Integral Boundary Value Problem for Fractional Order Differential Equations with Non-instantaneous Impulses

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Abstract
In this paper, we study the integral boundary value problem of fractional differential equations with non-instantaneous impulses. By using some fixed point theorems, we obtain sufficient conditions for existence of a unique solution, at least one solution, respectively. The main results are also demonstrated with examples.

Mathematics Subject Classifications: 34B15, 34B18, 34B37, 58E30

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1 Introduction
In the past years, differential equations with non-instantaneous impulsive effects are widely used to characterize the evolution processes in pharmacotherapy and ecological systems. This new type impulsive equations was introduced

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in [1] and the existence of solutions, stability and controllability problems for differential equations with non-instantaneous impulsive effects have been studied in [4, 5, 6, 7, 8, 9, 10, 12, 15]. There is a growing tendency nowadays for many experts to show their great enthusiasm for this aspect, and a lot of achievements have been made; see the monographs [4, 5, 6, 7, 8, 9, 10, 12, 15]. The differential equations with instantaneous impulse cannot explain some dynamics problems of evolution process. For example, the drug delivery in the bloodstream is a gradual and continuous process. However, the models with non-instantaneous impulses can explain these problems.

Based on the research on the changes of drugs in blood after human injection, Hernandez and O’regan [1] initially proposed the concept of non-instantaneous impulsive differential equation in 2013:

\[
\begin{cases}
  x'(t) = Ax(t) + f(t, x(t)), & t \in (s_i, t_{i+1}], i = 0, 1, 2, \ldots, m, \\
  x(t) = g_i(t, x(t)), & t \in (t_i, s_i], i = 1, 2, \ldots, m, \\
  x(0) = x_0 \in E,
\end{cases}
\]

where \(0 = s_0 < t_1 < s_1 < t_2 < \cdots < t_m < s_m < t_{m+1} = T\), \(A : D(A) \subseteq E \rightarrow E\) is the \(C_0\) semigroup operator in Banach space \(E\), \(f : C(J \times E, E)\), \(g_i \in C((t_i, s_i] \times E, E)\). In this paper, the author has given the conclusion of the only solution with the semi-group theory and the non-motion point.

In 2014, Wang [2] used the principle of compression mapping and Krasnosel’ski’s theorem to study the existence and uniqueness of periodic boundary value solutions for nonlinear fractional differential equations with non-transient pulses:

\[
\begin{cases}
  ^cD_{0, t}^q x(t) = f(t, x(t)), & t \in (s_i, t_{i+1}], i = 0, 1, 2, \ldots, m, q \in (0, 1), \\
  x(t) = g_i(t, x(t)), & t \in (t_i, s_i], i = 1, 2, \ldots, m, \\
  x(0) = x(T),
\end{cases}
\]

where \(0 = s_0 < t_1 < s_1 < t_2 < \cdots < t_m < s_m < t_{m+1} = T\) are pre-fixed numbers, \(f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}\) is continuous and \(g_i : [t_i, s_i] \times \mathbb{R} \rightarrow \mathbb{R}\) is continuous for all \(i = 1, 2, \cdots, m\).

In 2017, Yang [3] used the principle of compression mapping and Krasnosel’ski’s theorem to study the existence and uniqueness of solutions of nonlinear fractional differential equations with non-transient pulses:

\[
\begin{cases}
  ^cD_{0, t}^q x(t) = f(t, x(t)), & t \in (s_i, t_{i+1}], i = 0, 1, 2, \ldots, m, q \in (0, 1), \\
  x(t) = g_i(t, x(t)), & t \in (t_i, s_i], i = 1, 2, \ldots, m, \\
  x(0) = \int_0^1 x(s)ds,
\end{cases}
\]

where \(0 = s_0 < t_1 < s_1 < t_2 < \cdots < t_m < s_m < t_{m+1} = 1\) are pre-fixed numbers, \(f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}\) is continuous and \(g_i : [t_i, s_i] \times \mathbb{R} \rightarrow \mathbb{R}\) is continuous for all \(i = 1, 2, \cdots, m\).
Motivated by the discussion above, we mainly study the integral boundary value problem of fractional-order nonlinear differential equations with non-instantaneous impulses:

\[
\begin{aligned}
\begin{cases}
^{c}D_{0,t}^{q}x(t) = f(t, x(t)), & t \in [0, t_{1}] \cup (s_{i}, t_{i+1}], i = 1, 2, \ldots, m, q \in (0, 1), \\
x(t) = g_{i}(t, x(t)), & t \in (t_{i}, s_{i}], i = 1, 2, \ldots, m, \\
x(0) + \mu \int_{0}^{T} x(s)ds = x(T),
\end{cases}
\end{aligned}
\]  

(1.1)

where \(^{c}D_{0,t}^{q}\) denotes the Caputo fractional derivative of the order \(q\) with the lower limit zero, \(0 = s_{0} < t_{1} \leq t_{2} \leq \cdots \leq t_{m} < t_{m+1} = T\) are pre-fixed numbers, \(f: [0, T] \times \mathbb{R} \rightarrow \mathbb{R}\) is continuous and \(g_{i} : [t_{i}, s_{i}] \times \mathbb{R} \rightarrow \mathbb{R}\) is continuous for all \(i = 1, 2, \ldots, m\). \(\mu\) is a constant.

The following article is organized as follows: In Section 2, we will recall some known results for our consideration. Some lemmas and definitions are useful to our works. Section 3 is devoted to researching the existence and uniqueness of solutions for Eq.(1.1).

## 2 Preliminaries

In this section, we plan to introduce some basic definitions and lemmas which are used throughout this paper.

Let \(J = [0, T]\). Denote \(C(J, \mathbb{R})\) by the Banach space of all continuous functions from \(J\) into \(\mathbb{R}\) with the norm \(\|x\|_{C} := \max\{|x(t)| : t \in J\}\) for \(x \in C(J, \mathbb{R})\). Define the piecewise continuous space \(PC(J, \mathbb{R}) := \{x : J \rightarrow \mathbb{R} : x \in C((t_{i}, t_{i+1}], \mathbb{R}), i = 0, 1, 2, \ldots, m\) and there exist \(x(t_{i}^{-})\) and \(u(t_{i}^{+})\), \(i = 1, 2, \ldots, m\), with \(x(t_{i}^{-}) = x(t_{i})\}\) with the norm \(\|x\|_{PC} := \max\{|x(t)| : t \in J\}\). Set \(PC^{1}(J, \mathbb{R}) := \{x \in PC(J, \mathbb{R}) : x' \in PC(J, \mathbb{R})\}\) with \(\|x\|_{PC^{1}} := \max\{|x|_{PC}, \|x'\|_{PC}\}\). Clearly, \(PC^{1}(J, \mathbb{R})\) endowed with the norm \(\|\cdot\|_{PC^{1}}\) is a Banach space.

**Definition 2.1.** ([2]) The fractional integral of order with the lower limit zero for a function \(f : [0, \infty) \rightarrow \mathbb{R}\) is defined as \(I_{t}^{q}f(t) := \frac{1}{\Gamma(q)} \int_{0}^{t} \frac{f(s)}{(t-s)^{q-\alpha}}ds, t > 0, q > 0,\) provided the right side is point-wise defined on \([0, \infty)\), where \(\Gamma(\cdot)\) is the gamma function.

**Definition 2.2.** ([3]) The Riemann-Liouville derivative of order \(q\) with the lower limit zero for a function \(f : [0, \infty) \rightarrow \mathbb{R}\) can be written as \(D_{0,t}^{q}f(t) := \frac{d}{dt}I_{t}^{\alpha-q}f(t) = \frac{d}{dt}\int_{0}^{t} \frac{f(s)}{(t-s)^{q-\alpha}}ds, t > 0, 0 < q < 1.\)

**Definition 2.3.** ([23]) The generalized Caputo derivative of order \(q \in (0, 1)\) with the lower limit zero for a function \(f : [0, \infty) \rightarrow \mathbb{R}\) can be written as \(^{c}D_{0,t}^{q}f(t) := \frac{d}{dt}I_{t}^{\alpha-q}f(t) = \frac{d}{dt}D_{0,t}^{q}f(t), t > 0.\)
Definition 2.4. A function $x \in PC^1(J, \mathbb{R})$ is said to be a solution of the equation (2.1) if $x(t)$ satisfies (2.1).

We define

$$u_1 = \frac{1}{\Gamma(q)} \int_0^{t_1} \int_0^s (s - \tau)^{q-1}h(\tau)d\tau ds,$$

$$u_2 = \sum_{i=1}^{m} \int_{t_i}^{s_i} h_i(s)ds,$$

$$u_3 = \sum_{i=1}^{m} \int_{s_i}^{t_{i+1}} \left( h_i(s_i) - \frac{1}{\Gamma(q)} \int_0^{s_i} (s_i - \tau)^{q-1}h(\tau)d\tau ds + \frac{1}{\Gamma(q)} \int_0^{s} (s - \tau)^{q-1}h(\tau)d\tau \right) ds.$$

Lemma 2.5. A function $x \in PC^1(J, \mathbb{R})$ given by

$$x(t) = \begin{cases} 
  h_m(s_m) - \frac{1}{\Gamma(q)} \int_0^{s_m} (s_m - s)^{q-1}h(s)ds + \frac{1}{\Gamma(q)} \int_0^T (T-s)^{q-1}h(s)ds - \mu (u_1 + u_2 + u_3) \\
  + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}h(s)ds, & t \in [0, t_1], \\
  h_i(t), & t \in (t_i, s_i], i = 1, 2, \ldots, m, \\
  h_i(s_i) - \frac{1}{\Gamma(q)} \int_0^{s_i} (s_i - s)^{q-1}h(s)ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}h(s)ds, & t \in (s_i, t_{i+1}], i = 1, 2, \ldots, m 
\end{cases}$$

is a solution of the following system

$$\begin{cases} 
  cD_0^qx(t) = h(t), & t \in (s_i, t_{i+1}], i = 0, 1, 2, \ldots, m, q \in (0, 1), \\
  x(t) = h_i(t), & t \in (t_i, s_i], i = 1, 2, \ldots, m, \\
  x(0) + \mu \int_0^T x(s)ds = x(T).
\end{cases} \tag{2.1}$$

Proof. Assume that $x$ satisfies equation (2.1). For $t \in [0, t_1]$, integrating the first equation of (2.1) from zero to $t$, we have

$$x(t) = x(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}h(s)ds.$$

In addition, for $t \in (s_i, t_{i+1}], i = 1, 2, \ldots, m$ integrating the first equation of (2.1) from $s_i$ to $t$, we obtain

$$x(t) = x(s_i) - \frac{1}{\Gamma(q)} \int_0^{s_i} (s_i - s)^{q-1}h(s)ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}h(s)ds.$$

For $t \in (t_i, s_i]$, apply the impulsive condition of (2.1) we have

$$x(s_i) = h_i(s_i).$$
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So

\[ x(t) = h_i(s_i) - \frac{1}{\Gamma(q)} \int_0^{s_i} (s_i - s)^{q-1} h(s) ds + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} h(s) ds. \]

Now we remain to determine the initial value \( x(0) \). Note that

\[
\begin{align*}
  x(0) &= x(T) - \mu \int_0^T x(s) ds \\
  &= h_m(s_m) - \frac{1}{\Gamma(q)} \int_0^{s_m} (s_m - s)^{q-1} h(s) ds + \frac{1}{\Gamma(q)} \int_0^T (T - s)^{q-1} h(s) ds \\
  &\quad - \mu \left( \int_0^{t_1} x(0) ds + \frac{1}{\Gamma(q)} \int_0^{t_1} \int_0^s (s - \tau)^{q-1} h(\tau) d\tau ds + \sum_{i=1}^m \int_{s_i}^{t_{i+1}} h_i(s_i) ds \right) \\
  &\quad + \sum_{i=1}^m \int_{s_i}^{t_{i+1}} h_i(s_i) ds - \frac{1}{\Gamma(q)} \int_{s_i}^{t_{i+1}} \int_0^{s_i} (s_i - \tau)^{q-1} h(\tau) d\tau ds \\
  &\quad + \frac{1}{\Gamma(q)} \int_{s_i}^{t_{i+1}} \int_0^s (s - \tau)^{q-1} h(\tau) d\tau ds \\
  &= h_m(s_m) - \frac{1}{\Gamma(q)} \int_0^{s_m} (s_m - s)^{q-1} h(s) ds \\
  &\quad + \frac{1}{\Gamma(q)} \int_0^T (T - s)^{q-1} h(s) ds - \mu x(0)t_1 - \mu(u_1 + u_2 + u_3).
\end{align*}
\]

Thus, one can solve

\[
x(0) = \frac{1}{1 + \mu t_1} \left( h_m(s_m) - \frac{1}{\Gamma(q)} \int_0^{s_m} (s_m - s)^{q-1} h(s) ds + \frac{1}{\Gamma(q)} \int_0^T (T - s)^{q-1} h(s) ds \\
  - \mu(u_1 + u_2 + u_3) \right).
\]

The proof is completed. \( \square \)

Lemma 2.6. (Compression mapping principle) \( E \) is a complete metric space. If \( F \) is a compression map on \( E \), then map \( F \) has a unique fixed point on \( E \).

Theorem 2.7. ([3]) (Hölder inequality) If \( f(x) \) and \( g(x) \) are continuous non-negative functions on the closed interval \( a \leq x \leq b \), then

\[
\int_a^b f(x) g(x) dx \leq \left( \int_a^b f^p(x) dx \right)^{\frac{1}{p}} \left( \int_a^b g^q(x) dx \right)^{\frac{1}{q}},
\]

where \( \frac{1}{p} + \frac{1}{q} = 1 \), \( p \) and \( p_1 \) are real numbers greater than 1.
Theorem 3.1. Suppose that the conditions

(i) $P$ is compact and continuous,
(ii) $Q$ is a contraction mapping,

Then there exist $x \in E$ such that $Fx = P x + Q x$.

3 Main results

3.1 Existence of a unique solution

In this section we derive conditions under which system (1.1) admits a unique solution. Before stating and proving the results, we introduce the following hypotheses:

$(H_1)$ There is a positive constant $L_f$ such that $|f(t, x_1) - f(t, x_2)| \leq L_f |x_1 - x_2|$, for each $t \in [s_i, t_{i+1}]$, $i = 1, 2, \cdots, m$, and $x_1, x_2 \in \mathbb{R}$.

$(H_2)$ There is a positive constant $L_g$, such that $|g_i(t, x_1) - g_i(t, x_2)| \leq L_g |x_1 - x_2|$, for each $t \in [t_i, s_i]$, $i = 1, 2, \cdots, m$, and $x_1, x_2 \in \mathbb{R}$.

Next, we set $L_g = \max_{i=1,2,\cdots,m} L_{g_i}$, $\Theta_1 = \frac{1+\mu(T-t_1)}{1+\mu t_1}$ and $\Theta_2 = \frac{1}{1+\mu t_1} \left( \frac{\mu}{1+\mu t_1} \sum_{i=0}^{m} (t_{q+1} - s_i) + \frac{1}{r(q+1)} (s_{q+1} + T^q) \right) + \frac{1}{r(q+1)} t^q_1$.

Theorem 3.1. Suppose that the conditions $(H_1)$, and $(H_2)$ are satisfied. If $\triangle := \max \{ \Theta_1 L_g + \Theta_2 L_f, L_g + \frac{L_f}{r(q+1)} (t_{q+1} + s^q_1) \} < 1$, then the system (1.1) has a unique solution.

Proof. We turn the problem (1.1) into a fixed point problem. Define an operator $F : PC(J, \mathbb{R}) \rightarrow PC(J, \mathbb{R})$ by

$$(Fx)(t) = \begin{cases} 
\frac{1}{1+\mu t_1} (g_m(s_m, x(s_m)) - \frac{1}{r(q)} \int_0^{s_m} (s_m - s)^{q-1} f(s, x(s))ds \\
+ \frac{1}{r(q)} \int_0^{T} (T - s)^{q-1} f(s, x(s))ds - \mu (v_1 + v_2 + v_3)) \\
+ \frac{1}{r(q)} \int_0^{T} (t - s)^{q-1} f(s, x(s))ds, & t \in [0, t_1], \\
g_i(t, x(t)), & t \in (t_i, s_i], \\
g_i(s_i, x(s_i)) - \frac{1}{r(q)} \int_0^{s_i} (s_i - s)^{q-1} f(s, x(s))ds \\
+ \frac{1}{r(q)} \int_0^{t} (t - s)^{q-1} f(s, x(s))ds, & t \in (s_i, t_{i+1}], 
\end{cases}$$
where

\[
v_1 = \frac{1}{\Gamma(q)} \int_0^{t_1} \int_0^s (s - \tau)^{q-1} f(\tau, x(\tau)) d\tau ds,
\]

\[
v_2 = \sum_{i=1}^m \int_{t_i}^{s_i} g_i(s, x(s)) ds,
\]

\[
v_3 = \sum_{i=1}^m \int_{s_i}^{t_{i+1}} \left( g_i(s_i, x(s_i)) \right) - \frac{1}{\Gamma(q)} \int_0^{s_i} (s_i - \tau)^{q-1} f(\tau, x(\tau)) d\tau ds
\]

\[
+ \frac{1}{\Gamma(q)} \int_0^{s} (s - \tau)^{q-1} f(\tau, x(\tau)) d\tau ds
\]

Clearly, \( F \) is well defined and \( x \in PC(J, \mathbb{R}) \), for all \( Fx \in PC(J, \mathbb{R}) \). Next, we show that \( F \) is a contractive mapping.

Case 1. For \( x_1, x_2 \in PC(J, \mathbb{R}) \), and for each \( t \in [0, t_1] \), we have

\[
|Fx_1 - Fx_2|_{PC} \leq \frac{1}{1 + \mu t_1} \left( |g_m(s_m, x_1(s_m)) - g_m(s_m, x_2(s_m))| \right.
\]

\[
+ \frac{1}{\Gamma(q)} \int_0^{s_m} (s_m - s)^{q-1} |f(s, x_1(s)) - f(s, x_2(s))| ds
\]

\[
+ \frac{1}{\Gamma(q)} \int_0^{T} (T - s)^{q-1} |f(s, x_1(x)) - f(s, x_2(x))| ds + \mu(|v'_1 - v''_1| + |v'_2 - v''_2| + |v'_3 - v''_3|)
\]

\[
+ \frac{1}{\Gamma(q)} \int_0^{t} (t - s)^{q-1} |f(s, x_1(s)) - f(s, x_2(s))| ds,
\]

where

\[
|v'_1 - v''_1| = \left| \frac{1}{\Gamma(q)} \int_0^{t_1} \int_0^s (s - \tau)^{q-1} f(\tau, x_1(\tau)) d\tau ds - \frac{1}{\Gamma(q)} \int_0^{t_1} \int_0^s (s - \tau)^{q-1} f(\tau, x_2(\tau)) d\tau ds \right|
\]

\[
\leq \frac{1}{\Gamma(q)} \int_0^{t_1} \int_0^s (s - \tau)^{q-1} |f(\tau, x_1(\tau)) - f(\tau, x_2(\tau))| d\tau ds
\]

\[
\leq \frac{L_f}{\Gamma(q + 2)} t_1^{q+1} ||x_1 - x_2||_{PC},
\]

and

\[
|v'_2 - v''_2| = \left| \sum_{i=1}^m \int_{t_i}^{s_i} g_i(s, x_1(s)) ds - \sum_{i=1}^m \int_{t_i}^{s_i} g_i(s, x_2(s)) ds \right|
\]

\[
\leq \sum_{i=1}^m \int_{t_i}^{s_i} |g_i(s, x_1(s)) - g_i(s, x_2(s))| ds
\]

\[
\leq L_g \sum_{i=1}^m (s_i - t_i) ||x_1 - x_2||_{PC},
\]
\[
|v'_3 - v''_3| = \left| \sum_{i=1}^{m} \int_{s_i}^{t_{i+1}} g_i(s, x_1(s)) \, ds - \sum_{i=1}^{m} \int_{s_i}^{t_{i+1}} g_i(s, x_2(s)) \, ds \right|
\]
\[
+ \frac{1}{\Gamma(q)} \sum_{i=1}^{m} \int_{s_i}^{t_{i+1}} \int_{0}^{s_i} (s_i - \tau)^{q-1} f(\tau, x_1(\tau)) \, d\tau \, ds
\]
\[
- \frac{1}{\Gamma(q)} \sum_{i=1}^{m} \int_{s_i}^{t_{i+1}} \int_{0}^{s_i} (s_i - \tau)^{q-1} f(\tau, x_2(\tau)) \, d\tau \, ds
\]
\[
+ \frac{1}{\Gamma(q)} \sum_{i=1}^{m} \int_{s_i}^{t_{i+1}} \int_{0}^{s} (s - \tau)^{q-1} f(\tau, x_1(\tau)) \, d\tau \, ds
\]
\[
- \frac{1}{\Gamma(q)} \sum_{i=1}^{m} \int_{s_i}^{t_{i+1}} \int_{0}^{s} (s - \tau)^{q-1} f(\tau, x_2(\tau)) \, d\tau \, ds
\]
\[
\leq \sum_{i=1}^{m} \int_{s_i}^{t_{i+1}} \left| g_i(s, x_1(s_i)) - g_i(s, x_2(s_i)) \right| \, ds
\]
\[
+ \frac{1}{\Gamma(q)} \sum_{i=1}^{m} \int_{s_i}^{t_{i+1}} \int_{0}^{s_i} (s_i - \tau)^{q-1} \left| f(\tau, x_1(\tau)) - f(\tau, x_2(\tau)) \right| \, d\tau \, ds
\]
\[
+ \frac{1}{\Gamma(q)} \sum_{i=1}^{m} \int_{s_i}^{t_{i+1}} \int_{0}^{s} (s - \tau)^{q-1} \left| f(\tau, x_1(\tau)) - f(\tau, x_2(\tau)) \right| \, d\tau \, ds
\]
\[
\leq L_g \sum_{i=1}^{m} (t_{i+1} - s_i) |x_1 - x_2|_{PC}
\]
\[
+ \frac{L_f}{\Gamma(q+1)} \sum_{i=1}^{m} s^q_i (t_{i+1} - s_i) |x_1 - x_2|_{PC}
\]
\[
+ \frac{L_f}{\Gamma(q+2)} \sum_{i=1}^{m} (t_{i+1}^q - s_i^{q+1}) |x_1 - x_2|_{PC}
\]
\[
\leq \left( L_g \sum_{i=1}^{m} (t_{i+1} - s_i) + \frac{L_f}{\Gamma(q+1)} \sum_{i=1}^{m} s^q_i (t_{i+1} - s_i) + \frac{L_f}{\Gamma(q+2)} \sum_{i=1}^{m} (t_{i+1}^q - s_i^{q+1}) \right) |x_1 - x_2|_{PC}.
\]
Thus

\[
|Fx_1 - Fx_2| \\
\leq \frac{1}{1 + \mu t_1} \left( L_g + \frac{L_f q}{\Gamma(q + 1)} s(q + 1) + \frac{L_f}{\Gamma(q + 1)} T^q + \frac{\mu L_f}{\Gamma(q + 1)} t^{q+1} + \mu L_g (T - t_1) + \frac{\mu L_f}{\Gamma(q + 1)} \sum_{i=1}^{m} (s^q_i (t_{i+1} - s_i) + \frac{\mu L_f}{\Gamma(q + 1)} \sum_{i=1}^{m} (t^{q+1}_i s^q_i) \right) \|x_1 - x_2\|_{PC} + \frac{L_f q}{\Gamma(q + 1)} t^q_1 \|x_1 - x_2\|_{PC} \\
\leq (\Theta_1 L_g + \Theta_2 L_f) \|x_1 - x_2\|_{PC}.
\]

Case 2. For \( x_1, x_2 \in PC(J, \mathbb{R}) \), and for \( t \in (t_i, s_i] \), we have

\[
|Fx_1 - Fx_2| = |g_i(t_1, x_1(t_1)) - g_i(t_1, x_2(t_1))| \leq L_g \|x_1 - x_2\|_{PC}.
\]

Case 3. For \( x_1, x_2 \in PC(J, \mathbb{R}) \), and for \( t \in (s_i, t_{i+1}] \), we have

\[
|Fx_1 - Fx_2| \\
\leq L_g ||x_1 - x_2||_{PC} + \frac{L_f q}{\Gamma(q + 1)} ||x_1 - x_2||_{PC} + \frac{L_f t^{q+1}}{\Gamma(q + 1)} ||x_1 - x_2||_{PC} \\
\leq \left( L_g + \frac{L_f q}{\Gamma(q + 1)} (t^{q+1}_i + s^q_i) \right) ||x_1 - x_2||_{PC}.
\]

From above, we obtain \( |Fx_1 - Fx_2| \leq \Delta ||x_1 - x_2||_{PC} \), which implies that \( F \) is a contractive mapping and there exists a unique solution \( x \in PC(J, \mathbb{R}) \) of the system (1.1).

\[\square\]

**Example 3.1.** Consider the following equation

\[
\begin{align*}
\frac{d}{dt} D_0^\frac{q}{2} x(t) &= \frac{|x(t)|}{9 + 6t}, & t \in (0, 1] \cup (2, 3], \\
x(t) &= \frac{|x(t)|}{5 + 4t}, & t \in (1, 2], \\
x(0) - 2 \int_0^3 x(s)ds &= x(3).
\end{align*}
\]

Where \( q = \frac{1}{2}, J = [0, 3], \mu = -2 \) and \( 0 = s_0 < t_1 = 1 < s_1 = 2 < t_2 = 3 \).

Set \( f(t, x(t)) = \frac{|x(t)|}{9 + 6t}, g_i(t, x(t)) = \frac{|x(t)|}{5 + 4t} \). Let \( x_1, x_2 \in \mathbb{R} \) and \( t \in (0, 1] \cup (2, 3] \).

Then we have \( |f(t, x_1) - f(t, x_2)| \leq \frac{1}{15} |x_1 - x_2| \). Let \( x_1, x_2 \in \mathbb{R} \) and \( t \in (1, 2] \).

Then, we have \( |g_1(t, x_1) - g_1(t, x_2)| \leq \frac{1}{9} |x_1 - x_2| \). Set \( L_f = \frac{1}{15}, L_g = \frac{1}{10} \).

Hence, one can deduce \( \Delta = \Theta_1 L_g + \Theta_2 L_f = \frac{1}{3} \left( \frac{1}{2} + \frac{6(\sqrt{3} + \sqrt{2}) + 8((\sqrt{3})^3 - (\sqrt{2})^3)^2 + 4 \sqrt{2}}{45 \sqrt{\pi}} \right) + \frac{2 \sqrt{2}}{15 \sqrt{\pi}} < 0.7634 < 1 \), or \( \Delta = L_g + \frac{L_f q}{\Gamma(q + 1)} (t^{q+1}_i + s^q_i) = \frac{3 \sqrt{\pi} + 4 \sqrt{3} + 4 \sqrt{2}}{30 \sqrt{\pi}} < 0.2245 < 1 \).

Thus all the assumptions of theorem 3.1 are satisfied, our results can be applied to the problem (3.1).
3.2 Existence of at least one solution

In this section we derive conditions under which system (1.1) admits at least one solution. For this purpose, we introduce the following assumption.

\((H_3)\) There exists an integrable function \(\psi : [s_i, t_{i+1}] \to \mathbb{R}_+\), and an integrable, monotonous function \(\nu : \mathbb{R}_+ \to \mathbb{R}_+\) such that \(|f(t, x)| \leq \psi(t)\nu(||x||_{PC})\), for each \(t \in [s_i, t_{i+1}]\), and \(x \in \mathbb{R}\).

\((H_4)\) There exists an integrable function \(\varphi_i : [t_i, s_i] \to \mathbb{R}_+\), and an integrable, monotonous function \(\phi : \mathbb{R}_+ \to \mathbb{R}_+\) such that \(|g_i(t, x)| \leq \varphi_i(t)\nu(||x||_{PC})\), for each \(t \in [t_i, s_i], i = 1, 2, \ldots, m\), and \(x \in \mathbb{R}\).

Let \(M_i = \int_{t_i}^{s_i} \varphi_i(s)ds = ||\varphi_i||_{L^1_{[t_i, s_i]}}\), \(M = \max_{i=1,2,\ldots,m} M_i\), \(N = \max_{i=1,2,\ldots,m} \varphi_i(s_i)\).

Denote \(\Theta_3 = \frac{\mu}{1+\mu t_1} \left( \sum_{i=0}^{m} (t_{i+1}^{q+\frac{1}{p}} - s_i^{q+\frac{1}{p}}) + \sum_{i=1}^{m} s_i^{q-1+\frac{1}{p}} (t_{i+1} - s_i) \right) + \frac{1}{1+\mu t_1} \left( s_m^{q-1+\frac{1}{p}} + T^{q-1+\frac{1}{p}} \right) + t_1^{q-1+\frac{1}{p}}\). Let \((q-1)p+1 > 0\) for some \(p > 1\), and \(K = \frac{\|\psi\|_{L^p_{[0,T]}}}{(p(q-1)+1)^\frac{1}{p} \Gamma(q)}\), where \(\|\psi\|_{L^p_{[0,T]}} = \left( \int_0^T \psi(t)^\frac{1}{p} dt \right)^\frac{1}{p}\).

Now we are ready to give the following result.

**Theorem 3.2.** Assume that \((H_2)\), \((H_3)\) and \((H_4)\) hold. If \(p(q-1)+1 > 0\) for some \(p > 1\), \(L_g < 1\), and \(r \geq \max \left\{ N\phi(r) + K\nu(r) (t_{i+1}^{q-1\frac{1}{p}} + s_i^{q-1\frac{1}{p}}), \frac{N+2m\mu M}{1+\mu t_1} \phi(r) + K\nu(r)\Theta_3 \right\}\) holds for some \(r > 0\), then the system (1.1) has at least one solution.

**Proof.** Setting \(B_r = \{ x \in PC(J, \mathbb{R}) : ||x||_{PC} \leq r \}\), where \(r \geq \max \left\{ N\phi(r) + K\nu(r) (t_{i+1}^{q-1\frac{1}{p}} + s_i^{q-1\frac{1}{p}}), \frac{N+2m\mu M}{1+\mu t_1} \phi(r) + K\nu(r)\Theta_3 \right\}\). We define the operators \(P\) and \(Q\) on \(B_r\) as follows:

\[
(Px)(t) = \begin{cases} 
\frac{g_m(s_m,x(s_m))-\mu \left( \sum_{i=1}^{m} \int_{t_i}^{s_i} g_i(s,x(s))ds + \sum_{i=1}^{m} \int_{t_i}^{t_i+1} g_i(s_i,x(s_i))ds \right)}{1+\mu t_1}, & t \in [0, t_1], \\
g_i(t, x(t)), & t \in (t_i, s_i], i = 1, 2, \ldots, m, \\
g_i(s_i, x(s_i)), & t \in [s_i, t_{i+1}], i = 1, 2, \ldots, m, 
\end{cases}
\]
and

\[
(Qx)(t) = \begin{cases} 
\frac{1}{\Gamma(q)} \left[ \frac{1}{\Gamma(q)} \int_0^T (T-s)^{q-1} f(s, x(s)) ds 
- \frac{1}{\Gamma(q)} \int_0^{s_m} (s_m - s)^{q-1} f(s, x(s)) ds 
- \mu \left( \frac{1}{\Gamma(q)} \int_{t_{i-1}}^{t_i} \int_0^s (s - \tau)^{q-1} f(\tau, x(\tau)) d\tau ds \right) 
- \frac{1}{\Gamma(q)} \sum_{i=1}^m \int_{s_i}^{t_{i+1}} \int_0^{s_i} (s_i - \tau)^{q-1} f(\tau, x(\tau)) d\tau ds \right) 
+ \frac{1}{\Gamma(q)} \sum_{i=1}^m \int_{s_i}^{t_{i+1}} \int_0^{s_i} (s - \tau)^{q-1} f(\tau, x(\tau)) d\tau ds \right) 
+ \frac{1}{\Gamma(q)} \int_{t_{i-1}}^{t_i} \int_0^{s_i} (s_i - s)^{q-1} f(s, x(s)) ds, \quad t \in [0, t_1], \\
0, \quad t \in (t_i, s_i], i = 1, 2, \ldots, m, \\
\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s)) ds 
- \frac{1}{\Gamma(q)} \int_0^{s_i} (s_i - s)^{q-1} f(s, x(s)) ds, \quad t \in (s_i, t_{i+1}], i = 1, 2, \ldots, m. 
\end{cases}
\]

For the sake of convenience, we divide the proof into several steps.
Step 1. For any \( x \in B_r \), we prove that \( Px + Qx \in B_r \).
Case 1. For each \( t \in [0, t_1] \), we have

\[
|(Px + Qx)(t)| 
\leq \frac{1}{1 + \mu t_1} \left| g_m(s_m, x(s_m)) \right| 
+ \frac{1}{\Gamma(q)} \int_0^{s_m} (s_m - s)^{q-1} |f(s, x(s))| ds 
+ \frac{1}{\Gamma(q)} \int_0^T (T-s)^{q-1} |f(s, x(s))| ds 
+ \mu \left( \frac{1}{\Gamma(q)} \int_{t_{i-1}}^{t_i} \int_0^s (s - \tau)^{q-1} |f(\tau, x(\tau))| d\tau ds \right) 
+ \sum_{i=1}^m \int_{t_i}^{t_{i+1}} |g_i(s, x(s))| ds + \sum_{i=1}^m \int_{s_i}^{t_{i+1}} |g_i(s, x(s))| ds 
+ \frac{1}{\Gamma(q)} \sum_{i=1}^m \int_{s_i}^{t_{i+1}} \int_0^{s_i} (s_i - \tau)^{q-1} |f(s, x(s))| d\tau ds 
+ \frac{1}{\Gamma(q)} \sum_{i=1}^m \int_{s_i}^{t_{i+1}} \int_0^{s_i} (s - \tau)^{q-1} |f(\tau, x(\tau))| d\tau ds \right) 
+ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s, x(s))| ds 
\]
By using Theorem 2.7, we have

\[
\left| (P x + Q x)(t) \right| \\
\leq \frac{1}{1 + \mu t_1} \left( \varphi_m(s_m) \phi(r) + K \nu(r) s_m^{q-1 + \frac{1}{p}} + K \nu(r) T^{q-1 + \frac{1}{p}} \right) \\
+ \mu \left( 2 m M \phi(r) + \frac{p K \nu(r)}{pq + 1} t_1^{q + \frac{1}{p}} + K \nu(r) \sum_{i=1}^{m} s_i^{q-1 + \frac{1}{p}} (t_{i+1} - s_i) \right) \\
+ \frac{p K \nu(r)}{pq + 1} \sum_{i=0}^{m} \left( t_{i+1}^{q + \frac{1}{p}} - s_i^{q + \frac{1}{p}} \right) ) \right) + K \nu(r) t_1^{q-1 + \frac{1}{p}} \\
\leq \frac{1}{1 + \mu t_1} \left( \varphi_m(s_m) \phi(r) + K \nu(r) s_m^{q-1 + \frac{1}{p}} + K \nu(r) T^{q-1 + \frac{1}{p}} \right) \\
+ \mu \left( 2 m M \phi(r) + K \nu(r) \sum_{i=1}^{m} s_i^{q-1 + \frac{1}{p}} (t_{i+1} - s_i) \right) \\
+ \frac{p K \nu(r)}{pq + 1} \sum_{i=1}^{m} \left( t_{i+1}^{q + \frac{1}{p}} - s_i^{q + \frac{1}{p}} \right) ) \right) + K \nu(r) t_1^{q-1 + \frac{1}{p}} \\
\leq \frac{\phi(r)}{1 + \mu t_1} \left( N + 2 m \mu M \right) + K \nu(r) \Theta_3 \leq r.
\]

Case 2. For each \( t \in (t_i, s_i) \), we have

\[
\left| (P x + Q x)(t) \right| = \left| g_i(t, x(t)) \right| \leq \varphi_m(s_m) \mu(r) \leq r.
\]

Case 3. For each \( t \in (s_i, t_{i+1}) \), we have

\[
\left| (P x + Q x)(t) \right| \\
\leq \left| g_i(s_i, x(s_i)) \right| + \frac{1}{\Gamma(q)} \int_{0}^{s_i} (s_i - s)^{q-1} |f(s, x(s))| ds \\
+ \frac{1}{\Gamma(q)} \int_{0}^{t} (t - s)^{q-1} |f(s, x(s))| ds \\
\leq N \phi(r) + K \nu(r) (t_{i+1}^{q-1 + \frac{1}{p}} + s_i^{q-1 + \frac{1}{p}}) \leq r.
\]

So, we infer that \( P x + Q x \in B_r \).

Step 2. \( P \) is contractive mapping on \( B_r \).

Case 1. For \( x_1, x_2 \in B_r \), and for \( t \in [0, t_1] \), we have

\[
\left| (P x_1 - P x_2)(t) \right| \leq \frac{L_g + \mu L_g (T - t_1)}{1 + \mu t_1} ||x_1 - x_2||_{PC}.
\]

Case 2. For \( x_1, x_2 \in B_r \), and for \( t \in (t_i, s_i], i = 1, 2, ..., m \), we have

\[
\left| (P x_1 - P x_2)(t) \right| \leq L_g ||x_1 - x_2||_{PC}.
\]
Case 3. For \( x_1, x_2 \in B_r \), and for \( t \in (s_i, t_{i+1}] \), \( i = 1, 2, \ldots, m \), we have
\[
|(Px_1 - Px_2)(t)| \leq L_g ||x_1 - x_2||_{PC}.
\]
So, we can obtain \( |Px_1 - Px_2|_{PC} \leq L_g |x_1 - x_2|_{PC} \), which implies that \( P \) is a contractive.

Step 3. We show that \( Q \) is continuous.

Let \( \{x_n\} \) be a sequence such that \( x_n \to x \) in \( PC(J, \mathbb{R}) \).

Case 1. For each \( t \in (0, t_1] \), we have
\[
|(Qx_n)(t) - (Qx)(t)| \leq \int_{0}^{t} \left| \frac{T^q}{\Gamma(q + 1)} + \frac{s^q_m}{\Gamma(q + 1)} + \frac{\mu}{\Gamma(q + 2)} \sum_{i=0}^{m} (t_{i+1}^{q+1} - s_{i}^{q+1}) \right| ||f(\cdot, x_n(\cdot)) - f(\cdot, x(\cdot))|| \, ds.
\]

Case 2. For each \( t \in (t_i, s_i] \), and \( a < b \), \( x \in B_r \), we have
\[
|(Qx_n)(t) - (Qx)(t)| = 0.
\]

Case 3. For each \( t \in (s_i, t_{i+1}] \), we have
\[
|(Qx_n)(t) - (Qx)(t)| \leq \frac{t_{i+1}^{q} + s_{i}^{q}}{\Gamma(q + 1)} ||f(\cdot, x_n(\cdot)) - f(\cdot, x(\cdot))||.
\]

Since the function \( f \) is defined on finite dimensional spaces, from the continuity of the function \( f \) and the last inequality, we infer that \( \|Qx_n - Qx\| \to 0 \) \((n \to \infty)\).

Step 4. We show that \( Q \) is compact.

First, \( Q \) is uniformly bounded on \( B_r \).

Next, we show that \( Q \) maps bounded set into equicontinuous set of \( B_r \).

Case 1. For \( a, b \in [0, t_1] \), and \( a < b \), \( x \in B_r \), we have
\[
|(Qx)(a) - (Qx)(b)| \leq \int_{0}^{b} \left| (b - s)^{q-1} - (a - s)^{q-1} \right| ||f(s, x(s))|| \, ds
\]
\[
+ \int_{a}^{b} (b - s)^{q-1} ||f(s, x(s))|| \, ds
\]
\[
\leq \frac{3K\nu(r)}{(p(q - 1) + 1)^{\frac{1}{2}}} (b - a)^{q - 1 + \frac{1}{p}}.
\]

Case 2. For \( a, b \in (t_i, s_i] \), and \( a < b \), \( x \in B_r \), we have
\[
|(Qx)(a) - (Qx)(b)| = 0.
\]
Case 3. For \(a, b \in (s_i, t_{i+1}]\), and \(a < b\), \(x \in B_r\), we have

\[
|(Qx)(a) - (Qx)(b)| \leq \frac{3K\nu (r)}{(p(q - 1) + 1)\Gamma(q)}(b - a)^{q-1+\frac{1}{p}}.
\]

From above, we get \(|(Qx)(a) - (Qx)(b)| \to 0 (a \to b)\), so \(Q\) is equicontinuous. Now we know that \(Q : B_r \to B_r\) is continuous and compact. One can apply Theorem 2.8, \(F = P + Q\), which derive existence of solution of the system (1.1). This completes the proof. 

\(\Box\)

**Example 3.2.** Consider the following equation

\[
\begin{cases}
\epsilon D^{\frac{2}{3}}_{0,t} x(t) = \frac{|x(t)|}{9}, & t \in (0, 1] \cup (2, 3], \\
x(t) = \frac{|x(t)|}{9}, & t \in (1, 2], \\
x(0) - 2 \int_0^3 x(s)ds = x(3).
\end{cases}
\]

Where \(q = \frac{2}{3}\), \(\mu = -2\), \(J = [0, 3]\) and \(0 = s_0 < t_1 = 1 < s_1 = 2 < t_2 = 3\).

Set \(f(t, x(t)) = g_1(t, x(t)) = \frac{|x(t)|}{9}\). Let \(x_1, x_2 \in \mathbb{R}\) and \(t \in (0, 1] \cup (2, 3]\), the we have \(|f(t, x_1) - f(t, x_2)| \leq \frac{1}{9} |x_1 - x_2|\). Let \(x_1, x_2 \in \mathbb{R}\) and \(t \in (1, 2]\), the we have \(|g_1(t, x_1) - g_1(t, x_2)| \leq \frac{1}{9} |x_1 - x_2|\), \(L_g = L_{g_1} = \frac{1}{9} < 1\).

For all \(x \in \mathbb{R}\) and each \(t \in (0, 1] \cup (2, 3]\), we have \(|f(t, x)| \leq \frac{1}{9}||x||_{PC}\), \(\psi(t) = \frac{1}{9}\), \(\nu(t) = t\). For all \(x \in \mathbb{R}\) and each \(t \in (1, 2]\), we have \(|g_1(t, x)| \leq \frac{1}{9}||x||_{PC}\), \(N = \varphi_1(t) = \frac{1}{9}\), \(\phi(t) = t\). Moreover, \(M = M_1 = \int_1^2 \varphi_1(t)dt = \frac{1}{3}\).

Next, \(q = \frac{2}{3} p + 2\), then we have \((q-1)p+1 = \frac{1}{3} > 0\), \(K = \frac{1}{\sqrt{\frac{4}{3}\Gamma(\frac{1}{3})}} \simeq 0.1421\), \(\Theta_3 \simeq 4.258\), \(t_{i+1}^{q-1+\frac{1}{p}} + s_i^{q-1+\frac{1}{p}} \simeq 2.3234\). Obviously, we have inequality \(r > \max\{N\phi(r) + K\nu(r)(t_{i+1}^{q-1+\frac{1}{p}} + s_i^{q-1+\frac{1}{p}}), \frac{N+2\mu M}{\mu_1+\mu_1}\phi(r) + K\nu(r)\Theta_3\} \simeq 0.9557r\), which holds for every \(r > 0\).

Thus all the assumptions of theorem 3.2 are satisfied, our results can be applied to the problem (3.2).

**References**


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