Abstract

In this paper we solve additive $\beta$-functional inequalities with three variables and their Hyers-Ulam stability in non-Archimedean Banach spaces as well as in complex Banach spaces. It is shown that the solutions of first and second inequalities are additive mappings. Then Hyers-Ulam stability of these inequalities is studied and proven.

Keywords: additive $\beta$-functional equation; additive $\beta$-functional inequality; non-Archimedean normed space; complex Banach space; Hyers-Ulam stability

1. Introduction

Let $X$ and $Y$ be a normed spaces on the same field $\mathbb{K}$, and $f : X \rightarrow Y$. We use the notation $\| \cdot \|$ for all the norm on both $X$ and $Y$. In this paper, we investigate some additive $\beta$-functional inequalities when $X$ is non-Archimedean normed space and $Y$ is non-Archimedean Banach space, or $X$ is complex normed space and $Y$ is complex Banach space.
In fact, when $X$ is non-Archimedean normed space and $Y$ is non-Archimedean Banach space we solve and prove the Hyers-Ulam stability of following additive $\beta$-functional inequality. Let $|2| \neq 1$ and let $\beta$ be a non-Archimedean number $|\beta| < 1$.

\[
\left\| f\left( \frac{x+y}{2} \right) - f\left( \frac{x+y}{2} \right) - f(z) \right\| \leq \beta \left( 2f\left( \frac{x+y+z}{2} \right) - f\left( \frac{x+y}{2} \right) - f(z) \right) \\
(1.1)
\]

\[
\left\| 2f\left( \frac{x+y+z}{2} \right) - f\left( \frac{x+y}{2} \right) - f(z) \right\| \\
\leq \beta \left( f\left( \frac{x+y+z}{2} \right) - f\left( \frac{x+y}{2} \right) - f(z) \right) \\
(1.2)
\]

when $X$ is complex normed space and $Y$ is complex Banach spaces we solve and prove the Hyers-Ulam stability of following additive $\beta$-functional inequality. Let $\beta$ be a complex number with $|\beta| < 1$.

\[
\left\| f\left( \frac{x+y}{2} \right) - f\left( \frac{x+y}{2} \right) - f(z) \right\| \leq \beta \left( 2f\left( \frac{x+y+z}{2} \right) - f\left( \frac{x+y}{2} \right) - f(z) \right) \\
(1.3)
\]

\[
\left\| 2f\left( \frac{x+y+z}{2} \right) - f\left( \frac{x+y}{2} \right) - f(z) \right\| \\
\leq \beta \left( f\left( \frac{x+y+z}{2} \right) - f\left( \frac{x+y}{2} \right) - f(z) \right) \\
(1.4)
\]

The notions of non-Archimedean normed space and complex normed spaces will be reminded in the next section. The Hyers-Ulam stability was first investigated for functional equation of Ulam in [6] concerning the stability of group homomorphisms.

The functional equation

\[ f(x+y) = f(x) + f(y) \]

is called the Cauchy equation. In particular, every solution of the Cauchy equation is said to be an additive mapping.

The Hyers [7] gave first affirmative partial answer to the equation of Ulam in Banach spaces. After that, Hyers’ Theorem was generalized by Aoki [1] for additive mappings and by Rassias [8] for linear mappings considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Gavruta [4] by replacing the unbounded Cauchy difference by a general control function in the spirit
of Rassias’ approach.

\[ f\left(\frac{x+y}{2}\right) = \frac{1}{2}f(x) + \frac{1}{2}f(y) \]

is called the *Jensen equation*. See [2, 3, 9, 10] for more information on functional equations.

The Hyers-Ulam stability for functional inequalities have been investigated such as in [14, 15]. Gilany showed that if satisfies the functional inequality

\[ \|2f(x) + 2f(y) - f(xy^{-1})\| \leq \|f(xy)\| \]

Then \( f \) satisfies the Jordan-von Newman functional equation

\[ 2f(x) + 2f(y) - f(xy^{-1}) \] (1.5)


Choonkil Park [17] proved the Hyers-Ulam stability of additive \( \beta \)-functional inequalities. Recently, in [18, 19, 20] the authors studied the Hyers-Ulam stability for the following functional inequalities

\[ \|f(x+y) - f(x) - f(y)\| \leq \rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right) \] (1.6)

\[ \|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\| \leq \rho(f(x+y) - f(x) - f(y)) \] (1.7)

in Non-Archimedean Banach spaces and complex Banach spaces.

In this paper, we solve and prove the Hyers-Ulam stability for two \( \beta \)-functional inequalities (1.1)-(1.2), i.e. the \( \beta \)-functional inequalities with three variables [4, 13,14]. Under suitable assumptions on spaces \( X \) and \( Y \), we will prove that the mappings satisfying the \( \beta \)-functional inequalities (1.1) or (1.2). Thus, the results in this paper are generalization of those in [17] for \( \beta \)-functional inequalities with three variables.

The paper is organized as follows: In section preliminaries we remind some basic notations in [18, 19, 20] such as Non-Archimedean field, Non-Archimedean normed space and Non-Archimedean Banach space.

Section 3 is devoted to prove the Hyers-Ulam stability of the additive \( \beta \)-functional inequalities (1.1) and (1.2) when \( X \) Non-Archimedean normed space and \( Y \) Non-Archimedean Banach space.

Section 4 is devoted to prove the Hyers-Ulam stability of the additive \( \beta \)-functional inequalities (1.1) and (1.2) when \( X \) is complex normed space and \( Y \) is complex Banach space.
2. PRELIMINARIES

2.1. Non-Archimedean normed and Banach spaces. In this subsection we recall some basic notations [19, 20] such as Non-Archimedean fields, Non-Archimedean normed spaces and Non-Archimedean normed spaces.

A valuation is a function \(| \cdot |\) from a field \(K\) into \([0, \infty)\) such that 0 is the unique element having the 0 valuation,

\[
|r.s| = |r|.|s|, \forall r, s \in K
\]

and the triangle inequality holds, ie;

\[
|r + s| \leq |r| + |s|, \forall r, s \in K
\]

A field \(K\) is called a valued field if \(K\) carries a valuation. The usual absolute values of \(\mathbb{R}\) and \(\mathbb{C}\) are examples of valuation. Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the triangle inequality is replaced by

\[
|r + s| \leq \max\{|r|, |s|\}, \forall r, s \in K
\]

then the function \(| \cdot |\) is called a norm -Archimedean valuation, and field. Clearly \(|1| = |-1| = 1\) and \(|n| \leq 1, \forall n \in \mathbb{N}\). A trivial example of a non- Archimedean valuation is the function \(| \cdot |\) taking everything except for 0 into 1 and \(|0| = 0\) this paper, we assume that the base field is a non-Archimedean field, hence call it simply a field.

Definition 2.1.

Let be a vector space over a field \(K\) with a non-Archimedean \(| \cdot |\). A function \(\| \cdot \| : X \to [0, \infty)\) is said a non-Archimedean norm if it satisfies the following conditions:

1. \(\|x\| = 0\) if and only if \(x = 0\);
2. \(\|rx\| = |r|\|x\| (r \in K, x \in X)\);
3. the strong triangle inequality

\[
\|x + y\| \leq \max\{\|x\|, \|y\|\}, x, y \in X
\]

hold. Then \((X,\| \cdot \|)\) is called a norm -Archimedean norm space.
Definition 2.2.

(1) Let \( \{x_n\} \) be a sequence in a norm-Archimedean normed space \( X \). Then sequence \( \{x_n\} \) is called Cauchy if for a given \( \epsilon > 0 \) there a positive integer \( N \) such that
\[
\|x_n - x\| \leq \epsilon
\]
for all \( n, m \geq N \).

(2) Let \( \{x_n\} \) be a sequence in a norm-Archimedean normed space \( X \). Then sequence \( \{x_n\} \) is called cauchy if for a given \( \epsilon > 0 \) there a positive integer \( N \) such that
\[
\|x_n - x\| \leq \epsilon
\]
for all \( n, m \geq N \). The we call \( x \in X \) a limit of sequence \( x_n \) and denote
\[
\lim_{n \to \infty} x_n = x.
\]

(3) If every sequence Cauchy in \( X \) converger, then the norm-Archimedean normed space \( X \) is called a norm-Archimedean Banach space.

2.2. Solutions of the inequalities. The functional equation
\[
f(x + y) = f(x) + f(y)
\]
is called the cauchuy equation. In particular, every solution of the cauchuy equation is said to be an additive mapping.

3. Additive \( \beta \)-functional inequality in Non-Archimedean Banach space

Now, we first study the solutions of (1.1) and (1.2). Note that for these inequalities, \( X \) is non-Archimedean normed space and \( Y \) is non-Archimedean Banach spaces. Under this setting, we can show that the mapping satisfying (1.1) and (1.2) is additive. These results are given in the following.

Lemma 3.1. A mapping \( f:X \to Y \) saties
\[
\left\| f\left( \frac{x + y}{2} + z \right) - f\left( \frac{x + y}{2} \right) - f(z) \right\| \leq \beta \left( 2f\left( \frac{x + y}{2} + z \right) - f\left( \frac{x + y}{2} \right) \right)
\]
for all \( x, y, z \in X \) if and only if \( f:X \to Y \) is additive.

\[
\left(3.1\right)
\]
Proof. Assume that \( f : X \to Y \) satisfies (3.1)
Letting \( x = y = z = 0 \) in (3.1), we get \( \|f(0)\| \leq 0 \). So \( f(0) = 0 \)
Letting \( x = y = z \) in (3.1), we get \( \|f(2x) - 2f(x)\| \leq 0 \) and so \( f(2x) = 2f(x) \) for all \( x \in X \).
Thus
\[
\frac{f(x)}{2} = \frac{1}{2} f(x)
\tag{3.2}
\]
for all \( x \in X \) It follows from (3.1) and (3.2) that:
\[
\left\| f \left( \frac{x + y + z}{2} \right) - f \left( \frac{x + y}{2} \right) - f(z) \right\| \leq \beta \left\| 2f \left( \frac{x + y + z}{2} \right) - f \left( \frac{x + y}{2} \right) - f(z) \right\|
\]
\[
= \beta \left\| 2f \left( \frac{x + y}{2} \right) - f \left( \frac{x + y}{2} \right) - f(z) \right\|
\]
\[
\leq |\beta| \left\| f \left( \frac{x + y}{2} + z \right) - f \left( \frac{x + y}{2} \right) - f(z) \right\|
\]
and so
\[
f \left( \frac{x + y + z}{2} \right) = f \left( \frac{x + y}{2} \right) + f(z)
\]
for all \( x, y, z \in X \) The converse is obviously true. \( \square \)

Lemma 3.2. A mapping \( f : X \to Y \) satisfy \( f(0) = 0 \) and
\[
\left\| 2f \left( \frac{x + y + z}{2} \right) - f \left( \frac{x + y}{2} \right) - f(z) \right\| \leq \beta \left\| f \left( \frac{x + y + z}{2} \right) - f \left( \frac{x + y}{2} \right) - f(z) \right\|
\tag{3.3}
\]
for all \( x, y, z \in X \) if and if \( f : X \to Y \) is additive.

Proof. Assume that \( f : X \to Y \) (3.3).
Letting \( x = y = 0 \) in (3.3), we get
\[
\left\| 2f \left( \frac{z}{2} \right) - f(z) \right\| \leq 0
\tag{3.4}
\]
and so
\[
f \left( \frac{z}{2} \right) = \frac{1}{2} f(z)
\]
It follows from (3.3) and (3.4) that
\[
\left\| f \left( \frac{x + y + z}{2} \right) - f \left( \frac{x + y}{2} \right) - f(z) \right\| = \beta \left\| 2f \left( \frac{x + y + z}{2} \right) - f \left( \frac{x + y}{2} \right) - f(z) \right\|
\]
\[
= \beta \left\| 2f \left( \frac{x + y}{2} \right) - f \left( \frac{x + y}{2} \right) - f(z) \right\|
\]
\[
\leq |\beta| \left\| f \left( \frac{x + y}{2} + z \right) - f \left( \frac{x + y}{2} \right) - f(z) \right\|
\]
and so.
\[ f \left( \frac{x+y}{2} + z \right) = f \left( \frac{x+y}{2} \right) + f(z) \]
for all \( x, y, z \in X \). The converse is obviously true.

\[ \square \]

**Theorem 3.3.** Let \( \varphi : X^3 \to [0, \infty) \) be a function and let \( f : X \to Y \) be a mapping such that
\[
\psi(x, y, z) := \sum_{j=1}^{\infty} |2|^j \psi \left( \frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j} \right) < \infty \quad (3.5)
\]
\[
\left\| f \left( \frac{x+y}{2} + z \right) - f \left( \frac{x+y}{2} \right) - f(z) \right\| 
\leq \left\| \beta \left( 2f \left( \frac{x+y}{2^2} + \frac{z}{2} \right) - f \left( \frac{x+y}{2} \right) - f(z) \right) \right\| + \varphi(x, y, z) \quad (3.6)
\]
for all \( x, y, z \in X \).

Then there exists a unique mapping \( h : X \to Y \) such that
\[
\left\| f(x) - h(x) \right\| \leq \frac{1}{|2|} \psi(x, x, x) \quad (3.7)
\]
for all \( x \in X \).

**Proof.** Letting \( x = y = z \) in (3.6), we get
\[
\left\| f(2x) - 2f(x) \right\| \leq \varphi(x, x, x) \quad (3.8)
\]
for all \( x \in X \).

So
\[
\left\| f(x) - 2f \left( \frac{x}{2} \right) \right\| \leq \varphi \left( \frac{x}{2}, \frac{x}{2}, \frac{x}{2} \right)
\]
for all \( x \in X \).

Hence
\[
\left\| 2^l f \left( \frac{x}{2^l} \right) - 2^m f \left( \frac{x}{2^m} \right) \right\|
\leq \max \left\{ \left\| 2^l f \left( \frac{x}{2^l} \right) - 2^{l+1} f \left( \frac{x}{2^{l+1}} \right) \right\|, \ldots, \left\| 2^{m-1} f \left( \frac{x}{2^{m-1}} \right) - 2^m f \left( \frac{x}{2^m} \right) \right\| \right\}
\]
\[
= \max \left\{ \left\| 2^l \right\| \left\| f \left( \frac{x}{2^l} \right) - 2 f \left( \frac{x}{2^{l+1}} \right) \right\|, \ldots, \left\| 2^{m-1} \right\| \left\| f \left( \frac{x}{2^{m-1}} \right) - 2 f \left( \frac{x}{2^m} \right) \right\| \right\}
\]
\[
\leq \sum_{j=l}^{\infty} \left\| 2^j \right\| \varphi \left( \frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, \frac{x}{2^{j+1}} \right) \quad (3.9)
\]
for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in X \). It follows from (3.9) that the sequence \( \{ 2^n f \left( \frac{x}{2^n} \right) \} \) is a cauchy sequence for all \( x \in X \). Since \( Y \) is a
Non-Archimedean Banach space, the sequence \( \{2^n f\left(\frac{x}{2^n}\right)\} \) converges. So one can define the mapping \( h : X \to Y \) by

\[
h(x) := \lim_{n \to \infty} 2^n f\left(\frac{x}{2^n}\right)
\]

for all \( x \in X \). Moreover, letting \( l = 0 \) and passing the limit \( m \to \infty \) in (3.9), we get (3.7).

Now, let \( T : X \to Y \) be another additive mapping satisfy (3.7) then we have

\[
\|h(x) - T(x)\| = \left\| 2^n h\left(\frac{x}{2^n}\right) - 2^n T\left(\frac{x}{2^n}\right) \right\|
\leq \max\left\{ \left\| 2^n h\left(\frac{x}{2^n}\right) - 2^n f\left(\frac{x}{2^n}\right) \right\|, \left\| 2^n T\left(\frac{x}{2^n}\right) - 2^n f\left(\frac{x}{2^n}\right) \right\| \right\}
\leq |2|^{q-1} \psi\left(\frac{x}{2^{q-1}}, \frac{x}{2^{q-1}}, \frac{x}{2^{q-1}}\right)
\]

which tends to zero as \( q \to \infty \) for all \( x \in X \). So we can conclude that \( h(x) = T(x) \) for all \( x \in X \). The proves the uniqueness of \( h \). It follows from (3.5) and (3.6) that

\[
\left\| h\left(\frac{x+y+z}{2}\right) - h\left(\frac{x+y}{2}\right) - h(z) \right\| = \lim_{n \to \infty} \left\| 2^n \left( f\left(\frac{x+y+z}{2^{n+1}}\right) - f\left(\frac{x+y}{2^n}\right) - f\left(\frac{z}{2^n}\right) \right) \right\|
\leq \lim_{n \to \infty} \left\| 2^n \beta\left( f\left(\frac{x+y+z}{2^{n+2}}\right) - f\left(\frac{x+y}{2^{n+1}}\right) - f\left(\frac{z}{2^n}\right) \right) \right\|
\leq \lim_{n \to \infty} \left\| 2^n \psi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) \right\|
\]

for all \( x, y, z \in X \).

\[
\left\| h\left(\frac{x+y+z}{2}\right) - h\left(\frac{x+y}{2}\right) - h(z) \right\| \leq \beta\left( 2h\left(\frac{x+y+z}{2}\right) - h\left(\frac{x+y}{2}\right) - h(z) \right)
\]

for all \( x, y \in X \). By lemma 3.1, the mapping \( h : X \to Y \) is additive. \(\square\)

**Theorem 3.4.**

Let \( \varphi : X \to [0, \infty) \) be a function and let \( f : X \to Y \) be a mapping satisfying

\[
\psi(x, y, z) := \sum_{j=0}^{\infty} \frac{1}{|2|^j} \varphi(2^j x, 2^j y, 2^j z) < \infty
\]  

(3.12)
and
\[
\left\| f\left( \frac{x+y}{2} + z \right) - f\left( \frac{x+y}{2} \right) - f(z) \right\| \\
\leq \left\| \beta \left( 2f\left( \frac{x+y}{2^2} + \frac{z}{2} \right) - f\left( \frac{x+y}{2} \right) - f(z) \right) \right\| + \varphi(x, y, z)
\]
\hspace{1cm} (3.13)

for all \( x, y, z \in X \). Then there exists a unique additive mapping \( h : X \rightarrow Y \) such that
\[
\left\| f(x) - h(x) \right\| \leq \frac{1}{|2|} \psi(x, x, x)
\]
\hspace{1cm} (3.14)

for all \( x \in X \).

**Proof.** Letting \( x = y = z \) in (3.13), we get
\[
\left\| f(x) - \frac{1}{2} f(2x) \right\| \leq \frac{1}{|2|} \psi(x, x, x)
\]
for all \( x \in X \).

Hence
\[
\left\| \frac{1}{2^l} f(2^l x) - \frac{1}{2^m} f(2^m x) \right\|
\leq \max \left\{ \left\| \frac{1}{2^l} f(2^l x) - \frac{1}{2^{l+1}} f(2^{l+1} x) \right\|, \ldots, \left\| \frac{1}{2^{m-1}} f(2^{m-1} x) - \frac{1}{2^m} f(2^m x) \right\| \right\}
\]
\[
= \max \left\{ \frac{1}{|2|^l} \left\| f(2^l x) - \frac{1}{2} f(2^{l+1} x) \right\|, \ldots, \frac{1}{|2|^{m-1}} \left\| f(2^{m-1} x) - \frac{1}{2} f(2^m x) \right\| \right\}
\]
\[
\leq \sum_{j=1}^{\infty} \frac{1}{|2|^{j+1}} \varphi(2^j x, 2^j y, 2^j z)
\]
\hspace{1cm} (3.15)

for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in X \). It follows (3.15) that the sequence \( \left\{ \frac{1}{2^n} f(2^n x) \right\} \) is sequence for all \( x \in X \). Since \( Y \) is complete, the sequence \( \left\{ \frac{1}{2^n} f(2^n x) \right\} \) converges so one can define the mapping \( h : X \rightarrow Y \) by
\[
h(x) := \lim_{n \to \infty} \frac{1}{2^n} f(2^n x)
\]
for all \( x \in X \). Moreover, letting \( l = 0 \) and passing the limit \( m \to \infty \) in (3.15), we get (3.14). the rest of the proof is similar to the proof of the Theorem 3.3.

\[\square\]
Theorem 3.5.

Let $\varphi : X^3 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\psi(x, y, z) := \sum_{j=1}^{\infty} 2^j \psi \left( \frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j} \right) < \infty \quad (3.16)$$

$$\left\| 2f \left( \frac{x+y}{2^l} + \frac{z}{2^l} \right) - f \left( \frac{x+y}{2^l} \right) - f(z) \right\|
\leq \left\| \beta \left( f \left( \frac{x+y}{2^l} + z \right) - f \left( \frac{x+y}{2^l} \right) - f(z) \right) \right\| + \varphi(x, y, z) \quad (3.17)$$

for all $x, y, z \in X$. Then there exists a unique mapping $h : X \to Y$ such that

$$\| f(z) - h(z) \| \leq \psi(0, 0, z) \quad (3.18)$$

for all $z \in X$.

Proof. Letting $x = y = 0$ in (3.17)

$$\left\| 2f \left( \frac{z}{2^l} \right) - f(z) \right\| \leq \varphi(0, 0, z) \quad (3.19)$$

for all $x \in X$.

So

$$\left\| 2^l f \left( \frac{z}{2^l} \right) - 2^m f \left( \frac{z}{2^m} \right) \right\|
\leq \max\left\{ \left\| 2^l f \left( \frac{z}{2^l} \right) - 2^{l+1} f \left( \frac{z}{2^{l+1}} \right) \right\|, \ldots, \left\| 2^{m-1} f \left( \frac{z}{2^{m-1}} \right) - 2^m f \left( \frac{z}{2^m} \right) \right\| \right\}
= \max\left\{ |2|^l \left\| f \left( \frac{z}{2^l} \right) - 2 f \left( \frac{z}{2^{l+1}} \right) \right\|, \ldots, |2|^{m-1} \left\| f \left( \frac{z}{2^{m-1}} \right) - 2 f \left( \frac{z}{2^m} \right) \right\| \right\}
\leq \sum_{j=l+1}^{\infty} |2|^{j+l} \varphi(0, 0, \frac{z}{2^j}) < \infty \quad (3.20)$$

for all nonnegatives integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (3.20) that the sequence $\{2^k f \left( \frac{z}{2^k} \right) \}$ is Cauchy for all $x \in X$. Since $Y$ is a non-Archimedean Banach space, the sequence, $\{2^k f \left( \frac{z}{2^k} \right) \}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(z) := \lim_{k \to \infty} 2^k f \left( \frac{z}{2^k} \right)$$

for all $z \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.20), we get (3.18).
The rest of the proof is similar to the proof of the theorem (3.3).

\[ \square \]

**Theorem 3.6.**

Let \( \varphi : X^3 \to [0, \infty) \) be a function and let \( f : X \to Y \) be a mapping satisfying \( f(0) = 0 \) and

\[ \psi(x, y, z) := \sum_{j=1}^{\infty} \frac{1}{2^j} \varphi(2^j x, 2^j y 2^j z) < \infty; \quad (3.21) \]

\[ \left\| 2f \left( \frac{x+y+z}{2} \right) - f \left( \frac{x+y}{2} \right) - f(z) \right\| \leq \left\| \beta \left( f \left( \frac{x+y+z}{2} \right) - f \left( \frac{x+y}{2} \right) - f(z) \right) \right\| + \varphi(x, y, z) \quad (3.22) \]

for all \( x, y, z \in X \). Then there exists a unique mapping \( h : X \to Y \) such that

\[ \left\| f(x) - h(x) \right\| \leq \psi(0, 0, z) \quad (3.23) \]

for all \( z \in X \).

**Proof.** Letting \( x = y = 0 \) in (3.22)

\[ \left\| 2f \left( \frac{z}{2} \right) - f(z) \right\| \leq \varphi(0, 0, z) \quad (3.24) \]

for all \( z \in X \).

So

\[ \left\| f(z) - \frac{1}{2} f(2z) \right\| \leq \frac{1}{2} \varphi(0, 0, 2z) \]

for all \( z \in X \). Hence

\[ \left\| \frac{1}{2^l} f(2^l z) - \frac{1}{2^m} f(2^m z) \right\| \]

\[ \leq \max \left\{ \left\| \frac{1}{2^l} f(2^l z) - \frac{1}{2^{l+1}} f(2^{l+1} z) \right\|, \ldots, \left\| \frac{1}{2^{m-1}} f(2^{m-1} z) - \frac{1}{2^m} f(2^m z) \right\| \right\} \]

\[ = \max \left\{ \left\| \frac{1}{2^l} f(2^l z) - \frac{1}{2^{l+1}} f(2^{l+1} z) \right\|, \ldots, \left\| \frac{1}{2^{m-1}} f(2^{m-1} z) - \frac{1}{2^m} f(2^m z) \right\| \right\} \]

\[ \leq \sum_{j=l+1}^{\infty} \frac{1}{2^j} \varphi(0, 0, 2^j z) < \infty \quad (3.25) \]
for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( z \in X \). It follows from (3.25) that the sequence \( \{ 2^n f \left( \frac{z}{2^n} \right) \} \) is a cauchy sequence for all \( z \in X \). Since is complete, the sequence \( \{ 2^n f \left( \frac{z}{2^n} \right) \} \) coverges. So one can define the mapping \( h : X \to Y \) by
\[
h(z) := \lim_{n \to \infty} \frac{1}{2^n} f(2^n z)
\]
for all \( z \in X \). Moreover, letting \( l = 0 \) and passing the limit \( m \to \infty \) in (3.25), we get (3.24).

The rest of the proof is similar to the proof of theorem 3.3.

\[\square\]

**Corollary 3.7.**

let \( r < 1 \) and \( \theta \) be nonnegative real numbers, and let \( f : X \to Y \) be a mapping such that
\[
\left\| f \left( \frac{x+y}{2} + z \right) - f \left( \frac{x+y}{2} \right) - f(z) \right\|
\leq \left\| \beta \left( 2f \left( \frac{x+y}{2^2} + \frac{z}{2} \right) - f \left( \frac{x+y}{2} \right) - f(z) \right) \right\|
+ \theta (\|x\|^r + \|y\|^r + \|z\|^r)
\]
for all \( x, y, z \in X \).
Then there exits a unique additive maaping \( h : X \to Y \) such that
\[
\left\| f(x) - h(x) \right\| \leq \frac{2\theta}{|2^r - |2|^r|} \|x\|^r
\]
for all \( x \in X \)

**Corollary 3.8.**

let \( r > 1 \) and \( \theta \) be nonnegative real numbers, and let \( f : X \to Y \) be a mapping such that
\[
\left\| f \left( \frac{x+y}{2} + z \right) - f \left( \frac{x+y}{2} \right) - f(z) \right\|
\leq \left\| \beta \left( 2f \left( \frac{x+y}{2^2} + \frac{z}{2} \right) - f \left( \frac{x+y}{2} \right) - f(z) \right) \right\|
+ \theta (\|x\|^r + \|y\|^r + \|z\|^r)
\]
for all \( x, y, z \in X \).
Then there exits a unique additive maaping \( h : X \to Y \) such that
\[
\left\| f(x) - h(x) \right\| \leq \frac{2\theta}{|2^r - |2|^r|} \|x\|^r
\]
for all \( x \in X \).
Corollary 3.9.

Let $r < 1$ and $\theta$ be a nonnegative real number and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and
\[
\left\| 2f\left( \frac{x+y+z}{2} \right) - f\left( \frac{x+y}{2} \right) - f(z) \right\|
\leq \left\| \beta\left( f\left( \frac{x+y}{2} + z \right) - f\left( \frac{x+y}{2} \right) - f(z) \right) \right\| + \theta(\|x\|^r + \|y\|^r + \|z\|^r)
\]
for all $x, y, z \in X$. The there exists a unique additive mapping $h : X \to Y$ such that
\[
\left\| f(x) - h(x) \right\| \leq \frac{|2|^r \theta}{|2|^r - |2|} \|x\|^r
\]
for all $x \in X$.

Corollary 3.10.

Let $r > 1$ and $\theta$ be a nonnegative real number and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and
\[
\left\| 2f\left( \frac{x+y+z}{2} \right) - f\left( \frac{x+y}{2} \right) - f(z) \right\|
\leq \left\| \beta\left( f\left( \frac{x+y}{2} + z \right) - f\left( \frac{x+y}{2} \right) - f(z) \right) \right\| + \theta(\|x\|^r + \|y\|^r + \|z\|^r)
\]
for all $x, y, z \in X$. The there exists a unique additive mapping $h : X \to Y$ such that
\[
\left\| f(x) - h(x) \right\| \leq \frac{|2|^r \theta}{|2|^r - |2|} \|x\|^r
\]
for all $x \in X$.

4. Additive $\beta$-functional inequality in complex Banach space

Now, we study the solutions of (1.1) and (1.2). Note that for these inequalities, $X$ is complex normed space and $Y$ is complex Banach spaces. Under this setting, we can show that the mapping satisfying (1.1) and (1.2) is additive. These results are give in the following.

Lemma 4.1.

A mapping $f : X \to Y$ satisfies
\[
\left\| f\left( \frac{x+y+z}{2} \right) - f\left( \frac{x+y}{2} \right) - f(z) \right\| \leq \left\| \beta\left( 2f\left( \frac{x+y+z}{2} \right) - f\left( \frac{x+y}{2} \right) - f(z) \right) \right\|
\]
for all $x, y, z \in X$ if and only if $f : X \to Y$ is additive.

Proof. The proof is similar to the proof of lemma 3.1.
Lemma 4.2.

A mapping \( f: X \to Y \) satisfies \( f(0) = 0 \) and
\[
\|2f\left(\frac{x+y}{2^2} + \frac{z}{2}\right) - f\left(\frac{x+y}{2}\right) - f\left(\frac{x+y}{2} + z\right)\| \leq \|\beta\left(2f\left(\frac{x+y}{2^2} + \frac{z}{2}\right) - f\left(\frac{x+y}{2}\right) - f\left(\frac{x+y}{2} + z\right)\right)\| + \varphi(x, y, z)
\]
(4.2)
for all \( x, y, z \in X \) if and only if \( f: X \to Y \) is additive.

Proof. The proof is similar to the proof of lemma 3.2.

Theorem 4.3.

Let \( \varphi: X^3 \to [0, \infty) \) be a function and let \( f: X \to Y \) be a mapping such that
\[
\psi(x, y, z) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) < \infty
\]
(4.3)
\[
\|f\left(\frac{x+y}{2} + z\right) - f\left(\frac{x+y}{2}\right) - f\left(\frac{x+y}{2} + z\right)\| \leq \|\beta\left(2f\left(\frac{x+y}{2^2} + \frac{z}{2}\right) - f\left(\frac{x+y}{2}\right) - f\left(\frac{x+y}{2} + z\right)\right)\| + \varphi(x, y, z)
\]
(4.4)
for all \( x, y, z \in X \). Then there exists a unique mapping \( h: X \to Y \) such that
\[
\|f(x) - h(x)\| \leq \frac{1}{2} \psi(x, x, x)
\]
(4.5)
for all \( x \in X \).

Proof. Let \( x = y = z \) in (4.4), we get
\[
\|f(2x) - 2f(x)\| \leq \varphi(x, x, x)
\]
(4.6)
for all \( x \in X \). So
\[
\|f(x) - 2f\left(\frac{x}{2}\right)\| \leq \varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right)
\]
for all \( x \in X \). Hence
\[
\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\|
\leq \sum_{j=1}^{m-1} \left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \leq \sum_{j=1}^{m-1} \|2^j \varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right)\|
\]
(4.7)
for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in X \). It follows from (4.7) that the sequence \( \left\{2^m f\left(\frac{x}{2^m}\right)\right\} \) is a cauchy sequence for all \( x \in X \). Since
is complete, the sequence \( \left\{ 2^n f \left( \frac{x}{2^n} \right) \right\} \) converges. So one can define the mapping \( h : X \to Y \) by

\[
h(x) := \lim_{n \to \infty} \frac{1}{2^n} f \left( 2^n x \right)
\]

for all \( x \in X \). Moreover, letting \( l = 0 \) and passing the limit \( m \to \infty \) in (4.7), we get (4.5).

Now, let \( T : X \to Y \) be another additive mapping satisfying (4.5). Then we have

\[
\| h(x) - T(x) \| = \left\| 2^q h \left( \frac{x}{2^q} \right) - 2^q T \left( \frac{x}{2^q} \right) \right\|
\]

\[
\leq \left\| 2^q h \left( \frac{x}{2^q} \right) - 2^q f \left( \frac{x}{2^q} \right) \right\| + \left\| 2^q T \left( \frac{x}{2^q} \right) - 2^q f \left( \frac{x}{2^q} \right) \right\|
\]

\[
\leq 2^q \psi \left( \frac{x}{2^q}, \frac{x}{2^q}, \frac{x}{2^q} \right)
\]

which tends to zero as \( q \to \infty \) for all \( x \in X \). So we can conclude that \( h(x) = T(x) \) for all \( x \in X \). This proves the uniqueness of \( h \). It follows from (4.3) and (4.4) that

\[
\| h \left( \frac{x+y+z}{2} \right) - h \left( \frac{x+y}{2} \right) - h(z) \|
\]

\[
= \lim_{n \to \infty} \| 2^n \left( f \left( \frac{x+y}{2^{n+2}} + \frac{z}{2^{n+1}} \right) - f \left( \frac{x+y}{2^{n+1}} \right) - f \left( \frac{z}{2^n} \right) \right) \|
\]

\[
\leq \lim_{n \to \infty} \| 2^n \beta \left( 2f \left( \frac{x+y}{2^{n+3}} + \frac{z}{2^{n+2}} \right) - f \left( \frac{x+y}{2^{n+1}} \right) - f \left( \frac{z}{2^n} \right) \right) \|
\]

\[
+ \lim_{n \to \infty} 2^n \varphi \left( \frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n} \right)
\]

\[
\leq \beta \left( 2h \left( \frac{x+y}{2^2} + \frac{z}{2} \right) - h \left( \frac{x+y}{2} \right) - h(z) \right) \tag{4.8}
\]

for all \( x, y, z \in X \).

So

\[
\| h \left( \frac{x+y+z}{2} \right) - h \left( \frac{x+y}{2} \right) - h(z) \| \leq \beta \left( 2h \left( \frac{x+y}{2^2} + \frac{z}{2} \right) - h \left( \frac{x+y}{2} \right) - h(z) \right)
\]

for all \( x, y, z \in X \). By Lemma 4.1, the mapping \( h : X \to Y \) is additive. \( \square \)

**Theorem 4.4.**

Let \( \varphi : X^3 \to [0, \infty) \) be a function and let \( f : X \to Y \) be a mapping such that

\[
\psi(x, y, z) := \sum_{j=1}^{\infty} \frac{1}{2^j} \varphi(2^j x, 2^j y, 2^j z) < \infty \tag{4.9}
\]
Ly Van An

and

\[
\left\| f\left(\frac{x+y}{2} + z\right) - f\left(\frac{x+y}{2}\right) - f(z) \right\| \\
\leq \|\beta\left(2f\left(\frac{x+y}{2^2} + \frac{z}{2}\right) - f\left(\frac{x+y}{2}\right) - f(z)\right)\| + \phi(x, y, z)
\]

(4.10)

for all \(x, y, z \in X\). Then there exists a unique mapping \(h : X \to Y\) such that

\[
\left\| f(x) - h(x) \right\| \leq \frac{1}{2} \psi(x, x, x)
\]

(4.11)

for all \(x \in X\).

Proof. It follows from (4.10) that

\[
\left\| f(x) - \frac{1}{2} f(2x) \right\| \leq \frac{1}{2} \phi(x, x, x)
\]

for all \(x \in X\). Hence

\[
\left\| \frac{1}{2^l} f\left(2^l x\right) - \frac{1}{2^m} f\left(2^m x\right) \right\| \\
\leq \sum_{j=1}^{m-1} \left\| \frac{1}{2^j} f\left(2^j x\right) - \frac{1}{2^{j+1}} f\left(2^{j+1} x\right) \right\| \\
\leq \sum_{j=1}^{m-1} \frac{1}{2^j} \phi\left(2^j x, 2^j x, 2^j x\right)
\]

(4.12)

for all nonnegative integers \(m\) and \(l\) with \(m > l\) and all \(x \in X\). It follows from (4.12) that the sequence \(\left\{\frac{1}{2^m} f\left(2^n x\right)\right\}\) is a cauchy sequence for all \(x \in X\). Since is complete, the sequence \(\left\{\frac{1}{2^n} f\left(2^n x\right)\right\}\) coversges. So one can define the mapping \(h : X \to Y\) by

\[
h(x) := \lim_{n \to \infty} \frac{1}{2^n} f\left(2^n x\right)
\]

for all \(x \in X\). Moreover, letting \(l = 0\) and passing the limit \(m \to \infty\) in (4.12), we get (4.11).

The rest of the proof is similar to the proof of theorem 4.3.

\[\square\]

Theorem 4.5.

Let \(\phi : X^3 \to [0, \infty)\) be a function and let \(f : X \to Y\) be a mapping such that

\[
\psi(x, y, z) := \sum_{j=1}^{\infty} 2^j \phi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) < \infty
\]

(4.13)
\[ \left\| 2f\left( \frac{x+y}{2^2} + \frac{z}{2} \right) - f\left( \frac{x+y}{2} \right) - f(z) \right\| \leq \left\| \beta\left( f\left( \frac{x+y}{2} + z \right) - f\left( \frac{x+y}{2} \right) - f(z) \right) \right\| + \varphi(x, y, z) \]  

(4.14)

for all \( x, y, z \in X \). Then there exists a unique mapping \( h: X \rightarrow Y \) such that

\[ \left\| f(x) - h(x) \right\| \leq \frac{1}{2} \psi(x, x, x) \]  

(4.15)

for all \( x \in X \).

**Proof.** Letting \( y = x = 0 \) in (4.14), we get

\[ \left\| f(z) - 2f\left( \frac{z}{2} \right) \right\| = \left\| 2f\left( \frac{z}{2} \right) - f(z) \right\| \leq \varphi(0, 0, z) \]  

(4.16)

for all \( x \in X \). So

\[ \left\| 2^l f\left( \frac{z}{2^l} \right) - 2^m f\left( \frac{z}{2^m} \right) \right\| \leq \sum_{j=1}^{m-1} \left\| 2^j f\left( \frac{z}{2^j} \right) - 2^{j+1} f\left( \frac{z}{2^{j+1}} \right) \right\| \leq \sum_{j=1}^{m-1} 2^j \varphi(0, 0, \frac{z}{2^j}) \]  

(4.17)

for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in X \). It follows from (4.17) that the sequence \( \{2^n f\left( \frac{z}{2^n} \right) \} \) is a cauchy sequence for all \( z \in X \). Since is complete, the sequence \( \{2^n f\left( \frac{z}{2^n} \right) \} \) converges. So one can define the mapping \( h: X \rightarrow Y \) by

\[ h(z) := \lim_{n \rightarrow \infty} 2^n f\left( \frac{z}{2^n} \right) \]  

(4.18)

for all \( z \in X \). Moreover, letting \( l = 0 \) and passing the limit \( m \rightarrow \infty \) in (4.17), we get (4.15). The rest of the proof is similar to the proof of theorem 4.3. \( \square \)

**Theorem 4.6.**

Let \( \varphi: X^3 \rightarrow [0, \infty) \) be a function and let \( f: X \rightarrow Y \) be a mapping satisfying \( f(0) = 0 \), and

\[ \psi(x, y, z) := \sum_{j=1}^{\infty} \frac{1}{2^j} \varphi(2^j x, 2^j y, 2^j z) < \infty \]  

(4.19)

\[ \left\| 2f\left( \frac{x+y}{2^2} + \frac{z}{2} \right) - f\left( \frac{x+y}{2} \right) - f(z) \right\| \leq \left\| \beta\left( f\left( \frac{x+y}{2} + z \right) - f\left( \frac{x+y}{2} \right) - f(z) \right) \right\| + \varphi(x, y, z) \]  

(4.20)
for all $x, y, z \in X$. Then there exists a unique mapping $h : X \to Y$ such that
\[ \| f(z) - h(z) \| \leq \frac{1}{2} \psi(0, 0, z) \] (4.21)
for all $z \in X$

**Proof.** Letting $y = x = 0$ in (4.20), we get
\[ \| f(z) - 2f\left(\frac{z}{2}\right) \| = \| 2f\left(\frac{z}{2}\right) - f(z) \| \leq \varphi(0, 0, z) \] (4.22)
It follows from (4.20) that
\[ \| f(z) - \frac{1}{2} f(2z) \| \leq \frac{1}{2} \varphi(0, 0, 2z) \]
for all $z \in X$. Hence
\[ \left\| \frac{1}{2^l} f(2^l z) - \frac{1}{2^m} f(2^m z) \right\| \leq \sum_{j=1}^{m-1} \left\| \frac{1}{2^j} f(2^j z) - \frac{1}{2^{j+1}} f(2^{j+1} z) \right\| \leq \sum_{j=1}^{m-1} \frac{1}{2^j} \varphi(0, 0, 2^j x) \] (4.23)
for all nonnegative integers $m$ and $l$ with $m > l$ and all $z \in X$. It follows from (4.23) that the sequence $\{ \frac{1}{2^n} f(2^n z) \}$ is a cauchy sequence for all $z \in X$. Since is complete, the sequence $\{ \frac{1}{2^n} f(2^n z) \}$ covers. So one can define the mapping $h : X \to Y$ by
\[ h(z) := \lim_{n \to \infty} \frac{1}{2^n} f(2^n z) \] (4.24)
for all $z \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (4.23), we get (4.22). The rest of the proof is similar to the proof of theorem 4.3.

□

**Corollary 4.7.**

Let $r > 1$ and $\theta$ be positive real numbers, and let $f : X \to Y$ be a mapping such that
\[ \| f\left(\frac{x+y}{2} + z\right) - f\left(\frac{x+y}{2}\right) - f(z) \| \leq \beta \left(2f\left(\frac{x+y}{2^2} + \frac{z}{2}\right) - f\left(\frac{x+y}{2}\right) - f(z)\right) + \theta(\|x\|^r + \|y\|^r + \|z\|^r) \]
for all $x, y, z \in X$. Then there exists a unique additive mapping $h : X \to Y$ such that
\[ \| f(x) - h(x) \| \leq \frac{2\theta}{2^r - 2}\|x\|^r \]
for all \( x \in X \)

**Corollary 4.8.**

Let \( r < 1 \) and \( \theta \) be positive real numbers, and let \( f : X \to Y \) be a mapping such that
\[
\left\| \left( \frac{x + y}{2} + z \right) - f \left( \frac{x + y}{2} \right) - f(z) \right\| 
\leq \left\| \beta \left( 2f \left( \frac{x + y}{2^2} + \frac{z}{2} \right) - f \left( \frac{x + y}{2} \right) \right) \right\| + \theta (\| x \|^r + \| y \|^r + \| z \|^r)
\]
for all \( x,y,z \in X \). Then there exists a unique additive mapping \( h : X \to Y \) such that
\[
\left\| f(x) - h(x) \right\| \leq \frac{2\theta}{2^r - 2} \| x \|^r
\]
for all \( x \in X \)

**Corollary 4.9.**

Let \( \varphi : X^3 \to [0, \infty) \) be a function and let \( f : X \to Y \) be a mapping satisfying \( f(0) = 0 \),
\[
\psi(x,y,z) := \sum_{j=1}^{\infty} 2^j \varphi \left( \frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j} \right) < \infty
\]
\[
\left\| 2f \left( \frac{x + y}{2^2} + \frac{z}{2} \right) - f \left( \frac{x + y}{2} \right) \right\| 
- f(z) - \beta \left( f \left( \frac{x + y}{2} + z \right) - f \left( \frac{x + y}{2} \right) \right) \right\| 
\leq \varphi(x,y,z)
\]
for all \( x,y,z \in X \). Then there exists a unique mapping \( h : X \to Y \) such that
\[
\left\| f(z) - h(z) \right\| \leq \frac{1}{2} \psi(0,0,z)
\]
for all \( z \in X \)

**Corollary 4.10.**

Let \( r > 1 \) and \( \theta \) be positive real numbers, and let \( f : X \to Y \) be a mapping such that
\[\left\| 2f\left(\frac{x+y}{2} + \frac{z}{2}\right) - f\left(\frac{x+y}{2}\right) - f(z)\right\| \leq \left\| \beta\left(f\left(\frac{x+y}{2} + z\right) - f\left(\frac{x+y}{2}\right) - f(z)\right)\right\| + \theta\left(\|x\|^r + \|y\|^r + \|z\|^r\right)\]

for all \(x, y, z \in X\). Then there exists a unique additive mapping \(h : X \to Y\) such that

\[\left\| f(x) - h(x)\right\| \leq \frac{2\theta}{2^r - 2}\|x\|^r\]

for all \(x \in X\)

**Remark:** If \(\beta\) is a real number such that \(-1 < \beta < 1\) and is \(Y\) is a real Banach space, then all the assertions in this sections remain valid.

### 5. Conclusion

In this paper, it is shown that the solutions of the first and second \(\beta\)-functional inequalities are additive mappings and the Hyers-Ulam stability is proved. These are the main results of the paper, which are a generalization of the results in [17, 20].

### References


[17] Choonkil Park, Additive $\beta$-functional inequalities, *Journal of Nonlinear Science and Appl.*, **7** (2014), 296-310. [https://doi.org/10.22436/jnsa.007.05.02](https://doi.org/10.22436/jnsa.007.05.02)


Received: December 11, 2019; Published: June 12, 2020