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Existence of Solutions for Fractional Impulsive Anti-Periodic Boundary Value Problems [1]

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Abstract

In this paper, the existence of the solutions for an impulsive fractional equation of for nonlinear differential equations of order $\alpha \in (2, 3]$ with anti-periodic boundary value problem is discussed. Many of our conclusions are based on the fixed point theorems. At last, some examples are presented to illustrate the main results.

Mathematics Subject Classifications: 34B15, 34B18, 34B37, 58E30

Keywords: Fractional differential equations; Impulse; Anti-periodic boundary value; Fixed point theorems

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1 Introduction

As we all know, Caputo fractional differential equations are widely used in the fields of physics, chemistry, electrokinetics, polymer rheology, etc., therefore, it is more and more important to discuss the existence of Caputo fractional differential equations under certain conditions. In the extensive literature, the existence of solutions for boundary value problems of fractional differential equations by various nonlinear functional analysis methods is involved. For example, in recent literature about an fixed point theory, see[5, 6], the Mawhin continuation method, see[7], the Green function method, see[5, 10], the integral operator method, see[11, 12], the upper and lower solution method, see[14, 15], the numerical method, see[4, 18]. For the definition theorem of Caputo fractional order, see[1].

In [3, 13], the researcher got the existence and uniqueness of the solutions by using the fixed pint theorem. Then some researchers studied the solvability of anti-periodic boundary value problems under different assumptions, see[2, 4, 8, 9]. In reference[8], the author studied the $\alpha \in (1, 2)$ order antiperiodic impulsive boundary value problem with $u'(t)$, in reference[9], the author studied the $\alpha \in (2, 3)$ order antiperiodic impulsive boundary value problem without $u'(t)$. Motivated by [8, 9], we will going to discuss the existence and uniqueness of solutions for an anti-periodic boundary value problem of $\alpha \in (2, 3]$ with $u'(t)$ in this paper. Precisely, we consider

$$\begin{cases} {}^C D^\alpha u(t) = f(t, u(t), u'(t)), \qquad 2 < \alpha \leq 3,\ t \in J', \\ \Delta u(t_k) = I_k(u(t_k)), \Delta u'(t_k) = Q_k(u(t_k)), \Delta {}^C D^\beta u(t_k) = J_k(u(t_k)), \\ \quad 1 < \beta \leq 2, \quad k = 1, 2, \cdots, m, \\ u(0) = -u(T), u'(0) = -u'(T), {}^C D^\beta u(0) = -{}^C D^\beta u(T), \end{cases} \tag{1.1}$$

where ${}^C D^\alpha$ is a standard Caputo derivative, $f \in C(J \times \mathbb{R}, \mathbb{R}), I_k, Q_k, J_k \in C(\mathbb{R}, \mathbb{R}), J = [0, T], 0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T, J' = J \setminus t_1, t_2, \cdots, t_m, \Delta u(t_k) = u(t_k^+) - u(t_k^-), \Delta u'(t_k) = u'(t_k^+) - u'(t_k^-), \Delta {}^C D^\beta u(t_k) = {}^C D^\beta u(t_k^+) - {}^C D^\beta u(t_k^-)$, where $u(t_k^+)$ and $u(t_k^-)$ denote the right and the left limit of $u(t)$ at $t = t_k (k = 1, 2, \cdots, m)$, respectively. $\Delta u'(t_k)$ and $\Delta {}^C D^\beta u(t_k)$ have a similar meaning for $u'(t)$ and ${}^C D^\beta u(t)$ respectively. For convenience, let $J_0 = [0, t_1], J_1 = [t_1, t_2], \cdots, J_{m-1} = [t_{m-1}, t_m], J_m = [t_m, T]$.

The structure of this article is as follows. In Section 2, the definitions of the Caputo fractional order derivative and integral are given. In Section 3, by using fixed points and the existence of the solution is established. In Section 4, some examples are also presented to illustrate the main results.

2 Preliminaries

In this section, we introduce some notations and definitions which are used throughout this paper. Let $PC(J) = \{u : J \longrightarrow \mathbb{R} | u \in C(J_k), k = 0, 1, \cdots, m$, and $u(t_k^+)$ exist,$k = 1, 2, \cdots, m\}$,with the norm $\|u\| = \sup_{t\in J} |u(t)|$, and $PC^2(J, \mathbb{R}) = \{u \in C^2(J_k), k = 0, 1, \cdots, m$,and $u(t_k^+), u'(t_k^+), {}^C D^\beta u(t_k^+)$ exist, $k = 1, 2, \cdots, m\}$, with the norm $\|u\|_{PC^2} = \max\{\|u\|, \|u'\|, \|{}^C D^\beta u\|\}$. Clearly, $PC(J, \mathbb{R})$ and $PC^2(J, \mathbb{R})$ are Banach spaces.

Definition 2.1. *([19]) Given a function $f : [0, +\infty) \to \mathbb{R}$ on the interval $[a, b]$, the Caputo fractional order derivative is defined by*

$$^C D^\alpha f(t) = \int_a^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)} f^{(n)}(s)ds, \quad t > 0, n = [\alpha] + 1. \tag{2.1}$$

Where $[\alpha]$ denotes the intrger part of real number α, and Γ is the gamma function.

Definition 2.2. *([20]) The Riemann-Liouville integral of order α for a function $f : [0, +\infty) \to \mathbb{R}$ can be written as*

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > 0, n-1 < \alpha < n. \tag{2.2}$$

Definition 2.3. *A function $u \in PC^2(J, \mathbb{R})$ with its Caputo derivative of order α existing on J is a solution of (1.1) if it satisfies (1.1).*

Theorem 2.1. *([13]) Let E be a Banach space. Assume Ω is an open bounded subset of E with $\theta \in \Omega$ and let $T : \overline{\Omega} \to E$ be a completely continuous operator such that*

$$\|Tu\| \leq \|u\|, \quad \forall u \in \Omega.$$

Then T has a fixed point in $\overline{\Omega}$.

Theorem 2.2. *([13]) Let E be a Banach space. Assume $T : E \to E$ is a completely continuous operator and the set $W = \{u \in E | u = \mu Tu, 0 < \mu < 1\}$ is a bounded. The T has a fixed point in E.*

Lemma 2.1. *([16]) For $\alpha > 0$, the general solution of the fractional differential equation ${}^C D^\alpha u(t) = 0$ is given by*

$$u(t) = C_0 + C_1 t + C_2 t^2 + \cdots + C_{n-1} t^{n-1}, C_i \in \mathbb{R}, i = 0, 1, 2, \cdots, n-1, n = [\alpha] + 1, \tag{2.3}$$

where $[\alpha]$ denotes the integer part of the real number α.

Lemma 2.2. *([17]) In view of Lemma 2.1, it follows that*

$$I^{\alpha C} D^\alpha u(t) = u(t) + C_0 + C_1 t + C_2 t^2 + \cdots + C_{n-1} t^{n-1}, \tag{2.4}$$

for some $C_i \in \mathbb{R}, i = 0, 1, 2, \cdots, n-1, n = [\alpha] + 1$.

Lemma 2.3. *For a given $y \in C[0, T], 2 < \alpha < 3, 1 < \beta \leq 2$, a functional u is a solution of the following impulsive boundary value problem*

$$\begin{cases} {}^C D^\alpha u(t) = y(t), \qquad 2 < \alpha \leq 3, \ t \in J', \\ \Delta u(t_k) = I_k(u(t_k)), \Delta u'(t_k) = Q_k(u(t_k)), \Delta {}^C D^\beta u(t_k) = J_k(u(t_k)), \\ \quad 1 < \beta < 2, \quad k = 1, 2, \cdots, m, \\ u(0) = -u(T), u'(0) = -u'(T), {}^C D^\beta u(0) = -{}^C D^\beta u(T), \end{cases} \tag{2.5}$$

if and only if u is a solution of the impulsive fractional integral equation

$$\begin{aligned} u(t) = {} & \int_{t_k}^t \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1} y(s)ds + \sum_{i=1}^k \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + \sum_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} \frac{(t_k - t_i)(t_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds \\ & + \sum_{i=1}^k \int_{t_{i-1}}^{t_i} \frac{(t - t_k)(t_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds + \sum_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} \frac{\Gamma(3-\beta)(t-t_i)^2 (t_i - s)^{\alpha-\beta-1}}{2t_i^{2-\beta}\Gamma(\alpha-\beta)} y(s)ds \\ & + \sum_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} \frac{\Gamma(3-\beta)(t-t_k)(t_k - t_i)(t_i - s)^{\alpha-\beta-1}}{t_i^{2-\beta}\Gamma(\alpha-\beta)} y(s)ds + \sum_{i=1}^{k-1} \frac{\Gamma(3-\beta)(t_k - t_i)(t - t_k)}{t_i^{2-\beta}} J_i(u(t_i)) \\ & + \sum_{i=1}^k \int_{t_{i-1}}^{t_i} \frac{\Gamma(3-\beta)(t-t_i)^2 (t_i - s)^{\alpha-\beta-1}}{2t_i^{2-\beta}\Gamma(\alpha-\beta)} y(s)ds + \sum_{i=1}^{k-1} (t_k - t_i) Q_i(u(t_i)) + \sum_{i=1}^k I_i(u(t_i)) \\ & + \sum_{i=1}^k (t - t_k) Q_i(u(t_i)) + \sum_{i=1}^{k-1} \frac{\Gamma(3-\beta)(t_k - t_i)^2}{2t_i^{2-\beta}} J_i(u(t_i)) + \sum_{i=1}^k \frac{\Gamma(3-\beta)(t-t_i)^2}{2t_i^{2-\beta}} J_i(u(t_i)) \\ & + C_1 + C_2 t + C_3 t^2, \quad t \in J_k, k = 1, 2, \cdots, m, \end{aligned} \tag{2.6}$$

where

$$C_1 = -\Big\{ \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-1}}{2\Gamma(\alpha)} y(s)ds + \sum_{i=1}^{m-1} \int_{t_{i-1}}^{t_i} \frac{(t_m - t_i)(t_i - s)^{\alpha-2}}{2\Gamma(\alpha-1)} y(s)ds - \int_{t_m}^T \frac{(T-s)^{\alpha-2}}{4\Gamma(\alpha-1)} y(s)ds$$

$$
\begin{aligned}
&+\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\frac{(T-2t_m)(t_i-s)^{\alpha-2}}{4\Gamma(\alpha-1)}y(s)ds+\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\frac{\Gamma(3-\beta)(t_m^2-Tt_m)(t_i-s)^{\alpha-\beta-1}}{4t_i^{2-\beta}\Gamma(\alpha-\beta)}y(s)ds\\
&+\sum_{i=1}^{m-1}\int_{t_{i-1}}^{t_i}\frac{\Gamma(3-\beta)(T-t_m-t_i)(t_m-t_i)(t_i-s)^{\alpha-\beta-1}}{4t_i^{2-\beta}\Gamma(\alpha-\beta)}y(s)ds+\sum_{i=1}^{m}\frac{(T-2t_m)}{4}Q_i(u(t_i))\\
&+\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}\frac{(2T-1)T^{\beta}\Gamma(3-\beta)(t_i-s)^{\alpha-\beta-1}}{4\Gamma(\alpha-\beta)}y(s)ds+\sum_{i=1}^{m-1}\frac{(t_m-t_i)}{2}I_i(u(t_i))\\
&+\sum_{i=1}^{m-1}\frac{(t_m-t_i)}{2}Q_i(u(t_i))+\sum_{i=1}^{m-1}\frac{\Gamma(3-\beta)(3T-2t_m-2t_i)(t_m-t_i)}{8t_i^{2-\beta}}J_i(u(t_i))\\
&+\sum_{i=1}^{m-1}\frac{\Gamma(3-\beta)(t_m-t_i)(T-t_m)}{2t_i^{2-\beta}}J_i(u(t_i))+\sum_{i=1}^{m-1}\frac{\Gamma(3-\beta)(t_m-t_i)^2}{4t_i^{2-\beta}}J_i(u(t_i))\\
&+\sum_{i=1}^{m}\frac{T\Gamma(3-\beta)(T-t_m)(T-2t_m)}{8t_i^{2-\beta}}J_i(u(t_i))+\frac{T^3-T^2}{2}\sum_{i=1}^{m}J_i(u(t_i))\Big\},
\end{aligned}
$$

$$
\begin{aligned}
C_2=-\Big\{&\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{2\Gamma(\alpha-1)}y(s)ds+\sum_{i=1}^{m-1}\int_{t_{i-1}}^{t_i}\frac{\Gamma(3-\beta)(t_m-t_i)(t_i-s)^{\alpha-\beta-1}}{2t_i^{2-\beta}\Gamma(\alpha-\beta)}y(s)ds\\
&+\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\frac{\Gamma(3-\beta)(T-t_m)(t_i-s)^{\alpha-\beta-1}}{2t_i^{2-\beta}\Gamma(\alpha-\beta)}y(s)ds+\sum_{i=1}^{m}\frac{\Gamma(3-\beta)(T-t_m)}{4t_i^{2-\beta}}J_i(u(t_i))\\
&-\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}\frac{T^{\beta}\Gamma(3-\beta)(t_i-s)^{\alpha-\beta-1}}{2\Gamma(\alpha-\beta)}y(s)ds+\sum_{i=1}^{m}\frac{Q_i(u(t_i))}{2}-T^2\sum_{i=1}^{m}J_i(u(t_i))\\
&+\sum_{i=1}^{m-1}\frac{\Gamma(3-\beta)(t_m-t_i)}{4t_i^{2-\beta}}J_i(u(t_i))\Big\},
\end{aligned}
$$

and

$$C_3=-\Big\{\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}\frac{\Gamma(3-\beta)(t_i-s)^{\alpha-\beta-1}}{2T^{2-\beta}\Gamma(\alpha-\beta)}y(s)ds+\sum_{i=1}^{m}J_i(u(t_i))\Big\}.$$

Proof. Let u be a solution of (2.5). By(2.4), we have

$$u(t)=I^{\alpha}y(t)-c_1-c_2t-c_3t^2=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds-c_1-c_2t-c_3t^2,\quad t\in J_0, \tag{2.7}$$

for some $c_1,c_2,c_3\in\mathbb{R}$. Then

$$u'(t)=\int_0^t\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}y(s)ds-c_2-2c_3t,\quad t\in J_0,$$

$${}^CD^{\beta}u(t)=\int_0^t\frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}y(s)ds-\frac{2t^{2-\beta}}{\Gamma(3-\beta)}c_3,\quad t\in J_0,$$

where ${}^CD^{\beta}c_1=0,{}^CD^{\beta}c_2=0,{}^CD^{\beta}t^2=\frac{2t^{2-\beta}}{\Gamma(3-\beta)}$.
Furthermore, if $t\in J_1$, then

$$u(t)=\int_{t_1}^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds-d_1-d_2(t-t_1)-d_3(t-t_1)^2,$$

$$u'(t) = \int_{t_1}^{t} \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds - d_2 - 2d_3(t-t_1),$$

$$^C D^\beta u(t) = \int_{t_1}^{t} \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} y(s)ds - \frac{2t^{2-\beta}}{\Gamma(3-\beta)} d_3,$$

for some $d_1, d_2, d_3 \in \mathbb{R}$.

Thus, we have

$$u(t_1^-) = \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds - c_1 - c_2 t_1 - c_3 t_1^2, \quad u(t_1^+) = -d_1,$$

$$u'(t_1^-) = \int_0^{t} \frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds - c_2 - 2c_3 t_1, \quad u(t_1^+) = -d_2,$$

$$^C D^\beta u(t_1^-) = \int_0^{t} \frac{(t_1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} y(s)ds - \frac{2t^{2-\beta}}{\Gamma(3-\beta)} c_3, \quad ^C D^\beta u(t_1^+) = -\frac{2t^{2-\beta}}{\Gamma(3-\beta)} d_3.$$

Because of $\Delta u(t_1) = u(t_1^+) - u(t_1^-) = I_1(u(t_1)), \Delta u'(t_1) = u'(t_1^+) - u'(t_1^-) = Q_1(u(t_1))$ and $\Delta ^C D^\beta u(t_1) = {}^C D^\beta u(t_1^+) - {}^C D^\beta u(t_1^-) = J_1(u(t_1))$, we have

$$-d_1 = \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds - c_1 - c_2 t_1 - c_3 t_1^2 + I_1(u(t_1)),$$

$$-d_2 = \int_0^{t_1} \frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds - c_2 - 2c_3 t_1 + Q_1(u(t_1)),$$

$$-d_3 = \int_0^{t_1} \frac{\Gamma(3-\beta)(t_1-s)^{\alpha-\beta-1}}{2t_1^{2-\beta}\Gamma(\alpha-\beta)} y(s)ds - c_3 + \frac{\Gamma(3-\beta)}{2t_1^{2-\beta}} J_1(u(t_1)).$$

Consequently

$$\begin{aligned} u(t) = & \int_{t_1}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + \int_0^{t_1} \frac{(t-t_1)(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds \\ & + \int_0^{t_1} \frac{\Gamma(3-\beta)(t-t_1)^2(t_1-s)^{\alpha-\beta-1}}{2t_1^{2-\beta}\Gamma(\alpha-\beta)} y(s)ds + I_1(u(t_1)) + (t-t_1)Q_1(u(t_1)) \\ & + \frac{\Gamma(3-\beta)(t-t_1)^2}{2t_1^{2-\beta}} J_1(u(t_1)) - c_1 - c_2 t - c_3 t^2, \quad t \in J_1. \end{aligned}$$

Similarly, we get

$$\begin{aligned} u(t) = & \int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + \sum_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} \frac{(t_k-t_i)(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds \\ & + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t-t_k)(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds + \sum_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} \frac{\Gamma(3-\beta)(t-t_i)^2(t_i-s)^{\alpha-\beta-1}}{2t_i^{2-\beta}\Gamma(\alpha-\beta)} y(s)ds \end{aligned}$$

$$+\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}\frac{\Gamma(3-\beta)(t-t_k)(t_k-t_i)(t_i-s)^{\alpha-\beta-1}}{t_i^{2-\beta}\Gamma(\alpha-\beta)}y(s)ds+\sum_{i=1}^{k-1}(t_k-t_i)Q_i(u(t_i))$$
$$+\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{\Gamma(3-\beta)(t-t_i)^2(t_i-s)^{\alpha-\beta-1}}{2t_i^{2-\beta}\Gamma(\alpha-\beta)}y(s)ds+\sum_{i=1}^{k}I_i(u(t_i))+\sum_{i=1}^{k}(t-t_k)Q_i(u(t_i))$$
$$+\sum_{i=1}^{k-1}\frac{\Gamma(3-\beta)(t_k-t_i)^2}{2t_i^{2-\beta}}J_i(u(t_i))+\sum_{i=1}^{k-1}\frac{\Gamma(3-\beta)(t_k-t_i)(t-t_k)}{t_i^{2-\beta}}J_i(u(t_i))$$
$$+\sum_{i=1}^{k}\frac{\Gamma(3-\beta)(t-t_i)^2}{2t_i^{2-\beta}}J_i(u(t_i))-c_1-c_2t-c_3t^2,\quad t\in J_k,k=1,2,\cdots,m, \tag{2.8}$$

$$u'(t)=\int_{t_k}^{t}\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}y(s)ds+\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}y(s)ds+\sum_{i=1}^{k-1}\frac{\Gamma(3-\beta)(t_k-t_i)}{2t_i^{2-\beta}}J_i(u(t_i))$$
$$+\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{\Gamma(3-\beta)(t-t_k)(t_i-s)^{\alpha-\beta-1}}{t_i^{2-\beta}\Gamma(\alpha-\beta)}y(s)ds+\sum_{i=1}^{k}\frac{\Gamma(3-\beta)(t-t_k)}{2t_i^{2-\beta}}J_i(u(t_i))$$
$$+\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}\frac{\Gamma(3-\beta)(t_k-t_i)(t_i-s)^{\alpha-\beta-1}}{t_i^{2-\beta}\Gamma(\alpha-\beta)}y(s)ds+\sum_{i=1}^{k}Q_i(u(t_i))-c_2-2c_3t, \tag{2.9}$$

and

$$^{C}D^{\beta}u(t)=\int_{t_k}^{t}\frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}y(s)ds+\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}y(s)ds+\sum_{i=1}^{k}J_i(u(t_i))$$
$$-\frac{2t^{2-\beta}}{\Gamma(3-\beta)}c_3,\quad t\in J_k,k=1,2,\cdots,m. \tag{2.10}$$

Use the boundary condition $u(0)=-u(T), t\in J_k, k=1,2,\cdots,m$, we have

$$2c_1+c_2T+c_3T^2=\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds+\sum_{i=1}^{m-1}\int_{t_{i-1}}^{t_i}\frac{\Gamma(3-\beta)(t_m-t_i)^2(t_i-s)^{\alpha-\beta-1}}{2t_i^{2-\beta}\Gamma(\alpha-\beta)}y(s)ds$$
$$+\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\frac{(T-t_m)(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}y(s)ds+\sum_{i=1}^{m-1}\int_{t_{i-1}}^{t_i}\frac{(t_m-t_i)(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}y(s)ds$$
$$+\sum_{i=1}^{m-1}\int_{t_{i-1}}^{t_i}\frac{\Gamma(3-\beta)(T-t_m)(t_m-t_i)(t_i-s)^{\alpha-\beta-1}}{t_i^{2-\beta}\Gamma(\alpha-\beta)}y(s)ds+\sum_{i=1}^{m}I_i(u(t_i))$$
$$+\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\frac{\Gamma(3-\beta)(T-t_m)^2(t_i-s)^{\alpha-\beta-1}}{2t_i^{2-\beta}\Gamma(\alpha-\beta)}y(s)ds+\sum_{i=1}^{m-1}(t_m-t_i)Q_i(u(t_i))$$
$$+\sum_{i=1}^{m}(T-t_m)Q_i(u(t_i))+\sum_{i=1}^{m-1}\frac{\Gamma(3-\beta)(t_m-t_i)(T-t_m)}{t_i^{2-\beta}}J_i(u(t_i))$$
$$+\sum_{i=1}^{m-1}\frac{\Gamma(3-\beta)(t_m-t_i)^2}{2t_i^{2-\beta}}J_i(u(t_i))+\sum_{i=1}^{m}\frac{\Gamma(3-\beta)(T-t_m)^2}{2t_i^{2-\beta}}J_i(u(t_i)), \tag{2.11}$$

by the condition $u'(0) = -u'(T)$, we have

$$\begin{aligned}2c_2 + 2c_3T^2 =& \sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}y(s)ds + \sum_{i=1}^{m-1}\int_{t_{i-1}}^{t_i}\frac{\Gamma(3-\beta)(t_m-t_i)(t_i-s)^{\alpha-\beta-1}}{t_i^{2-\beta}\Gamma(\alpha-\beta)}y(s)ds\\ &+\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\frac{\Gamma(3-\beta)(T-t_m)(t_i-s)^{\alpha-\beta-1}}{t_i^{2-\beta}\Gamma(\alpha-\beta)}y(s)ds + \sum_{i=1}^{m}Q_i(u(t_i))\\ &+\sum_{i=1}^{m-1}\frac{\Gamma(3-\beta)(t_m-t_i)}{2t_i^{2-\beta}}J_i(u(t_i)) + \sum_{i=1}^{m}\frac{\Gamma(3-\beta)(T-t_m)}{2t_i^{2-\beta}}J_i(u(t_i)). \quad (2.12)\end{aligned}$$

Combining (2.10),(2.11) and (2.12) with the condition ${}^CD^\beta u(0) = -{}^CD^\beta u(T)$, we find that

$$c_3 = \sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}\frac{\Gamma(3-\beta)(t_i-s)^{\alpha-\beta-1}}{2T^{2-\beta}\Gamma(\alpha-\beta)}y(s)ds + \sum_{i=1}^{m}J_i(u(t_i)),$$

$$\begin{aligned}c_2 =& \sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{2\Gamma(\alpha-1)}y(s)ds + \sum_{i=1}^{m-1}\int_{t_{i-1}}^{t_i}\frac{\Gamma(3-\beta)(t_m-t_i)(t_i-s)^{\alpha-\beta-1}}{2t_i^{2-\beta}\Gamma(\alpha-\beta)}y(s)ds\\ &+\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\frac{\Gamma(3-\beta)(T-t_m)(t_i-s)^{\alpha-\beta-1}}{2t_i^{2-\beta}\Gamma(\alpha-\beta)}y(s)ds + \sum_{i=1}^{m}\frac{Q_i(u(t_i))}{2}\\ &-\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}\frac{T^\beta\Gamma(3-\beta)(t_i-s)^{\alpha-\beta-1}}{2\Gamma(\alpha-\beta)}y(s)ds + \sum_{i=1}^{m}\frac{\Gamma(3-\beta)(T-t_m)}{4t_i^{2-\beta}}J_i(u(t_i))\\ &-T^2\sum_{i=1}^{m}J_i(u(t_i)) + \sum_{i=1}^{m-1}\frac{\Gamma(3-\beta)(t_m-t_i)}{4t_i^{2-\beta}}J_i(u(t_i)),\end{aligned}$$

and

$$\begin{aligned}c_1 =& \sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-1}}{2\Gamma(\alpha)}y(s)ds + \sum_{i=1}^{m-1}\int_{t_{i-1}}^{t_i}\frac{(t_m-t_i)(t_i-s)^{\alpha-2}}{2\Gamma(\alpha-1)}y(s)ds - \int_{t_m}^{T}\frac{(T-s)^{\alpha-2}}{4\Gamma(\alpha-1)}y(s)ds\\ &+\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\frac{(T-2t_m)(t_i-s)^{\alpha-2}}{4\Gamma(\alpha-1)}y(s)ds + \sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\frac{\Gamma(3-\beta)(t_m^2-Tt_m)(t_i-s)^{\alpha-\beta-1}}{4t_i^{2-\beta}\Gamma(\alpha-\beta)}y(s)ds\\ &+\sum_{i=1}^{m-1}\int_{t_{i-1}}^{t_i}\frac{\Gamma(3-\beta)(T-t_m-t_i)(t_m-t_i)(t_i-s)^{\alpha-\beta-1}}{4t_i^{2-\beta}\Gamma(\alpha-\beta)}y(s)ds + \sum_{i=1}^{m}\frac{(T-2t_m)}{4}Q_i(u(t_i))\\ &+\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}\frac{(2T-1)T^\beta\Gamma(3-\beta)(t_i-s)^{\alpha-\beta-1}}{4\Gamma(\alpha-\beta)}y(s)ds + \sum_{i=1}^{m-1}\frac{(t_m-t_i)}{2}I_i(u(t_i))\\ &+\sum_{i=1}^{m-1}\frac{(t_m-t_i)}{2}Q_i(u(t_i)) + \sum_{i=1}^{m-1}\frac{\Gamma(3-\beta)(3T-2t_m-2t_i)(t_m-t_i)}{8t_i^{2-\beta}}J_i(u(t_i))\\ &+\sum_{i=1}^{m-1}\frac{\Gamma(3-\beta)(t_m-t_i)(T-t_m)}{2t_i^{2-\beta}}J_i(u(t_i)) + \sum_{i=1}^{m-1}\frac{\Gamma(3-\beta)(t_m-t_i)^2}{4t_i^{2-\beta}}J_i(u(t_i))\end{aligned}$$

$$+\sum_{i=1}^{m}\frac{T\Gamma(3-\beta)(T-t_m)(T-2t_m)}{8t_i^{2-\beta}}J_i(u(t_i))+\frac{T^3-T^2}{2}\sum_{i=1}^{m}J_i(u(t_i)).$$

Substituting the value of $c_i(i=1,2,3)$ in (2.7), (2.8) and letting $C_1 = -c_1, C_2 = -c_2, C_1 = -c_3$, we obtain (2.6). Conversely, assume that u is a solution of (2.6), then by a direct computation, it follows that the solution given by (2.6) satisfies (2.5). The proof is complete. □

3 Main results

Define an operator $T : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ as

$$\begin{aligned}
Tu(t) =& \int_{t_k}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s),u'(s))ds+\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{(t-t_k)(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,u(s),u'(s))ds\\
&+\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}\frac{(t_k-t_i)(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,u(s),u'(s))ds+\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s),u'(s))ds\\
&+\sum_{i=1}^{k-1}\frac{\Gamma(3-\beta)(t_k-t_i)(t-t_k)}{t_i^{2-\beta}}J_i(u(t_i))+\sum_{i=1}^{k-1}(t_k-t_i)Q_i(u(t_i))+\sum_{i=1}^{k}(t-t_k)Q_i(u(t_i))\\
&+\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}\frac{\Gamma(3-\beta)(t-t_k)(t_k-t_i)(t_i-s)^{\alpha-\beta-1}}{t_i^{2-\beta}\Gamma(\alpha-\beta)}f(s,u(s),u'(s))ds+\sum_{i=1}^{k}I_i(u(t_i))\\
&+\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{\Gamma(3-\beta)(t-t_i)^2(t_i-s)^{\alpha-\beta-1}}{2t_i^{2-\beta}\Gamma(\alpha-\beta)}f(s,u(s),u'(s))ds+\sum_{i=1}^{k-1}\frac{\Gamma(3-\beta)(t_k-t_i)^2}{2t_i^{2-\beta}}J_i(u(t_i))\\
&+\sum_{i=1}^{k}\frac{\Gamma(3-\beta)(t-t_i)^2}{2t_i^{2-\beta}}J_i(u(t_i))+\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}\frac{\Gamma(3-\beta)(t-t_i)^2(t_i-s)^{\alpha-\beta-1}}{2t_i^{2-\beta}\Gamma(\alpha-\beta)}f(s,u(s),u'(s))ds\\
&+p_1+p_2t+p_3t^2, \quad t\in J_k, k=1,2,\cdots,m, \qquad (3.1)
\end{aligned}$$

where $p_1 = C_1, p_2 = C_2, p_3 = C_3$. By the Lemma 2.3 with $y(t) = f(t,u(t),u'(t))$, problem (1.1) has a solution if and only if the operator T has a fixed point.

Theorem 3.1. *Let* $\lim\limits_{u\to 0}\frac{f(t,u,u')}{u} = 0, \lim\limits_{u\to 0}\frac{I_k(u)}{u} = 0, \lim\limits_{u\to 0}\frac{Q_k(u)}{u} = 0, \lim\limits_{u\to 0}\frac{J_k(u)}{u} = 0,$ *then problem (1.1) has at least one solution.*

Proof. Firstly, we construction that the operator $T : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ is completely continuous and that T is continuous under the continuity of $f, I_k(u), Q_k(u)$ and $J_k(u)$.

Let $\Omega \subset PC(J,\mathbb{R})$ be bounded and that there exist positive constant $L_i > 0(i = 1,2,3,4)$, such that $|f(t,u,u')| \le L_1, I_i(u(t_i)) \le L_2, Q_k(u(t_k)) \le L_3$ and $J_k(u(t_k)) \le L_4$. Thus, $\forall u \in \Omega$, we have

$$p_1 \le \sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-1}}{2\Gamma(\alpha)}|f(s,u(s),u'(s))|ds+\sum_{i=1}^{m-1}\int_{t_{i-1}}^{t_i}\frac{|t_m-t_i|(t_i-s)^{\alpha-2}}{2\Gamma(\alpha-1)}|f(s,u(s),u'(s))|ds$$

$$
\begin{aligned}
&+\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\frac{|T-2t_m|(t_i-s)^{\alpha-2}}{4\Gamma(\alpha-1)}|f(s,u(s),u'(s))|ds-\int_{t_m}^{T}\frac{(T-s)^{\alpha-2}}{4\Gamma(\alpha-1)}|f(s,u(s),u'(s))|ds\\
&+\sum_{i=1}^{m-1}\int_{t_{i-1}}^{t_i}\frac{\Gamma(3-\beta)|T-t_m-t_i||t_m-t_i|(t_i-s)^{\alpha-\beta-1}}{4t_i^{2-\beta}\Gamma(\alpha-\beta)}|f(s,u(s),u'(s))|ds\\
&+\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\frac{\Gamma(3-\beta)|t_m^2-Tt_m|(t_i-s)^{\alpha-\beta-1}}{4t_i^{2-\beta}\Gamma(\alpha-\beta)}|f(s,u(s)u'(s))|ds+\sum_{i=1}^{m}\frac{|T-2t_m|}{4}|Q_i(u(t_i))|\\
&+\frac{(2T-1)T^\beta\Gamma(3-\beta)}{4\Gamma(\alpha-\beta)}\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-\beta-1}|f(s,u(s),u'(s))|ds+\sum_{i=1}^{m-1}\frac{|t_m-t_i|}{2}|Q_i(u(t_i))|\\
&+\sum_{i=1}^{m-1}\frac{\Gamma(3-\beta)|t_m-t_i|^2}{4t_i^{2-\beta}}|J_i(u(t_i))|+\sum_{i=1}^{m-1}\frac{\Gamma(3-\beta)|3T-2t_m-2t_i||t_m-t_i|}{8t_i^{2-\beta}}|J_i(u(t_i))|\\
&+\sum_{i=1}^{m-1}\frac{\Gamma(3-\beta)|t_m-t_i||T-t_m|}{2t_i^{2-\beta}}|J_i(u(t_i))|+\sum_{i=1}^{m}\frac{T\Gamma(3-\beta)|T-t_m||T-2t_m|}{8t_i^{2-\beta}}|J_i(u(t_i))|\\
&+\frac{T^3}{2}\sum_{i=1}^{m}|J_i(u(t_i))|+\sum_{i=1}^{m-1}\frac{|t_m-t_i|}{2}|I_i(u(t_i))|\\
\le&\sum_{i=1}^{m+1}\frac{L_1}{2\Gamma(\alpha+1)}+\sum_{i=1}^{m-1}\frac{TL_1}{2\Gamma(\alpha)}+\sum_{i=1}^{m}\frac{TL_1}{4\Gamma(\alpha)}+\sum_{i=1}^{m-1}\frac{T^2\Gamma(3-\beta)L_1}{2\Gamma(\alpha-\beta+1)}+\sum_{i=1}^{m}\frac{T^2\Gamma(3-\beta)L_1}{4\Gamma(\alpha-\beta+1)}\\
&+\sum_{i=1}^{m+1}\frac{T^\beta(2T-1)\Gamma(3-\beta)L_1}{4\Gamma(\alpha-\beta+1)}+\frac{L_1}{4\Gamma(\alpha)}+\frac{(m-1)L_2}{2}+\frac{(2m-1)L_3T}{4}+\frac{(m-1)T^2\Gamma(3-\beta)L_4}{2}\\
&+\frac{m(\Gamma(3-\beta)+4)T^3L_4}{8}
\end{aligned}
$$

$$
\begin{aligned}
p_2\le&\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{2\Gamma(\alpha-1)}|f(s,u(s),u'(s))|ds+\sum_{i=1}^{m}\frac{|Q_i(u(t_i))|}{2}+\sum_{i=1}^{m-1}\frac{\Gamma(3-\beta)|t_m-t_i|}{4t_i^{2-\beta}}|J_i(u(t_i))|\\
&+\sum_{i=1}^{m-1}\int_{t_{i-1}}^{t_i}\frac{\Gamma(3-\beta)|t_m-t_i|(t_i-s)^{\alpha-\beta-1}}{2t_i^{2-\beta}\Gamma(\alpha-\beta)}|f(s,u(s),u'(s))|ds+\sum_{i=1}^{m}\frac{\Gamma(3-\beta)|T-t_m|}{4t_i^{2-\beta}}|J_i(u(t_i))|\\
&+\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\frac{\Gamma(3-\beta)|T-t_m|(t_i-s)^{\alpha-\beta-1}}{2t_i^{2-\beta}\Gamma(\alpha-\beta)}|f(s,u(s),u'(s))|ds+T^2\sum_{i=1}^{m}|J_i(u(t_i))|\\
&+\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}\frac{T^\beta\Gamma(3-\beta)(t_i-s)^{\alpha-\beta-1}}{2\Gamma(\alpha-\beta)}|f(s,u(s)u'(s))|ds\\
\le&\frac{(m+1)L_1}{2\Gamma(\alpha)}+\frac{m\Gamma(3-\beta)T^\beta L_1}{\Gamma(\alpha-\beta+1)}+\frac{m\Gamma(3-\beta)TL_1}{2\Gamma(\alpha-\beta+1)}+\frac{mL_3}{2}+\frac{(m-1)\Gamma(3-\beta)TL_4}{4}\\
&+\frac{m\Gamma(3-\beta)TL_4}{4}+mT^2L_4
\end{aligned}
$$

and

$$
\begin{aligned}
p_3\le&\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}\frac{\Gamma(3-\beta)(t_i-s)^{\alpha-\beta-1}}{2T^{2-\beta}\Gamma(\alpha-\beta)}|f(s,u(s),u'(s))|ds+\sum_{i=1}^{m}|J_i(u(t_i))|\\
\le&\frac{(m+1)T^{\beta-2}\Gamma(3-\beta)L_1}{2\Gamma(\alpha-\beta+1)}+mL_4.
\end{aligned}
$$

Therefore,

$$
|Tu(t)|\le\int_{t_k}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s)u'(s))|ds+\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}\frac{|t_k-t_i|(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s),u'(s))|ds
$$

$$
\begin{aligned}
&+\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}\frac{\Gamma(3-\beta)|t-t_k||t_k-t_i|(t_i-s)^{\alpha-\beta-1}}{t_i^{2-\beta}\Gamma(\alpha-\beta)}|f(s,u(s),u'(s))|ds+\sum_{i=1}^{k}I_i(u(t_i))\\
&+\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{|t-t_k|(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s)u'(s))|ds+\sum_{i=1}^{k-1}\frac{\Gamma(3-\beta)|t_k-t_i||t-t_k|}{t_i^{2-\beta}}|J_i(u(t_i))|\\
&+\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}\frac{\Gamma(3-\beta)|t-t_i|^2(t_i-s)^{\alpha-\beta-1}}{2t_i^{2-\beta}\Gamma(\alpha-\beta)}|f(s,u(s),u'(s))|ds+\sum_{i=1}^{k-1}|t_k-t_i||Q_i(u(t_i))|\\
&+\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{\Gamma(3-\beta)|t-t_i|^2(t_i-s)^{\alpha-\beta-1}}{2t_i^{2-\beta}\Gamma(\alpha-\beta)}|f(s,u(s)u'(s))|ds+\sum_{i=1}^{k-1}\frac{\Gamma(3-\beta)|t_k-t_i|^2}{2t_i^{2-\beta}}|J_i(u(t_i))|\\
&+\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s),u'(s))|ds+\sum_{i=1}^{k}|t-t_k||Q_i(u(t_i))|\\
&+\sum_{i=1}^{k}\frac{\Gamma(3-\beta)|t-t_i|^2}{2t_i^{2-\beta}}|J_i(u(t_i))|+|p_1|+|p_2|T+|p_3|T^2\\
\le&\frac{(2m+3)L_1}{4\Gamma(\alpha+1)}+\frac{(m+1)L_1}{4\Gamma(\alpha)}+\frac{(13m-4)TL_1}{4\Gamma(\alpha)}+\frac{9mT^2\Gamma(3-\beta)L_1}{4\Gamma(\alpha-\beta+1)}+\frac{3(m+1)T^{\beta}\Gamma(3-\beta)L_1}{4\Gamma(\alpha-\beta+1)}\\
&+\frac{mT^{\beta+1}\Gamma(3-\beta)L_1}{4\Gamma(\alpha-\beta+1)}+\frac{3mL_2}{2}+(3m-1)TL_3+\frac{(11m-5)T^2\Gamma(3-\beta)L_4}{4}\\
&+\frac{(m-1)T^3(\Gamma(3-\beta)+4)L_4}{8},
\end{aligned}
\tag{3.2}
$$

which implies that

$$
\begin{aligned}
\|Tu\|\le&\frac{(2m+3)L_1}{4\Gamma(\alpha+1)}+\frac{(m+1)L_1}{4\Gamma(\alpha)}+\frac{(13m-4)TL_1}{4\Gamma(\alpha)}+\frac{9mT^2\Gamma(3-\beta)L_1}{4\Gamma(\alpha-\beta+1)}+\frac{3(m+1)T^{\beta}\Gamma(3-\beta)L_1}{4\Gamma(\alpha-\beta+1)}\\
&+\frac{mT^{\beta+1}\Gamma(3-\beta)L_1}{4\Gamma(\alpha-\beta+1)}+\frac{3mL_2}{2}+(3m-1)TL_3+\frac{(11m-5)T^2\Gamma(3-\beta)L_4}{4}\\
&+\frac{(m-1)T^3(\Gamma(3-\beta)+4)L_4}{8}:=L.
\end{aligned}
$$

Next, for any $t\in J_k, 0\le k\le m$, we get

$$
\begin{aligned}
|(Tu)'(t)|\le&\int_{t_k}^{t}\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s),u'(s))|ds+\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s),u'(s))|ds\\
&+\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}\frac{\Gamma(3-\beta)(t_k-t_i)(t_i-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}|f(s,u(s),u'(s))|ds+\sum_{i=1}^{k}|Q_i(u(t_i))|\\
&+\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{\Gamma(3-\beta)(t-t_k)(t_i-s)^{\alpha-\beta-1}}{t_i^{2-\beta}\Gamma(\alpha-\beta)}|f(s,u(s),u'(s))|ds+\sum_{i=1}^{k}\frac{\Gamma(3-\beta)(t-t_k)}{2}|J_i(u(t_i))|\\
&+\sum_{i=1}^{k-1}\frac{\Gamma(3-\beta)(t_k-t_i)}{2t_i^{2-\beta}}|J_i(u(t_i))|+|p_2|+2|p_3|T\\
\le&\frac{3(m+1)L_1}{2\Gamma(\alpha)}+\frac{(5m-2)TL_1}{4\Gamma(\alpha)}+\frac{mT^{\beta}\Gamma(3-\beta)L_1}{\Gamma(\alpha-\beta+1)}+\frac{(m+1)T^{\beta-1}L_1}{\Gamma(\alpha-\beta+1)}\\
&+\frac{3mTL_3}{2}+\frac{(5m-3)T^2\Gamma(3-\beta)L_4}{4}+2mTL_4+mT^2L_4:=\bar{L}.
\end{aligned}
$$

Therefore, for $t_1,t_2\in J_k$ with $t_1<t_2, 0\le k\le m$, we have

$$
|(Tu)(t_2)-(Tu)(t_1)|\le\int_{t_1}^{t_2}|(Tu)'(s)|ds\le\bar{L}(t_2-t_1).
$$

This means that T is equicontinuous on all subintervals $J_k (k = 0, 1, 2, \cdots, m)$. Hence, by Arzela-Ascoli Theorem, it obtained that T is completely continuous.

Now, because of $\lim\limits_{u\to 0} \frac{f(t,u,u')}{u} = 0, \lim\limits_{u\to 0} \frac{I_k(u)}{u} = 0, \lim\limits_{u\to 0} \frac{Q_k(u)}{u} = 0$ and $\lim\limits_{u\to 0} \frac{J_k(u)}{u} = 0$, there exists a constant $z > 0$ such that $|f(t,u,u')| \le \rho_1|u|, |I_k(u)| \le \rho_2|u|, |Q_k(u)| \le \rho_3|u|$ and $|J_k(u)| \le \rho_4|u|$ for $0 < |u| < z$, where $\rho_i > 0 (i = 1, 2, 3, 4)$ satisfy

$$\begin{aligned}
&\frac{(2m+3)\rho_1}{4\Gamma(\alpha+1)} + \frac{(m+1)\rho_1}{4\Gamma(\alpha)} + \frac{(13m-4)T\rho_1}{4\Gamma(\alpha)} + \frac{9mT^2\Gamma(3-\beta)\rho_1}{4\Gamma(\alpha-\beta+1)} + \frac{3(m+1)T^\beta\Gamma(3-\beta)\rho_1}{4\Gamma(\alpha-\beta+1)} \\
&+ \frac{mT^{\beta+1}\Gamma(3-\beta)\rho_1}{4\Gamma(\alpha-\beta+1)} + \frac{3m\rho_2}{2} + (3m-1)T\rho_3 + \frac{(11m-5)T^2\Gamma(3-\beta)\rho_4}{4} \\
&+ \frac{(m-1)T^3(\Gamma(3-\beta)+4)\rho_4}{8} \le 1
\end{aligned} \tag{3.3}$$

Define $\Omega = \{u \in PC(J,\mathbb{R}) | \|u\| < r\}$ and take $u \in PC(J,\mathbb{R})$ such that $\|u\| = z$ so that $u \in \partial\Omega$. Then, we have

$$\begin{aligned}
|Tu(t)| \le &\Big\{ \frac{(2m+3)\rho_1}{4\Gamma(\alpha+1)} + \frac{(m+1)\rho_1}{4\Gamma(\alpha)} + \frac{(13m-4)T\rho_1}{4\Gamma(\alpha)} + \frac{9mT^2\Gamma(3-\beta)\rho_1}{4\Gamma(\alpha-\beta+1)} + \frac{3(m+1)T^\beta\Gamma(3-\beta)\rho_1}{4\Gamma(\alpha-\beta+1)} \\
&+ \frac{mT^{\beta+1}\Gamma(3-\beta)\rho_1}{4\Gamma(\alpha-\beta+1)} + \frac{3m\rho_2}{2} + (3m-1)T\rho_3 + \frac{(11m-5)T^2\Gamma(3-\beta)\rho_4}{4} \\
&+ \frac{(m-1)T^3(\Gamma(3-\beta)+4)\rho_4}{8} \Big\} \|u\|
\end{aligned} \tag{3.4}$$

which implies that $\|Tu\| \le \|u\|, u \in \partial\Omega$.

Hence, by the Theorem (2.1), the operator T has at least one fixed point and the problem (1.1) has at least one solution $u \in \overline{\Omega}$. □

Theorem 3.2. *Assume that*
(H_1) *there exist positive constants* $G_i (i = 1, 2, 3, 4)$ *such that*
$|f(t,u,u') - f(t,v,v')| \le G_1(|u-v| + |u'-v'|), \quad |I_k(u) - I_k(v)| \le G_2|u-v|,$
$|Q_k(u) - Q_k(v)| \le G_3|u-v|, \quad |J_k(u) - J_k(v)| \le G_4|u-v|,$
where $t \in J, u \in \mathbb{R}, k = 1, 2, \cdots, m$.
Then problem (1.1) has a unique solution if

$$\begin{aligned}
\Lambda = &\frac{(2m+3)G_1}{4\Gamma(\alpha+1)} + \frac{(m+1)G_1}{4\Gamma(\alpha)} + \frac{(13m-4)TG_1}{4\Gamma(\alpha)} + \frac{9mT^2\Gamma(3-\beta)G_1}{4\Gamma(\alpha-\beta+1)} + \frac{3(m+1)T^\beta\Gamma(3-\beta)G_1}{4\Gamma(\alpha-\beta+1)} \\
&+ \frac{mT^{\beta+1}\Gamma(3-\beta)G_1}{4\Gamma(\alpha-\beta+1)} + \frac{3mG_2}{2} + (3m-1)TG_3 + \frac{(11m-5)T^2\Gamma(3-\beta)G_4}{4} \\
&+ \frac{(m-1)T^3(\Gamma(3-\beta)+4)G_4}{8} < 1.
\end{aligned} \tag{3.5}$$

Proof. For $u, v \in C(J, R)$, we have

$$|(Tu)(t) - (Tv)(t)| \le \frac{1}{\Gamma(\alpha)} \int_{t_m}^{T} (t-s)^{\alpha-1} |f(s,u(s),u'(s)) - f(s,v(s),v'(s))| ds$$

$$+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}|f(s,u(s),u'(s))-f(s,v(s),v'(s))|ds$$
$$+\sum_{i=1}^{m-1}\frac{(t_m-t_i)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}|f(s,u(s),u'(s))-f(s,v(s),v'(s))|ds$$
$$+\sum_{i=1}^{m}\frac{(t-t_m)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}|f(s,u(s),u'(s))-f(s,v(s),v'(s))|ds$$
$$+\sum_{i=1}^{m-1}\frac{(t_m-t_i)}{2\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}|f(s,u(s)u'(s))-f(s,v(s),v'(s))|ds$$
$$+\sum_{i=1}^{m-1}\frac{\Gamma(3-\beta)(t-t_i)^2}{2t_i^{2-\beta}\Gamma(\alpha-\beta)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-\beta-1}|f(s,u(s),u'(s))-f(s,v(s),v'(s))|ds$$
$$+\sum_{i=1}^{m}\frac{(T-2t_m)}{4\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}|f(s,u(s),u'(s))-f(s,v(s),v'(s))|ds$$
$$+\sum_{i=1}^{m-1}\frac{\Gamma(3-\beta)T^2}{4t_i^{2-\beta}\Gamma(\alpha-\beta)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-\beta-1}|f(s,u(s)u'(s))-f(s,v(s),v'(s))|ds$$
$$+\sum_{i=1}^{m-1}\frac{\Gamma(3-\beta)(t-t_m)(t_m-t_i)}{t_i^{2-\beta}\Gamma(\alpha-\beta)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-\beta-1}|f(s,u(s),u'(s))-f(s,v(s),v'(s))|ds$$
$$+\sum_{i=1}^{m}\frac{\Gamma(3-\beta)(t_m^2-Tt_m)}{4t_i^{2-\beta}\Gamma(\alpha-\beta)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-\beta-1}|f(s,u(s),u'(s))-f(s,v(s),v'(s))|ds$$
$$+\frac{1}{4\Gamma(\alpha-1)}\int_{t_m}^{T}(T-s)^{\alpha-2}|f(s,u(s),u'(s))-f(s,v(s),v'(s))|ds$$
$$+\sum_{i=1}^{m}\frac{\Gamma(3-\beta)(t-t_i)^2}{2t_i^{2-\beta}\Gamma(\alpha-\beta)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-\beta-1}|f(s,u(s),u'(s))-f(s,v(s),v'(s))|ds$$
$$+\frac{1}{2\Gamma(\alpha)}\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}|f(s,u(s),u'(s))-f(s,v(s),v'(s))|ds$$
$$+\frac{T^{\beta+1}\Gamma(3-\beta)}{2\Gamma(\alpha-\beta)}\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}T(t_i-s)^{\alpha-\beta-1}|f(s,u(s),u'(s))-f(s,v(s),v'(s))|ds$$
$$+\sum_{i=1}^{m-1}\frac{T\Gamma(3-\beta)(t_m-t_i)}{2t_i^{2-\beta}\Gamma(\alpha-\beta)}\int_{t_{i-1}}^{t_i}T(t_i-s)^{\alpha-\beta-1}|f(s,u(s)u'(s))-f(s,v(s),v'(s))|ds$$
$$+\frac{T^{\beta}\Gamma(3-\beta)}{2\Gamma(\alpha-\beta)}\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}T^2(t_i-s)^{\alpha-\beta-1}|f(s,u(s),u'(s))-f(s,v(s),v'(s))|ds$$
$$+\frac{T}{2\Gamma(\alpha-1)}\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}T(t_i-s)^{\alpha-2}|f(s,u(s),u'(s))-f(s,v(s),v'(s))|ds$$
$$+\sum_{i=1}^{m}\frac{T\Gamma(3-\beta)(T-t_m)}{2t_i^{2-\beta}\Gamma(\alpha-\beta)}\int_{t_{i-1}}^{t_i}T(t_i-s)^{\alpha-\beta-1}|f(s,u(s),u'(s))-f(s,v(s),v'(s))|ds$$
$$+\frac{(2T-1)T^{\beta}\Gamma(3-\beta)}{4\Gamma(\alpha-\beta)}\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-\beta-1}|f(s,u(s),u'(s))-f(s,v(s),v'(s))|ds$$
$$+\sum_{i=1}^{m-1}\frac{(t_m-t_i)}{2}I_i(u(t_i))+\sum_{i=1}^{m}\frac{(T-2t_m)}{4}|Q_i(u(t_i)-v(t_i))|+\sum_{i=1}^{k}|I_i(u(t_i)-v(t_i))|$$
$$+\sum_{i=1}^{m-1}\frac{(t_m-t_i)}{2}|Q_i(u(t_i)-v(t_i))|+\sum_{i=1}^{m-1}\frac{\Gamma(3-\beta)(t_m-t_i)^2}{4t_i^{2-\beta}}|J_i(u(t_i)-v(t_i))|$$
$$+\sum_{i=1}^{k}\frac{\Gamma(3-\beta)(t-t_i)^2}{2t_i^{2-\beta}}|J_i(u(t_i)-v(t_i))|+\sum_{i=1}^{m-1}\frac{\Gamma(3-\beta)T^2}{2t_i^{2-\beta}}|J_i(u(t_i)-v(t_i))|$$

$$
\begin{aligned}
&+\sum_{i=1}^{m-1}\frac{\Gamma(3-\beta)(3T-2t_m-2t_i)(t_m-t_i)}{8t_i^{2-\beta}}|J_i(u(t_i)-v(t_i))|+\sum_{i=1}^{m}\frac{T|Q_i(u(t_i)-v(t_i))|}{2}\\
&+\sum_{i=1}^{m}\frac{T\Gamma(3-\beta)T^2}{8t_i^{2-\beta}}|J_i(u(t_i)-v(t_i))|+\frac{T^3-T^2}{2}\sum_{i=1}^{m}|J_i(u(t_i)-v(t_i))|\\
&+\sum_{i=1}^{m-1}\frac{T\Gamma(3-\beta)(t_m-t_i)}{4t_i^{2-\beta}}|J_i(u(t_i)-v(t_i))|+\sum_{i=1}^{m}\frac{\Gamma(3-\beta)(T-t_m)}{4t_i^{2-\beta}}T|J_i(u(t_i)-v(t_i))|\\
&+T^3\sum_{i=1}^{m}|J_i(u(t_i)-v(t_i))|+\sum_{i=1}^{m}|J_i(u(t_i)-v(t_i))|+\sum_{i=1}^{m}(t-t_m)|Q_i(u(t_i)-v(t_i))|\\
&+\sum_{i=1}^{m-1}(t_m-t_i)|Q_i(u(t_i)-v(t_i))|+\sum_{i=1}^{m-1}\frac{\Gamma(3-\beta)(t_m-t_i)^2}{2t_i^{2-\beta}}|J_i(u(t_i)-v(t_i))|\\
&+\sum_{i=1}^{m-1}\frac{\Gamma(3-\beta)(t_m-t_i)(t-t_m)}{t_i^{2-\beta}}|J_i(u(t_i)-v(t_i))|\\
&\le\bigg\{\frac{(2m+3)L_1}{4\Gamma(\alpha+1)}+\frac{(m+1)L_1}{4\Gamma(\alpha)}+\frac{(13m-4)TL_1}{4\Gamma(\alpha)}+\frac{9mT^2\Gamma(3-\beta)G_1}{4\Gamma(\alpha-\beta+1)}\\
&+\frac{3(m+1)T^{\beta}\Gamma(3-\beta)G_1}{4\Gamma(\alpha-\beta+1)}+\frac{mT^{\beta+1}\Gamma(3-\beta)G_1}{4\Gamma(\alpha-\beta+1)}+\frac{3mG_2}{2}+(3m-1)TG_3\\
&+\frac{(11m-5)T^2\Gamma(3-\beta)G_4}{4}+\frac{(m-1)T^3(\Gamma(3-\beta)+4)G_4}{8}\bigg\}\|u-v\|\\
&\le\Lambda\|u-v\|,
\end{aligned}
$$

where Λ is given by (3.5). Hence, T is a contraction mapping principle and problem (1.1) has a unique solution. □

4 Examples

Example 4.1. Consider the following anti-periodic fractional boundary value problem with $2<\alpha\le 3, \beta=\frac{3}{2}, T=[0,1]$ and

$$
\begin{cases}
{}^CD^{\alpha}u(t)=(t^3+u'(t))\mathrm{arctan}^2u(t)+e^tu^3(t), \qquad 0<t<1, t\neq\frac{1}{2}\\
\Delta u(\frac{1}{2})=1-\mathrm{cos}u(t), \Delta u'(\frac{1}{2})=\sin^2u(t), \Delta^CD^{\frac{3}{2}}u(\frac{1}{2})=\ln(1+u^2(t)),\\
u(0)=-u(1), u'(0)=-u'(1), {}^CD^{\frac{3}{2}}u(0)=-{}^CD^{\frac{3}{2}}u(1),
\end{cases}
\tag{4.1}
$$

where we have taken $m=1$.

By $\lim\limits_{u\to 0}\frac{f(t,u,u')}{u}=0, \lim\limits_{u\to 0}\frac{I_k(u)}{u}=0, \lim\limits_{u\to 0}\frac{Q_k(u)}{u}=0, \lim\limits_{u\to 0}\frac{J_k(u)}{u}=0$, we have

$$
\lim_{u(t)\to 0}\frac{f((t^3+u'(t))\mathrm{arctan}^2u(t)+e^tu^3(t))}{u(t)}=0, \qquad \lim_{u(t)\to 0}\frac{1-\mathrm{cos}u(t)}{u(t)}=0,
$$

and

$$
\lim_{u(t)\to 0}\frac{\sin^2u(t)}{u(t)}=0, \qquad \lim_{u(t)\to 0}\frac{\ln(1+u^2(t))}{u(t)}=0.
$$

All the assumptions of Theorem 3.1 are satisfied. Thus, the impulsive anti-periodic impulsive fractional boundary value problem (4.1) has at least one solution on $[0, 1]$.

Example 4.2. At last, consider the following anti-periodic impulsive fractional boundary value problem with $\alpha = \frac{5}{2}, \beta = \frac{3}{2}, T = [0, 2\pi]$ and

$$\begin{cases} {}^C D^{\frac{5}{2}} u(t) = (t+5)^3(\sin u + \cos u), \qquad 0 < t < 1, t \neq \frac{1}{t_1} \\ \Delta u(\frac{1}{t_1}) = \frac{1}{120}, \Delta u'(\frac{1}{t_1}) = \frac{1}{80}, \Delta {}^C D^{\frac{3}{2}} u(\frac{1}{t_1}) = \frac{1}{50}, \\ u(0) = -u(2\pi), u'(0) = -u'(2\pi), {}^C D^{\frac{3}{2}} u(0) = -{}^C D^{\frac{3}{2}} u(2\pi), \end{cases} \tag{4.2}$$

where we have taken $m = 1$. Let $u, v, u', v' \in \mathbb{R}$, and $G_1 = \frac{1}{200}, G_2 = \frac{1}{110}, G_3 = \frac{1}{70}, G_2 = \frac{1}{40}$. Thus,

$$|f(t,u,u') - f(t,v,v')| \leq \frac{1}{200}(|u-v| + |u'-v'|), \quad |I_1(u) - I_1(v)| \leq \frac{1}{110},$$
$$|Q_1(u) - Q_1(v)| \leq \frac{1}{70}, \quad |J_1(u) - J_1(v)| \leq \frac{1}{40}.$$

By the conclusion (3.5), we have

$$\begin{aligned}\Lambda =& \frac{(2m+3)G_1}{4\Gamma(\alpha+1)} + \frac{(m+1)G_1}{4\Gamma(\alpha)} + \frac{(13m-4)TG_1}{4\Gamma(\alpha)} + \frac{9mT^2\Gamma(3-\beta)G_1}{4\Gamma(\alpha-\beta+1)} + \frac{3(m+1)T^{\beta}\Gamma(3-\beta)G_1}{4\Gamma(\alpha-\beta+1)} \\ &+ \frac{mT^{\beta+1}\Gamma(3-\beta)G_1}{4\Gamma(\alpha-\beta+1)} + \frac{3mG_2}{2} + (3m-1)TG_3 + \frac{(11m-5)T^2\Gamma(3-\beta)G_4}{4} \\ &+ \frac{(m-1)T^3(\Gamma(3-\beta)+4)G_4}{8} \\ =& \frac{\frac{5}{200}}{4\Gamma(\frac{7}{2})} + \frac{\frac{2}{50}}{4\Gamma(\frac{5}{2})} + \frac{(2\pi)\times\frac{9}{200}}{4\Gamma(\frac{5}{2})} + \frac{(2\pi)^2\times\Gamma(\frac{1}{2})\times\frac{9}{200}}{4\Gamma(2)} + \frac{(2\pi)^{\frac{3}{2}}\times\Gamma(\frac{3}{2})\times\frac{6}{200}}{4\Gamma(2)} + \frac{(2\pi)^{\frac{5}{2}}\times\Gamma(\frac{3}{2})\times\frac{1}{200}}{4\Gamma(2)} \\ &+ \frac{3\times\frac{1}{110}}{2} + 4\pi\times\frac{1}{70} + \frac{6\times(2\pi)^2\times\Gamma(\frac{3}{2})\times\frac{1}{40}}{4} + 0 \\ \approx& 0.818 < 1.\end{aligned}$$

Therefore, all the assumptions of Theorem 3.2 are satisfied. Thus, by the conclusion Theorem 3.2, the anti-periodic impulsive fractional boundary value problem (4.2) has a unique solution on $[0, 2\pi]$.

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