Controllability of Nonlocal Neutral Impulsive Differential Equations with Measure of Noncompactness

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Abstract

In this paper, we study the existence and uniqueness of the controllability of neutral impulsive differential equations with non local conditions. First, we establish a property of measure of non-compactness in the space of piecewise continuous functions. Then, by using this property and Darbo-Sadovskii’s fixed point theorem, we get the controllability of non local neutral impulsive differential equation under compactness conditions, Lipschitz conditions and mixed-type conditions, respectively.

keywords: Controllability, impulsive neutral differential equation, non local conditions, mild solution, measure of non compactness, fixed point theorem

1 Introduction

The theory of impulsive differential equation has been emerging as an important area of investigation in recent years, because, all the structures of its emergence have deep physical
background and realistic mathematical models. Impulsive differential equations appears as a natural description of several real processes subject to certain perturbations whose duration is negligible in comparison with the duration of the process. It has seen considerable development in the last decade, see the monographs of Bainov and Simenov [2], Lakshmikantham et al. [1], and the papers of [4, 5, 6, 7], where the numerous properties of their solutions are studied and detailed bibliographies are given.

The notion of controllability is of great importance in mathematical control theory. Many basic problems of control theory namely; pole-assignment, structural engineering, and optimal control; may be solved under the assumption that the system is controllable. The concept of controllability plays an crucial role in both finite and infinite dimensional spaces, that is systems represented by ordinary differential equation and partial differential equation respectively. In recent years, significant progress has been made in the controllability of linear and nonlinear deterministic systems [8, 9, 10, 11, 12, 13]. The objective of this paper is to prove the uniqueness of controllability result for impulsive neutral functional differential equations defined in [9] and also verify the above result by using measure of non-compactness.

The intention of this paper is to discuss the controllability of neutral impulsive differential equations with nonlocal conditions in a real Banach space X of the form:

\[
\frac{d}{dt} \{u(t) + g(t, u(t))\} = Au(t) + f(t, u(t)) + Bv(t), t \in J = [0, b], t \neq t_i,
\]

\[u(0) = h(u),\]

\[\Delta u(t_i) = I_i(u(t_i)), i = 1, 2, \ldots, s,\]

(1.1)

where \(A : D(A) \subseteq X \to X\) is the infinitesimal generator of a strongly continuous semigroup \(T(t), t \geq 0\) in a Banach space \(X\), \(B : U \subseteq X \to X\) is a bounded linear operator; the control function \(v(.)\)is given in \(L^2(J, U)\) with \(U\) as a Banach space; \(f, g\) are appropriate continuous functions to be specified later; \(I_i : X \to X\) is a nonlinear map, \(\Delta u(t_i) = u(t^+_i) - u(t^-_i)\), for all \(i = 1, 2, \ldots, s\), \(0 = t_0 < t_1 < t_2 < \ldots < t_s < t_s + 1 = b\), where \(u(t^-_i), u(t^+_i)\) denote the left and right limit of \(u\) at \(t = t_i\) respectively.

The study of abstract nonlocal conditions was initiated by Byszewski, and the importance of the problem consists in the fact that it is more general and has better effect than the classical initial conditions \(u(0) = u_0\). Therefore, it has been studied extensively under various conditions. Here, we utilize some papers [7, 8, 9, 10], where they studied impulsive differential equations with nonlocal conditions. In particular, the measure of noncompactness has been used as an important tool to deal with some similar functional differential and integral equations; see [3, 8, 14, 15].

From the viewpoint of theory and practical, it is natural for mathematics to combine impulsive conditions, nonlocal conditions and controllability. Recently, the controlability of nonlocal neutral impulsive differential problem of type (1.1) has been discussed in the papers of Anguraj [5, 7] and Chalishajar et al. [9].

2 Preliminaries

In this intention, we introduce notations, definitions and preliminaries which are used throughout this paper.

Let \((X, \|\cdot\|)\) and \((U, \|\cdot\|)\) be real Banach spaces. \(T(t)\) is a strongly continuous semigroup on \(X\),
with generator $A$ is $A : D(A) \to X$. We denote by $C([0, b]; X)$ the space of $X$-valued continuous functions on $[0, b]$ with the norm
\[
\|u\| = \sup \{\|u(t)\| : t \in [0, b]\}.
\]
A measurable function $f : [0, b] \to X$ is Bochner integrable if and only if $|f|$ is Lebesgue integrable. For properties of the Bochner integrable, see for instance, Yosida [8]. By $L^1([0, b]; X)$ the space of $X$-valued Bochner integrable functions on $[0, b]$ with the norm
\[
\|f\|_{L^1} = \int_0^b \|f(t)\| \, dt.
\]
The semigroup $T(t)$ is said to be equicontinuous if \{T(t)x : x \in B\} is equicontinuous at $t > 0$ for any bounded subset $B \subset X$ (cf.27). Obviously, if $T(t)$ is a compact semigroup, it must be equicontinuous. And the converse of the relation is not correct. Throughout this paper, we suppose that

(H1) The semigroup $T(t) : t \geq 0$ generated by $A$ is equicontinuous. Moreover, there exists a positive number $M$ such that
\[
M = \sup_{0 \leq t \leq b} \|T(t)\|.
\]
For the sake of simplicity, we put $J = [0, b]; J_0 = [0, t_1]; J_i = (t_i, t_i + 1], i = 1, 2, \ldots, s$. In order to define a mild solution of the problem (1.1), we introduce the set $PC([0, b]; X) = \{u : [0, b] \to X : u is continuous at \, t \neq t_i and left continuous at \, t = t_i, and the right limit u(t_i^+) exists, i = 1, 2, \ldots, s\}$. It is easy to verify that $PC([0, b]; X)$ is a Banach space with the norm
\[
\|u\|_{PC} = \sup_{t \in [0, b]} \|u(t)\|.
\]
Consider the infinite-dimensional linear control system
\[
\begin{align*}
u'(t) &= Au(t) + Bv(t), \quad t \in J = [0, b], \\
u(0) &= u_0,
\end{align*}
\]
where $v(t) \in L^2(J, U), A : X \to X, B : U \to X$.

(H2) Let $B \in L(U, X)$ and $b \geq 0$. The linear operator $W : L^2(J, U) \to X$ is defined by
\[
Wv = \int_0^b (b - s)Bv(s)ds
\]
such that

(i) $W$ has an invertible operator $W^{-1}$ which takes values in $L^2(J, U)/\ker W$ and there exist positive constants $M_1$ and $M_2$ such that $\|B\| \leq M_1$ and $\|W^{-1}\| \leq M_2$.

(ii) There is $K_w \in L^{-1}(J, R^+)$ such that, for every bounded set $Q \subset X$
\[
\beta(W^{-1}Q)(t) \leq K_w(t)\beta(Q).
\]
We define the control
\[ v(t) = W^{-1}\{u_1 - T(b)[h(u) + g(0, h(u))] + g(t, u(t)) + \int_0^b AT(b-s)g(s, u(s))ds \]
\[ - \int_0^b T(b-s)f(s, u(s))ds - \sum_{0 < t_i < t} T(t-t_i)I_i[u(t_i)]\}(t) \]

**Definition 2.1.** A function \( u \in PC([0, b]; X) \) is a mild solution of the problem (1.1) if the following condition holds:
\[ u(t) = T(t)\{h(u) + g(0, h(u))\} - g(t, u(t)) - \int_0^t AT(t-s)g(s, u(s))ds + \int_0^t T(t-s)f(s, u(s))ds \]
\[ + \int_0^t T(t-s)Bv(s)ds + \sum_{0 < t_i < t} T(t-t_i)I_i[u(t_i)] \]
for all \( t \in [0, b] \)

Next, we introduce the Hausdorff measure of non-compactness (in short MNC) defined by
\[ \beta(\Omega) = \inf\{\epsilon > 0 : \Omega \text{ has a finite } \epsilon - \text{net in } X\} \]
for each bounded subset \( \Omega \) in a Banach space \( X \).

Some basic properties of the Hausdorff measure of non-compactness \( \beta(.) \) are given in the following lemma:

**Lemma 2.2** ([3]). Let \( X \) be a real Banach space and \( E, F \subseteq X \) be bounded. Then the following properties hold:

1. \( E \) is precompact if and only if \( \beta(E) = 0 \);
2. \( \beta(E) = \beta(\bar{E}) = \beta(\text{conv}E) \), where \( \bar{E} \) and \( \text{conv}E \) means the closure and convex hull of \( E \) respectively;
3. \( \beta(E) \leq \beta(F) \), when \( E \subseteq F \);
4. \( \beta(E + F) \leq \beta(E) + \beta(F) \), where \( E + F = \{x + y : x \in E, y \in F\} \);
5. \( \beta(E \cup F) \leq \max\{\beta(E), \beta(F)\} \);
6. \( \beta(\lambda E) \leq ||\lambda||\beta(E) \), for any \( \lambda \in R \);
7. If the map \( Q : D(Q) \subseteq X \to Z \) is Lipschitz continuous with constant \( k \), then \( \beta_z(QE) \leq k\beta(E) \) for any bounded subset \( E \subseteq D(Q) \), where \( Z \) is a Banach space.

The map \( Q : D \subseteq X \to X \) is said to be \( \beta \)-condensing, if \( Q \) is continuous and bounded, and for any non-compact bounded subset \( B \subseteq D \), we have \( \beta(QE) < \beta(E) \), where \( X \) is a Banach space.

**Lemma 2.3** ([3], Darbo-Sadovskii). If \( D \subseteq X \) is bounded, closed and convex, the continuous map \( Q : D \to D \) is \( \beta \)-condensing, then \( Q \) has at least one fixed point in \( D \).
Definition 2.4. A countable set \( \{ f_n \}_{n=1}^{\infty} \subset L^1([0,b];X) \) is said to be semi-compact if the sequence \( \{ f_n(t) \}_{n=1}^{\infty} \) is compact in \( X \) for a.e. \( t \in [0,b] \) and if there is a function \( \mu \in L^1([0,b];\mathbb{R}) \) satisfying \( \sup_{n \geq 1} \| f_n(t) \| \leq \mu(t) \) for a.e. \( t \in [0,b] \).

Definition 2.5. We define the operator \( G : L^1([0,b];X) \to PC([0,b];X) \) by

\[
(Gu)(t) = T(t)[h(u) + g(0,h(u))] - g(t,u(t)) - \int_0^t AT(t-s)g(s,u(s))ds + \int_0^t T(t-s)f(s,u(s))ds \\
+ \int_0^t T(t-s)Bv(s)ds + \sum_{0 < t_i < t} T(t-t_i)I_i[u(t_i)]
\]

for all \( t \in [0,b] \).

3 Main Results

In this section, we prove the controllability of non-local neutral impulsive problem (1.1).

Let \( r \) be a finite positive constant. Consider the sets

\[
B_r = \{ x \in X | \| x \| \leq r \}, W_r = \{ u \in PC([0,b];X) | u(t) \in B_r, \forall t \in [0,b] \}
\]

Lemma 3.1 ([4]). Suppose that the following conditions hold:

(1) Let \( 0 < \alpha \leq 1 \). Then \( X_\alpha \) is a Banach space.

(2) If \( 0 < \gamma \leq \alpha \), then \( X_\alpha \to X_\gamma \) is continuous.

(3) For every constant \( C_\alpha > 0 \) such that

\[
\|(A)^\alpha T(t)\| \leq \frac{C_\alpha}{\rho^\alpha},
\]

for every \( t > 0 \).

Now, we introduce the following hypothesis:

(H3) A function \( f : [0,b] \times X \to X \) satisfies the following conditions:

(i) if \( f : [0,b] \times X \to X \), for a.e. \( t \in [0,b] \), then the function \( f(t,.) : X \to X \) is continuous for all \( x \in X \) and the function \( f(t,.) : [0,b] \to X \) is measurable.

(ii) Moreover, for any \( l > 0 \), there exists a function \( \rho_l \in L^1([0,b];\mathbb{R}) \) such that

\[
\|f(t,x)\| \leq \rho_l(t)
\]

for a.e. \( t \in [0,b] \) and all \( x \in B_l \).

(iii) There exists a function \( m \in L^1([0,b];\mathbb{R}) \) and a continuous non-decreasing function \( \psi : [0,\infty) \to (0,\infty) \) such that

\[
\|f(t,x)\| \leq m(t)\psi(\|x\|)
\]

for all \( x \in X, t \in [0,b] \).
(H4) There exists a function $\phi \in L^1([0,b]; R)$ such that for every bounded $D \subseteq W_r$,

$$\beta(f(t,D)) \leq \phi(t)\beta(D)$$

for a.e. $t \in [0, b]$, where $\beta$ is the Hausdorff measure of non-compactness.

(H5) There exists $\gamma \in (0,1)$ and positive constant $L_{\gamma}, L_f, L_k, k = 1, 2, \ldots, s, g$ is a $x_\gamma$-valued function and the following Lipschitz conditions hold:

(i) $\|(-A)^\gamma g(s, u(s)) - (-A)^\gamma g(s, w(s))\| \leq L_\gamma \|u - w\|_{PC}$.

(ii) $\|f(s, u(s)) - f(s, w(s))\| \leq L_f \|u - w\|_{PC}$.

(iii) $\|(-A)^\gamma g(t, u(t))\| \leq L_g \|u\|_{PC}$.

for $u, w \in PC([0, b]; X_\gamma), t \in [0, b]$.

(H6) If $h$ is continuous on $[0, b]$, then there exists a positive constant $L_h > 0$ such that

(i) $\|h(u) - h(w)\| \leq L_h \|u - w\|_{PC}$.

(ii) $\|h(u)\| \leq L_h \|u\|_{PC}$.

for $u \in PC([0, b]; X), t \in [0, b]$.

Theorem 3.2. Assume that the hypotheses (H1)-(H6) are satisfied. Then the non-local impulsive problem (1.1) has at least one mild solution on $[0, b]$ provided that

$$NL_0 + MNK\|h(0)\| + MN\|p_t\|_{L^1} + \wedge \leq r$$

(3.1)

where,

$$N = 1 + MM_1M_2b$$

$$K = 1 + L_g\|(-A)^\gamma\|$$

$$L_0 = L_g\|(-A)^\gamma\|(1 + ML_h) + M \sum_{i=1}^{s} L_i + ML_h + \frac{C_{1-\gamma} b^\gamma L_g}{\gamma}$$

$$\wedge = MM_1M_2b\|u_1\|$$

Proof. Let us define the operator $G : PC([0, b]; X) \to PC([0, b]; X)$ by

$$(Gu)(t) = T(t)[h(u) + g(0, h(u))] - g(t, u(t)) - \int_0^t AT(t-s)g(s, u(s))ds + \int_0^t T(t-s)f(s, u(s))ds$$

$$+ \int_0^t T(t-s)Bv(s)ds + \sum_{0 < t_i < t} T(t - t_i)I_i(u(t_i))$$

(3.2)

for all $t \in [0, b]$.

It is easy to prove that the fixed point of $G$ is the mild solution of the non local impulsive problem (1.1). Consequently, we will verify that $G$ has a fixed point, by 2.2.
First, we have to Earlier, we have proved that the mapping \( G \) is continuous on \( PC([0,b];X) \). For this intention, let \( \{u_n\}_{n=1}^{\infty} \) be a sequence in \( PC([0,b];X) \) with \( \lim_{n \to \infty} u_n = u \) in \( PC([0,b];X) \). By the continuity of \( f \), we deduce that for each \( s \in [0,b] \), \( f(s,u_n(s)) \) converges to \( f(s,u(s)) \) in \( X \) and we have

\[
\|Gu_n - Gu\| \leq \left\| T(t) \right\| \|h(u_n) - h(u)\| + \|g(0,h(u_n)) - g(0,h(u))\| + \|g(t,u_n(t)) - g(t,u(t))\|
+ \int_0^t \|(-A)T(t-s)\| |g(s,u_n(s)) - g(s,u(s))| ds
+ \int_0^t \|T(t-s)\| |f(s,u_n(s)) - f(s,u(s))| ds
+ \int_0^t \|T(t-s)\| |v_n(s) - v(s)| ds
+ \sum_{i=1}^s \|T(t-t_i)\| \|I_i[u_n(t_i)] - I_i[u(t_i)]\|
\leq M \{\|h(u_n) - h(u)\| + \|(-A)^{-\gamma}\| \|(-A)^\gamma g(0,h(u_n)) - (-A)^\gamma g(0,h(u))\|\}
+ \|(-A)^{-\gamma}\| \|(-A)^\gamma g(t,u_n(t)) - (-A)^\gamma g(t,u(t))\|
+ \int_0^t \|(-A)^{1-\gamma}T(t-s)\| \|(-A)^\gamma g(s,u_n(s)) - (-A)^\gamma g(s,u(s))\| ds
+ M \int_0^t \|f(s,u_n(s)) - f(s,u(s))\| ds
+ MM_1 \int_0^t \|v_n(s) - v(s)\| ds
+ M \sum_{i=1}^s \|I_i[u_n(t_i)] - I_i[u(t_i)]\|
\leq \{ML_h + ML_g \|(-A)^{-\gamma}\| \|h(u_n) - h(u)\| + L_g \|(-A)^{-\gamma}\|\}
+ L_g \int_0^b \|(-A)^{1-\gamma}T(t-s)\| ds \|u_n - u\|_{PC} + ML.fb \|u_n - u\|_{PC}
+ MM_1 \sqrt{b} \|v_n - v\|_{L^2} + M \sum_{i=1}^s L_i \|u_n - u\|_{PC}
\leq N \{L_0 + ML.fb\} \|u_n - u\|_{PC}
\]

where

\[
\|v_n - v\|_{L^2} \leq M_2 [ML_h + ML_g L_h \|(-A)^{-\gamma}\| + L_g \|(-A)^{-\gamma}\|] + L_g \frac{C_1 \gamma}{\gamma} b^\gamma
+ MbL_f + M \sum_{i=1}^s L_i \|u_n - u\|_{PC}
\]

Since, \( \lim_{n \to \infty} u_n = u \) in \( PC([0,b];X) \), we get

\[
\lim_{n \to \infty} Gu_n = Gu
\]
which implies that the mapping $G$ is continuous on $PC([0,b];X)$.

Next, we claim that $GW_r \subseteq W_r$. For any $u \in W_r \subseteq PC([0,b];X)$ by hypothesis (H3), (3.1) and we have

$$
\|Gu(t)\| \leq \|T(t)\|\{\|h(u)\| + \|g(0,h(u))\|\} + \|g(t,u(t))\| + \int_0^t \|(-A)T(t-s)\|\|g(s,u(s))\|ds
+ \int_0^t \|T(t-s)\|\|f(s,u(s))\|ds + \int_0^t \|T(t-s)\|\|B\|\|v(s)\|ds + \sum_{i=1}^s \|T(t-t_i)\|\|I_i[u(t_i)]\|
\leq M\{\|h(u) - h(0)\| + \|h(0)\| + M\|(-A)^{-\gamma}\|\|(-A)^{\gamma}g(0,h(u))\|\}
+ \|(-A)^{-\gamma}\|\|(-A)^{\gamma}g(t,u(t))\| + \int_0^t \|(-A)^{1-\gamma}T(t-s)\|\|(-A)^{\gamma}g(s,u(s))\|ds
+ M\int_0^t \|f(s,u(s))\|ds + MM_1\int_0^t \|v(s)\|ds + M\sum_{i=1}^s \|I_i[u(t_i)]\|
\leq NL_0 + MNK\|h(0)\| + MN\|\rho_1\|_{L^1} + \wedge
\leq r
$$

which implies that $GW_r \subseteq W_r$.

At present, according to Lemma 2.2, it remains to verify that $G$ is an $\beta$-condensing in $W_r$. By using conditions (H5) and (H6), we get that the mapping $G : W_r \rightarrow PC([0,b];X)$ is Lipschitz continuous with constant $N\{L_0 + ML_1b\} < 1$. In fact, for any $u, w \in W_r$, we have

$$
\|Gu - Gw\| \leq N\{L_0 + ML_1b\}\|u - w\|_{PC}
$$

Thus Lemma 2.2 (7), we obtain that

$$
\beta(GW_r) \leq N\{L_0 + ML_1b\}\beta(W_r)
$$

Since $N\{L_0 + ML_1b\} < 1$, we get

$$
\beta(GW_r) \subseteq \beta(W_r).
$$

The mapping $G$ is an $\beta$-condensing in $W_r$. By Darbo Sadovskii’s fixed point theorem, the operator $G$ has a fixed point in $W_r$, which is the mild solution of the non-local impulsive problem (1.1). This completes the proof.

\[\Box\]

References


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