) \frac{1}{\lambda} \, dx$$

$$\leq \frac{1}{\lambda^{\alpha+1}} \int_\mathbb{R} \varphi_\alpha (\log(1 + \lambda |f^*(x)|)) \, dx$$

$$< \frac{\eta}{\lambda^{\alpha+1}} = K = \text{constant}$$

for all $f \in L$. Therefore, we get condition (i).

(ii) For any number $\varepsilon > 0$, choose $\eta$ satisfying $\eta < \varepsilon/2^\alpha+3$. Next we take a number $\lambda = \lambda(\varepsilon)$, $0 < \lambda < 1$, so that (1) holds for all $f \in L$. Then for each measurable set $E \subset \mathbb{R}$, we obtain

$$\int_E \varphi_\alpha (\log(1 + |f^*(x)|)) \, dx < \int_E \varphi_\alpha \left( \log \left( \frac{1}{\lambda} + |f^*(x)| \right) \right) \, dx$$

$$= \int_E \varphi_\alpha \left( \log \left( \frac{1}{\lambda} + \log(1 + \lambda |f^*(x)|) \right) \right) \, dx$$

$$\leq 2^{\alpha+2} \left( \int_E \varphi_\alpha \left( \log \frac{1}{\lambda} \right) + \varphi_\alpha (\log(1 + \lambda |f^*(x)|)) \right) \, dx$$

$$< 2^{\alpha+2} \left( \varphi_\alpha (\log \frac{1}{\lambda}) |E| + \int_\mathbb{R} \varphi_\alpha (\log(1 + \lambda |f^*(x)|)) \, dx \right)$$

$$= 2^{\alpha+2} \left( \varphi_\alpha (\log \frac{1}{\lambda}) |E| + \eta \right),$$

where we use the inequality

$$\varphi_\alpha (x + y) \leq 2^{\alpha+2} (\varphi_\alpha (x) + \varphi_\alpha (y)) \quad (x, \, y \geq 0).$$

Taking $\delta > 0$ as $\delta < \frac{\varepsilon}{2^{\alpha+3} \varphi_\alpha (\log(1/\lambda))}$, we have
\[
\int_E \varphi_\alpha (\log(1 + |f^*(x)|)) \, dx < 2^{\alpha+2} \left( \delta \varphi_\alpha \left( \log \frac{1}{\lambda} \right) + \frac{\varepsilon}{2^{\alpha+3}} \right)
\]
\[
< 2^{\alpha+2} \left( \frac{\varepsilon}{2^{\alpha+3}} + \frac{\varepsilon}{2^{\alpha+3}} \right) = \varepsilon
\]

for all \( f \in L \) and any measurable set \( E \subset \mathbb{R}, |E| < \delta \). Thus condition (ii) is satisfied.

**Sufficiency.** Let conditions (i) and (ii) hold for a subset \( L \) of \( N \log^\alpha N(D), \alpha > 0 \). Let \( V \) be a neighborhood of zero:

\[
V = \{ g \in N \log^\alpha N(D) : d_{N \log^\alpha N(D)}(g, 0) < \eta \}.\]

We take \( \varepsilon > 0 \) as \( \varepsilon < \eta/3 \). According to (ii), there exists a number \( \delta > 0 \) such that

\[
\int_E \varphi_\alpha (\log(1 + |f^*(x)|)) \, dx < \varepsilon < \frac{\eta}{3} \quad (2)
\]

for all \( f \in L \) and any measurable set \( E \subset \mathbb{R}, |E| < \delta \). Next there is a finite constant \( K > 0 \) such that condition (i) holds for all \( f \in L \). Applying Chebyshev’s inequality to the Lebesgue measure of the set \( E_f = \{ x \in \mathbb{R} | \varphi_\alpha (\log(1 + |f^*(x)|)) > K/\delta \} \) for \( f \in L \), we have the estimate

\[
|E_f| \leq \frac{\delta}{K} \int_\mathbb{R} \varphi_\alpha (\log(1 + |f^*(x)|)) \, dx < \delta.
\]

We may presume \( E = E_f \) and \( |f^*(x)| > \exp(\varphi^{-1}(K/\delta)) - 1 = C \) in inequality (2), namely, \( |f^*(x)|/C < 1 \) for all \( x \in \mathbb{R}\setminus E_f \). Therefore, for any number \( \lambda (0 < \lambda < 1) \) and all \( f \in L \), it follows that

\[
\int_\mathbb{R} \varphi_\alpha (\log(1 + \lambda|f^*(x)|)) \, dx
\]
\[
= \int_{E_f} \varphi_\alpha (\log(1 + \lambda|f^*(x)|)) \, dx + \int_{\mathbb{R}\setminus E_f} \varphi_\alpha (\log(1 + \lambda|f^*(x)|)) \, dx
\]
\[
< \int_{E_f} \varphi_\alpha (\log(1 + |f^*(x)|)) \, dx
\]
\[
+ \int_{E_1} \varphi_\alpha (\log(1 + \lambda|f^*(x)|)) \, dx + \int_{E_2} \varphi_\alpha (\log(1 + \lambda|f^*(x)|)) \, dx, \quad (3)
\]
where \( R \setminus E_f = E_1 \cup E_2 \), \( E_1 = \{ x \in R \mid |f^*(x)| < 1 \} \) and \( E_2 = \{ x \in R \mid 1 \leq |f^*(x)| < C \} \). Utilizing the inequality \( 1 + nt \leq (1 + t)^{2n} \) \((0 \leq t < 1, 0 < n < 1/2)\) to the second integral in (3), using (2) and taking

\[
\lambda = \min \left( \frac{1}{2}, \frac{\eta}{6K}, \frac{1}{C} \left( \exp \left( \frac{\eta \varphi_\alpha(\log \frac{2}{3K})}{\sqrt{\log(1 + \lambda |f^*(x)|)} - 1} \right) \right) \right),
\]

we obtain

\[
\int_R \varphi_\alpha \left( \log(1 + \lambda |f^*(x)|) \right) dx < \frac{\eta}{3} + 2\lambda K + \frac{\eta}{3K} \int_{E_2} \varphi_\alpha \left( \log(1 + 1) \right) dx
\]

\[
< \frac{\eta}{3} + \frac{\eta}{3} + \frac{\eta}{3K} \int_R \varphi_\alpha \left( \log(1 + |f^*(x)|) \right) dx
\]

\[
< \eta.
\]

Hence we gain \( \lambda L \subset V \) and the set \( L \) is bounded in \( N \log^\alpha N(D) \), which concludes the proof.

\[\Box\]

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