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The Action Principle for Streams

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Abstract

This paper discusses the integrability conditions of dynamical systems of particle streams of charges and masses (charged and neutral).

Keywords: Dynamical systems for streams and fields, Lagrange equations

1 The problem domain

In classical mechanics, the dynamical system is commonly defined through the Lagrange function $L : (t, q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n) \mapsto L(t, q, \dot{q}) \in \mathbb{R}$, which is defined and continuously differentiable (in all coordinates) on some open set $I \times U \subset \mathbb{R} \times \mathbb{R}^n$ of time and location coordinates $t, q_1, \ldots, q_n$, where $\dot{q}_j := \frac{dq_j}{dt}$, $(1 \leq q_j \leq n$ are the according velocity coordinates. The Lagrange function itself is not deliberate, but the difference $L := V - T$ of a potential energy $V : (t, q, \dot{q}) \mapsto V(t, q, \dot{q}) \in \mathbb{R}$ and the kinetic energy $T := \sum_j \frac{p^2}{2m_j}$ of the system, where $E := T + V$ is to be the system's energy. That makes it possible to substitute the velocity coordinates $\dot{q}_j$ with the conjugated momenta $p_j := \partial L / \partial \dot{q}_j$ in what is called a Legendre transformation, which transforms L to a function $L = p(t) \cdot \dot{q} - H(t, q, p)$ of q, p, and t, with H being the energy of the system (see e.g.: [6, Ch.7, 40] or online [7, Ch.7, 40]). This defines the differential 1-form $p(t) \cdot dq - H(t, q, p)dt$. And with this, the problem of extremal action reduces to find local, if not global integrals of that 1-form (see [1], which also discusses the main theorem of differential forms, known as Poincaré's Lemma, in Chapter 2.13). If it exists, then it is unique up to an additive constant, so if the open region $U \subset \mathbb{R}^n$ is simply connected or even convex, then a local integral for some neighbourhood of $q \in U$ can be continued onto U.

In this paper, the aim is is to carry over the results for n-particle systems to relativistic 4-vector fluxes $j(t,x) = (j_0(t,x), \ldots, j_3(t,x))$ on space-time regions, for both mass and charges.

2 Non-relativistic fluxes (only masses)

As above, let $I \subset \mathbb{R}$ be a (non-empty) open (time-)interval, $U \subset \mathbb{R}^3$ be a (non-empty) open region, and let $j = (j_0, \ldots, j_3)$ be the quadrupel of energy and momentum densities (called flux), all defined as continuously differentiable functions $j_k : I \times U \ni (t,x) \mapsto j_\nu(t,x) \in \mathbb{R}$, $(0 \le k \le 3)$.
That is not yet defining the dynamics of the system: Like $E = H(t,q,p)$ for mass points, we need to express the energy density j_0 as a function $j_0 = H(\rho, \vec{j})$ to get things going, where $\rho : I \times U \ni (t, \vec{x}) \mapsto \rho(t,x) \in \mathbb{R}$ stands for the mass density of the system.
That raises a major problem upfront:

The densities of dynamical units are not appropriate for a calculus of streams: They do not constitute an algebra: For instance, it is common to write $\vec{j} = \rho\vec{v}$ for the fluxes, where ρ is the mass density. In here, $\vec{v}$ is not a vector, but a vector field, and to get at that field, given $\vec{j}$, we need to divide $\vec{j}$ by ρ (for each space-timevalue $(t, \vec{x})$. We would then divide the densities j_k by another density, ρ, which is not well-defined, even if we could assume that $\rho(t, \vec{x}) > 0$ whenever $j_k(t, vecx) \neq 0$. Similarly, given the vector field v and the mass density ρ, $T := (1/2)\rho v^2$, would well-define the kinetic energy, but $T = \frac{1}{2\rho}\vec{j}^2$ or even $\vec{j}^2$ are undefined. So, we have to keep from densities when dealing with streams in dynamical systems.
This leads to the following convention, posed as definition:

Definition 2.1 (Streams)**.** *Let u be a scalar real-valued quantity of a physical unit (like meter, energy etc.). Then the **stream** for that unit will be defined asa function $u : I \times U \ni (t, \vec{x}) \mapsto u(t, \vec{x}) \in \mathbb{R}$, where $I \subset \mathbb{R}$ is an (open) interval and $U \subset \mathbb{R}^3$ an open region of space. Analogously, for vector quantities $\vec{u}$ like momentum $\vec{p}$ and velocity $\vec{v}$, the stream for these quantities is defined a mapping $\vec{u} : (t, \vec{x}) \mapsto \vec{u}(t, \vec{x})$ from $I \times U$ to that real vector space $\mathbb{R}^n$, say. And if the units are complex-valued, then the target space of the approriate streams will be $\mathbb{C}$-, or $\mathbb{C}^n$-valued functions. The density of these streams is defined separately by a time curve $\mu : I \ni t \mapsto \mu(t)$ of finite Borel measures $d\mu(t)$ on U (which then may define the pointwise positions of the n particles at given time t if the measures are discrete finite sums of Dirac-distributions).*

Remark 2.2 (Non-technical explanation)**.** *Streams set the stage for the dynamical system, for instance at which location in space-time the system will have which potential energy, or what is the supposed velocity of the pointlike particles to be at which location. Then the (finite) set of particles are initially* *placed for time t_0 and $\vec{x} \in U \subset \mathbb{R}^3$. That determines the density of the particles all over space at t_0, which defines the measure $\mu(t_0)$ on U, w.r.t. which it will be possible to integrate (instead of having to count up the distinct values). And the goal is to set the rules of motion of these particles and to determine the change $\mu : t \mapsto \mu(t)$ over time from start time t_0 to end time t_1.*

With this, the quadrupel $j = (j_0, \ldots, j_3)$ rewrites to the energy-momentum stream $(E, \vec{p})$, where in case of missing interaction (free dynamical system), E is given as $E = H(t, p, \vec{x}) = \frac{1}{2m}p^2 = \frac{m}{2}v^2$, with m, $\vec{v}$, and $v^2 := v_1^2 + \cdots + v_3^2$ being streams, where $\frac{1}{2m(t,\vec{x})}p_j^2(t, \vec{x}) := 0$ is set, whenever $p_j(t, \vec{x}) = 0$. And non-free, interacting theories will be added a potential field stream V, which may even come with a velocity stream in its argument.

Then the dynamics problem reduces to integrate the 1-form of streams $p \cdot dx - H(t, \vec{x}, p)dt$ at least locally, if not globally, because, if it integrates to a function $S(t, \vec{x})$, then

(i) $p(t, \vec{x}) = \frac{\partial S(t,\vec{x})}{\partial x}$,

(ii) $E(t, \vec{x}) = -\frac{\partial S(t,\vec{x})}{\partial t}$, and

(iii) $\dot{p}(t, \vec{x}) = -F(t, \vec{x})$, where $F := \frac{\partial E(t,\vec{x})}{\partial x}$.

Thus, given n mass points at start time t_0, $\vec{x}_1(t_0), \ldots, \vec{x}_n(t_0)$, say, their motion is constrained on a geodesic of S, until, of course, they reach the boundary of U (, which is why one would like to continue S onto a convex set $U' \supset U$, if not onto all of $\mathbb{R}^3$). Clearly, the direction of the particles' trajectories at time t_0 are determined by their momentum, which has to be set at start time t_0, too.

Remark 2.3 (Collision problem)**.** *There is still a severe and subtle problem: What happens, when the particles collide? For now, I would like to let the particles pass through eachother "freely" to settle that problem. (Reason: when two particles of equal mass m collide in an overhead collision with inverted velocity $vecv_0$ and $-vecv_0$, the we'd get a particle pair with relative momentum of zero, while the parity-invariant kinetic energy would add to mv_0^2. Put differently: In a liquid, we might have an equal amount of particles colliding constantly over time, one half moving to the left, the other to the right. And the total momentum could be zero at all times: we'd then created an "ether of free energy": heat.)*

The condition of integrability of $p \cdot dq - Hdt$ is Poincar'e's lemma: Given an open time interval $I \in \mathbb{R}$ and an open simply connected or even convex region $U \subset \mathbb{R}^3$, on which the streams H and $\vec{p}$ are continuously differentiable. Then $p \cdot dq - Hdt$ is integrable on $I \times U$ if and only if the derivative of the vector stream (H, p_1, p_2, p_3) is a symmetric matrix for all $(t, \vec{x} \in I \times U)$, i.e.: if and only if (H, p_1, p_2, p_3) is curl-free on that domain, see: [1]).

3 Relativistic streams for masses and charges

Because the electromagnetic force is a factor $2 \times 10^{4}3$ stronger than the gravitational force, relativistic effects dominate electrodynamics even for small velocities. That is why it plays no role in non-relativistic dynamical systems, but takes a predominant role in relativistical dynamical systems.

Although the mass will be seen to be equivalent to a neutral conglomerate of charges in the following, let us begin with masses under relativistic conditions:

Again, we are well-advised not to base calculus on energy and momentum densities $j = (j_0, \dots, j_3)$, but on the quadrupel of streams $p := (p_0, \dots, p_3)$ as in 2.1, where p_0 stands for the energy stream E. Now, p is not a 4-vector, but a mapping from time and space 4-vectors to 4-vectors, and, according to relativity, p is demanded to be Lorentz-covariant: That means that for each Lorentz transformation Λ of the time-space vectors p must transform to a 4-vector field p', such that the Minkowski square $p_0'^2 - \dots - p_3'^2 = p_0^2 - \dots - p_3^2$ is conserved.

In here, one can seemingly avoid the squares by expressing the covariance condition in terms of the Lorentz transformations themselves: $p(x) := p(t, \vec{x})$ has to transform under Λ to $p'(x') = \Lambda p(\Lambda^{-1} x)$.

Then the basic relativistic equation for a free mass particle,

$$E = \sqrt{m_0^2 c^4 + \vec{p}^2 c^2},$$

where $m_0 c^2$ is the rest (mass) energy and $c \equiv 1$ the speed of light, demands that the stream H must be formally the same. This square root is a formidable complication in comparison to the non-relativistic limit, which is the 2^{nd} order approximation of the Taylor series expansion of this square root.

Note that this square root complication is luring also behind time and space axes itself: we have $t^2 - \vec{x}^2 = const$, either, so $t^2 = const + \vec{x}^2$, and it only be due to a smart, yet unmentioned physical principle that we must replace that constant with 0, that we may simply identify $|t|$ with $|\vec{x}|$.

From a mathematical view, the above problem is about how to map Euclidean spacetime bijectively onto Minkowski spacetime. And there is such a mapping: With the help of the four Dirac matrices $\gamma_0, \dots, \gamma_3$ (see: e.g.: [5]), we may idenify the Minkowski space $\mathcal{X}$ with, or better, explicitly define it as:

Definition 3.1 (Minkowsi space). *The Minkowski space $\mathcal{X}$ is the vector space of all $\sum_{0 \le \nu \le 3} \gamma_\nu x_\nu$ with $x = (x_0, \dots, x_3) \in \mathbb{R}^4$. Alternatively, we'll also write: $(x_0\gamma_0, \dots x_3\gamma_3) := \sum_\nu \gamma_\nu x_\nu$ for its elements.*

Then $\Theta : \mathbb{R}^4 \ni x \mapsto \sum_\nu \gamma_\nu x_\nu \in \mathcal{X}$ becomes a vector space isomorphism of the Euclidean spacetime $\mathbb{R}^4$ onto the Minkowski space $\mathcal{X}$.

Remark 3.2 (Uniqueness). *Note that the Minkowski space $\mathcal{X}$ and the mapping Θ are uniquely defined up to an isomorphism of vector spaces: this is, because the group generated by the γ_ν has only unity as its invariant subgroup, i.e. only the unit element (which is the unit 4×4-matrix in this case) commutes with all the other group members).*

It now evident that we need to transform both, the 4-vector coordinates and space-time coordinates of the Euclidean stream $x \mapsto p(x)$ to the Minkowski space, in order to get these set up covariantly. So, $x \mapsto p(x)$ becomes $\sum_\nu x_\nu\gamma_\nu \mapsto p(x_0\gamma_0, \dots, x_3\gamma_3)$, and the tuple (dx, dt) path integral differentials , will transform into $(\gamma_1 dx, \dots, \gamma_3 dx_3, \gamma_0 dt)$, while H for a free system becomes $H(\gamma\cdot x) = m_0\gamma_0 + \sum_{1\le k\le 3} p_\nu\gamma_0\gamma_\nu$.

As immediate consequence we get:

Proposition 3.3 (Integrable Lorentz covariant streams). *Let $p : I \times U : x \mapsto (p_0(x), \dots, p_3(x)) \in \mathbb{R}^4$ be a continuously differentiable energy-momentum stream in the Euclidean spacetime region $I \times U$, where I is an open time interval and $U \subset \mathbb{R}^3$ is an open and convex spatial region. Then its image Θp is integrable in the Minkowski space $\mathcal{X}$ if and only if the derivative Dp is anti-symmetric (for all $x \in I \times U$).*

Proof. Because the Dirac matrices anti-commute, the p_ν in Minkowski space come with the additional factors γ_ν, and integration in there along the ν^{th} component transforms to $\gamma_\nu dx_\nu$, Poincar'e's lemma mandates the anti-symmetry of Dp on $I \times U$. And because $\gamma_0^2 = -\gamma_1^2 = \cdots = -\gamma_1^2 = 1_4$, (where 1_4 denotes the unit matrix), the resulting action function S can be identified with a real-valued function. □

The step from masses to electric charges is minimal:
The mass m will be replaced by the charge e, the momentum $\vec{p} = m\vec{v}$ becomes the current $e\vec{v}$ and will be given in the SI unit of Ampere. That is all.

Interestingly, the electromagnetic field tensor, which is the Euclidean derivative $D(eA)$ of the electromagnetic 4-vector potential eA (e being the electronic charge) is always anti-symmetric (see e.g.:[2, Vol. II]), which implies the anti-symmetry $eD\Box A$, where $\Box := -(\partial^2/\partial x_0^2 - \cdots - \partial^2/\partial x_3^2)$ is the d'Alembert operator. But $e\Box A = j$ according to the covariant Maxwell equations, where j is the 4-vector current density, so Dj always is anti-symmetric on all of $\mathbb{R}^4$. So, in practically all cases, where is continuously differentiable, the above proposition applies and implies the global integrability of the electromagnetic streams in the Minkowski space.

Now, irrespective of whether a relativistic theory of charges or masses: From the above, we know that if the streams are globally integrable, the 4-vector stream $p : x \mapsto p(x)$ can be integrated in Minkowski space to a scalar action function. This is then a 0-form. And this 0-form can be integrated again to a vector field. The path integration has to be in the Minkowski space: that is: S is to be integrated from a fixed point, the origin, say, along the x_0-component to $\gamma_0 y_0$ with $\gamma_0 dx_0$, and for $\nu = 1, 2, 3$ the x_ν^{th} component from 0 to $\gamma_\nu y_\nu$ with the differential $\gamma_\nu dx_\nu$. This gives the vector function

$$\mathcal{A} : \gamma \cdot y := (\gamma_0 y_0, \ldots, \gamma_3 y_3) \mapsto (\gamma_0 A_0(\gamma \cdot y), \ldots, \gamma_3 A_3(\gamma \cdot y)) = \sum_\nu \gamma_\nu A_\nu(\gamma \cdot y),$$

where charge or mass factor e or m are included into A for simplicity.

Since $\left(\sum_\nu \gamma_\nu \frac{\partial}{\partial_\nu}\right)^2 = \Box$, we have $\Box \mathcal{A}(\gamma \cdot x) = \sum_\nu p_\nu(\gamma \cdot x)$, and mapping this back to the Euclidean space time $\mathbb{R}^{\not 4}$ we get

Proposition 3.4. *The action function of a relativistic covariant dynamic system spreads from the stream of charges or masses at the speed of light, which makes it equivalent to light. Twice covariant integration of the action the Minkowski space yields the* ***electomagnetic 4-vector potential*** *(density) A from the 4-vector fluxes j up to the addition of* ***plane waves***

$$\{\Psi : x \mapsto \mathbb{C} \mid \Box \Psi \equiv 0\},$$

which is the kernel of $\Box$. But moreover, it now also follows that this electromagnetic potential A must affect the relativistic theory of (mechanical) masses in the very same way. This strongly points to the spatial (or magnetic) part $\vec{A} := (A_1, A_2, A_3)$ as the responsible for ***thermal radiation*** *in a relativistic theory of masses (see remark 2.3), which is well-known to be electromagnetic in origin.*

Applying of what is known in (quantum) physics as the "principle of minimal coupling", it will be straightforward to add A as an "external field" to the given "internal", otherwise free electromagnetic or mechanical 4-vector stream. However, that principle has never been proved with rigor: it just works by experimental proof (see: [2, Vol. III, Ch. 21-1]). So, it's worth to have a closer look at this.

4 Gravitational and electromagnetic interaction

As was shown in the previous section, the electromagnetic 4-potential A, in particular, derives from the source streams (or flux densities) by convariantly integrating these twice, and for this to be possible, the derivative of the sources in the Minkowski space must be a symmetric matrix. Though, Theoretical Physics does this in one step through what is called the "Green's function" (see e.g.: [2, Vol. II, Ch. 21]) $G(t,\vec{x}) := \frac{\delta(t-|\vec{x}|)}{4\pi|\vec{x}|}$, such that $A(t,\vec{x}) = \int \frac{1}{4\pi|\vec{x}-\vec{x'}|} j(t - |\vec{x}-\vec{x'}|, \vec{x'}) d^3x'$ solves $\Box A(x) = j(x)$, and it does this without respect to symmetry or anti-symmetry of the derivative Dj. So, how can this be?

Let's examine the derivation of that Green's function:
A Green's function for a linear differential operator $\mathcal{D}$ is mathematically nothing but a distribution, F, say, for which $\mathcal{D}F = \delta(x)$ holds, where δ is the Dirac distribution and x an element of an open subset of $\mathbb{R}^n$. The elements of the kernel of $\mathcal{D}$ are called solutions of the homogenous equation, so that, given the well-definedness of the convolution $T * g$, the solution of the inhomogenous equation $\mathcal{D}f = g$ is - up to the addition of homogenous solutions (short: modulo $kern(\mathcal{D})$), given by $f = T * g$ (see: [8] for details).
So, head on, the proper Green's function for the d'Alembert operator $\Box$ would solve $\Box G(x) = \delta(x)$, where $x \in \mathbb{R}^4$. But that's not what the above Green's function is about: it's solving for a subset:
Given that $-\Delta\frac{1}{|\vec{x}|} = 4\pi\delta(\vec{x})$, where $\Delta := \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \partial^2/\partial x_2^2$ is the Laplace operator, with the help of the equation $\delta f(x) = \delta(x)/f'(x)$, with f' being the derivative of (from e.g. [3]), we see that this function G is about to solve $\Box G(x) = \delta(t^2 - vecx^2)$ on the forward light cone w.r.t. the location coordinates with just a factor $\frac{1}{2}$ being off.
Indeed, we have $\delta(t^2 - r^2) = \frac{\delta(t+r)}{2r} + \frac{\delta(t-r)}{2r}$ (see e.g: [3]) as a distribution in t. G evidently derives from that equation. That also explains the missing factor $\frac{1}{2}$, because the previous equation splits $\delta(t^2 - r^2)$ in two halves: one with support on the forward cone (half of the retarded propagator) plus the other half on the backward light cone (half of the advanced propagator).

The intention is clear: we want to come up with the Coulomb law as to charges, the potential energy of which goes with e/r, which then will be propagated via G from the external charged source at the speed of light to the (internal) target, with which light is to interact. Plus: the separation of time from space, which concentrates on a partial solution is smart, not only because of its simplicity, but because it concentrates on long ranges over a long time, which is exactly where one will expect gravitation and electromagnetism to live in.

However, this idea brings several physical problems: The first one is that G has support only on the forward cone, but there also must be a past, so G is lacking the solution on the backward light cone. So we should replace G with $\delta(t^2 - r^2) = \frac{\delta(t+r)}{2r} + \frac{\delta(t-r)}{2r}$. But then, even that solution is missing time symmetry: In that equation the light all comes in from the past (on the backward cone) and goes out along the forward light cone (w.r.t. the time direction). And if time is inverted, then light will come from the future and further propagate to the past. And that is considered to be unphysical, because being "irrational": it would conflict the causality principle (made just to enforce exclusion of light from propagating from the future to the past). With this principle in place, the result is the breakdown of the immediate interaction of particles and fields, known as "actio=reactio": particle A emits light (at A's time t_0) that hits particle B at a later, retarded time. When particle B gets hit at time t_1 by something from the past, it immediately starts to react. In the meantime A is without reaction from B at B's local time t_1 and travels on some time without B's reaction at t_1... That is the beginning of difficulties. And it is all unnecessary: Because all the discussion is w.r.t. Euclidean time, but not Minkowski time: time stands still along the light cone, and so the interaction of A and B on the light cone occurs at the same instance of Minkowski time, i.e.: instantly to and fro.

Now we can fix that problem swiftly:
We have $\delta(t^2 - r^2) = \delta(t^2 - r^2) * \delta(t^2 - r^2)$, so we'll get time symmetry by convoluting the the advanced propagator $\frac{\delta(t+r)}{r}$ with the retarded propagator $\frac{\delta(t-r)}{r}$ w.r.t. the time coordinate (so its a one-dimentional convolution), and since $\delta(t + r) * \delta(t - r) = \delta(t)$, we end up with

$$G(t, \vec{x}) := \delta(t^2 - \vec{x}^2) = \frac{\delta(t)}{\vec{x}^2}. \tag{4.1}$$

So, it appears that we fall short of one factor $\frac{1}{r}$, because we expect the potentials to go with $\frac{e}{r}$ for charges rather than $\frac{e}{r^2}$. However, we're not: There is no real error in our calculation, there is a quirk in the very definition of both electromagnetic and gravitational fields: The Green's function G operates on the (charged) 4-vector source stream by convolution, so as the source comes with the common charge factor e, then the resulting field also has that proportionality factor e. We expect that result to be the electromagnetic 4-vector field A. However, by convention, A is defined per charge, so it drops the charge by (internally) dividing by it. (This is, why the potential of charge/mass is proportional to the product of charges/masses, rather than the charge/mass itself, and it is the reason, why the charge in the fine structure constant $e^2/\hbar c$ is squared rather than in terms of $|e|$.) So, we are actually short of the additional factor e, and everything seems fine: we just need to take the square root of $\frac{e^2}{r^2}$. But then the problem is, how to get at the mutual potential $\pm\frac{q_1 q_2}{r}$ for masses/charges q_1 and q_2.

To get at its root, we would need to integrate the scalar field $\Phi : \mathbb{R}^3 \ni \vec{x} \mapsto \frac{1}{\vec{x}^2}$ over a sphere via an orientated surface integral (which is - as might not be obvious - what Cauchy did for 2 dimensions). Rather than citing [4] for a general abstract analysis, let's shortly work up what is needed for now:

5 Cauchy surface integrals in 3 dimensions

There are two reasons that make Cauchy path integrals fundamentally different from ordinary path integration within $\mathbb{R}^2$: The first is that the vector space isomorphism $I : \mathbb{R}^2 \ni (x, y) \mapsto x + iy \in \mathbb{C}$ extends on side of $\mathbb{C}$ to an algebra: the product of two complex numbers is in fact "some kind of a vector product", as the result is again a vector. The second one is that a complex function is, when seen from within $\mathbb{R}^2$, not a scalar function, but a vector function. And the result of that is, that the integration in $\mathbb{C}$ of a real-valued scalar field in a closed path is orientated, while the integral within $\mathbb{R}^2$ is (by definition) un-orientated:
For example, let $\Phi : \mathbb{R}^2 \ni (x, y) \mapsto 1 \in \mathbb{R}$ be the constant field. If we integrate that in a circle of radius 1 around the origin in $\mathbb{R}^2$, the expected result is the length of that circle, which is 2π. This is, because the parity flip that occurs, when going from left to right or from top to bottom, is cancelled (for good) by taking the lengths of the path differentials to their absolute values. However, within $\mathbb{C}$, the integral over the circle is zero: the (orientated Cauchy path) integral of Φ from a to b, which is a vector function, depends only on the start a and endpoint b. As strange as that Cauchy integral might be in this case, it has its virtues: Given two discs $D1$ and $D2$ in $\mathbb{R}^2$, then the Cauchy integrals over their boundaries add for $D1 \cup D2$, even if the intersection $D1 \cap D2 \neq \emptyset$. This is what enables analytic continuation... But let's proceed with 3 dimensions:

Remark 5.1. *Note that Cauchy integrals are equivalent to path integrals of closed paths in a vector field: Given scalar function Φ and path ω, one can select a $\vec{a} \in \mathbb{R}^2$ as a reference point outside the image of ω. Further for any $\vec{x} \neq \vec{a}$, let $\vec{e}_{\vec{x}}$ be the unit vector of $\vec{x} - \vec{a}$. Then $\Psi : \vec{x} \mapsto \Phi(\vec{x})\vec{e}_{\vec{x}} \in \mathbb{R}^2$ defines a vector field, for which the path integral $\int_\gamma \Psi(\vec{x}) \wedge d\vec{x}$ is orientated and delivers equivalent results, if only one defines $\vec{x} \wedge \vec{y} := sin(\vec{x}, \vec{y})|\vec{x}||\vec{y}|$ with $sin(\vec{x}, \vec{y})$ being the sinus between $\vec{x}, \vec{y} \in \mathbb{R}^2$. As such, Cauchy theorem is related to Stokes theorem in 3 dimensions, while it is in essence the 2-dimensional version of Gauß law (see: section 6).*

The surface integral in three dimensions over a scalar field is a direct extension: The differentials of surface integrals in $\mathbb{R}^3$ are always differentials of 2-forms $dx_1 \wedge dx_2, dx_2 \wedge dx_3$, and $dx_3 \wedge dx_1$, w.r.t. which a given (potential) scalar function $\Phi : \vec{x} := (x_1, x_2, x_3) \mapsto \Phi(\vec{x})$ is to be integrated over a (piecewise) smooth boundary of some contained region. That is automatically orientated, because of anti-symmetry of the exterior derivatives $dx_k \wedge dx_l = -dx_l \wedge dx_k$ for all k, l. As such, as it is oriented, the integral of the constant function $\Phi \equiv 1$ over a sphere again is zero. If we wanted to get at the "correct" surface of the sphere, again we have to get away with the parity flips, which is done by taking not the (signed) determinants that occur in that integral, but their absolute values. That way, we are integrating unorientatedly (which is usually defined as the proper and only way to integrate scalar fields over closed boundaries of region). Keeping with Cauchy means integrating with orientation: That said, the surface integral over the boundaries of two balls is again additive, so given a bounded region Ω in $\mathbb{R}^3$ with a sufficiently smooth boundary, for every $\epsilon > 0$ we can find a finite set of ϵ-balls in Ω such that the boundary of the union of the balls approximates the surface of R, such that the maximal distance of both boundaries is smaller than 2ϵ. Now, given that the function Φ is continuously differentiable on an open set U containing the closure $\overline{\Omega}$ of Ω, then we even can cover the the closed region by finitely many ϵ-balls, which will spare us to deal with irregular boundaries. Because the surface integral of Φ over each ϵ-ball is smaller than $4\pi \sup_{\vec{x}\in\overline{\Omega}}|\Phi(\vec{x})|\epsilon^2$, we get two things: The limit of the surface integrals of the ϵ-balls uniformly converges to zero (even in the second order), as $\epsilon \to 0$, and this holds for all ϵ balls around $x \in \overline{\Omega}$. That gives the zero function on $\overline{\Omega}$, so the volume integral over Ω is zero. Secondly, whatever shape the boundary of Ω has, it must be continuous, hence it is Borel measurable. And since the values on that boundary are all zero, all Borel measures on the boundary will vanish, too. (If the smoothness was only given on Ω itself, we would have needed to restrict on ϵ-balls in its interior, and would have needed an estimate to prove that the surface integral of the balls converges to the surface integral of the boundary of Ω.)

Now, there is one particular family of functions, for which that Cauchy surface integral does not vanish: that is $\Phi_{\vec{a}} : \mathbb{R}^3 \setminus \{\vec{a}\} \ni \vec{x} \mapsto \frac{1}{\|\vec{x}-\vec{a}\|^2}$, $(a \in \mathbb{R}^3)$: Integration in positive orientation of these functions over a sphere (or cube) containing the pole $\vec{a}$ in its interior, always will give 4π, irrespective the size of sphere or cube (and -4π, when integrating with negative orientation). If however the pole $\vec{a}$ is outside the sphere, then $\Phi_{\vec{a}}$ is continuosly differentiable in a neighbourhood of that sphere and its internal, and then the Cauchy surface integral vanishes. And that gives the final hint:

6 Conclusion

Let q denote either charge e or mass m. Then $\Phi : \vec{x} \mapsto \frac{1}{4\pi}\frac{q^2}{(\vec{x}-\vec{a})^2}$ equivalently represents a particle with square unit q^2 at the location $\vec{a}$, which we can locate via the Cauchy surface integral above. Moreover, given n such particles within the interior of a sufficiently smooth boundary, the squares q_k^2 add up as their fields Φ_k add linearly through superposition, and also as do the their Cauchy surface integrals. If the unit for q was supposed to be additive, we'd need the mixed terms $2q_kq_l$, such that with $Q = \sum_k q_k$, $Q^2 = \sum_k q_k^2 + \sum_{k\neq l} q_kq_l$ will hold. Then these mixed terms are to be delivered by the propagator $\frac{\delta(t)}{r^2}$

as interaction Φ_{kl} between the factors Φ_k and Φ_l. Let's work out, what that Φ_{kl} is: Both q_k and q_l are to be positioned on the light cone to interact, so they share the same time Minkowski time $t = 0$, say, but different location coodinates $\vec{a}_k \neq \vec{a}_l$. Then

$$\Phi_{kl}(\vec{x}) = \int_{\mathbb{R}^3} \frac{1}{(4\pi)^2} \frac{q_k q_l}{(\vec{x} - \vec{a}_k)^2 (\vec{a}_k - \vec{y})^2 (\vec{y} - \vec{a}_l)^2} d^3 y.$$

Now, integration over $\mathbb{R}^3$ is the limit $\epsilon \to 0$ of the integration of $\mathbb{R}^3 \setminus B_\epsilon(\vec{a}_l)$, where $B_\epsilon(\vec{a}_l)$ is the ϵ-ball around $\vec{a}_l$, and we are in the lucky situation of even parity of the surface integral (for which orientation is irrelevant). So, integration over $\mathbb{R}^3 \setminus B_\epsilon(\vec{a}_l)$ is given by the surface integral over the ϵ-sphere $S_\epsilon(\vec{a}_l)$ around $\vec{a}_l$, followed by the radial integration from ϵ to ∞. Integration of the radial component by parts then gives:

$$\begin{aligned}
\Phi_{kl}(\vec{x}) &= lim_{\epsilon \to 0} \frac{1}{(4\pi)^2} \int_{S_\epsilon(\vec{a}_l)} \frac{q_k q_l}{(\vec{x} - \vec{a}_k)^2 |\vec{a}_k - \vec{a}_l| \, (\vec{y} - \vec{a}_l)^2} dS \\
&- lim_{\epsilon \to 0} \frac{1}{(4\pi)^2} \int_\epsilon^\infty \int_{S_\epsilon(\vec{0})} \frac{q_k q_l}{(\vec{x} - \vec{a}_k)^2 |\vec{a}_k - \vec{a}_l - \vec{r}|} \frac{\partial}{\partial r} \frac{1}{\vec{r}^2} dS dr \\
&= \frac{1}{(4\pi)} \frac{q_k q_l}{(\vec{x} - \vec{a}_k)^2 |\vec{a}_k - \vec{a}_l|},
\end{aligned}$$

where the 2^{nd} summand is zero, because $\frac{\partial}{\partial r} \frac{1}{\vec{r}^2} = -\frac{2}{r^3}$, and the (orientated) surface integrals $\int_{S_\epsilon(\vec{0})} \frac{-2}{|\vec{a}_k - \vec{a}_l - \vec{r}|} \frac{1}{r^3} dS$ vanish for $\epsilon > 0$: the integrand has only a pole of degree 1 in $\vec{a}_k - \vec{a}_l$ and a pole of degree 3 in $\vec{0}$, but lacks a pole of degree 2. And for $r \to \infty$, the surface integral vanishes again. So, and the radial integral of that term vanishes along with its (orientated) surface integrals.
The subsequent integration of $\vec{x}$ on a sphere around $\vec{a}_k$ finally results in $\frac{q_k q_l}{|\vec{a}_k - \vec{a}_l|}$.

Thus, we derived Coulomb and gravitational potential from the propagator $\frac{\delta(t)}{r^2}$, albeit as a square of energy: it is the sum of the mixed terms $\sum_{k \neq l} E_k E_l$, which supplements $\sum E_k^2$ to the square of sums $E^2 = (\sum_k E_k)^2$. Therefore, it appears that the physical convention to declare gravitational and electromagnetic potential per mass and per charge, is a means to be able to deal with a second order energy potential as if it was a first order one.

So far, we have $\Phi = \sum_k \Phi_k + \sum_{k \neq l} \Phi_{kl}$. But that is still incomplete for a free system: The square of the kinetic momentum $p^2 c^2$ still needs to be added (because $E^2 = m^2 c^4 + p^2 c^2$). And again, we get the appropriate field from the sources $c\vec{p}_k$ via the propagator $\frac{\delta(t)}{r^2}$ as $\Psi(0, \vec{x}) = \sum_k \Psi_k(0, \vec{x}) + \sum_{k \neq l} \Psi_{kl}(0, \vec{x})$ with $\Psi_k = \frac{c^2}{4\pi} \frac{p_k^2}{|\vec{x} - \vec{a}_k|^2}$ and $\Psi_{kl} = \frac{c^2 p_k p_l}{|\vec{a}_k - \vec{a}_l|}$. Again - up to sign of charge and a different value of the coupling constant - there is no difference between masses and charges, only that the Ψ_{kl} for charges would be called "magnetic field energy". (Actually, the Ψ_{kl} are well-known to be responsible for the transversal character of light, which affects both: charges and neutral masses.)

With that, the total square energy field Φ_{tot} of the system is given as $\Phi_{tot} = \sum_k(\Phi_k + \Psi_k) + \sum_{k\neq l}(\Phi_{kl} + \Psi_{kl})$, where it is essential that the Φ_{kl} and Ψ_{kl} are of square energy, because then $\sum_{k\neq l}(\Phi_{kl} + \Psi)$ will define an oscillator, while $\sum_k \Phi_k$ will be the energy square of its ground state.

References

[1] H. Cartan, *Differential Forms*, Herman Kershaw, 1971.

[2] R. P. Feynman, *Lectures on Physics*, Vol. I-III, Addison Wesley, 1977.

[3] I. M. Gel'fand and G. E. Shilov, *Generalized Functions*, Vol I+II, Academic Press, 1964. https://doi.org/10.1090/chel/377

[4] H. D. Huettenbach, Analytic functions for Clifford Algebras, *International Journal of Mathematical Analysis*, **15** (1) (2021), 61-69. https://doi.org/10.12988/ijma.2021.912140

[5] H. Kleinert, *Particles and Quantum Fields*, 2014. http://users.physik.fu-berlin.de/ kleinert/b6/

[6] L. D. Landau, E. M. Lifschitz, *Lehrbuch der Theoretischen Physik*, Vol. I, Akademie-Verlag, 1973.

[7] L. D. Landau, E. M. Lifschitz, *Course of Theoretical Physics*, Vol. 1, 1973. https://fdocuments.in/document/mechanics-3rd-edition-vol-1-landau-lifshitz.html

[8] François Trèves, *Topological Vector Spaces, Distributions, and Kernels*, Academic Press, 1967.

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