Existence of Homoclinic Orbits and Conditions for the Onset of Chaotic Behavior in a Perturbed Double-Well Oscillator

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Abstract

In this paper, the existence of homoclinic orbits and conditions for the onset of chaotic behavior in a perturbed double-well Duffing Oscillator were examined under
parametric excitations using Melnikov method and Lyapunov exponents. The results indicated that as the method of the Lyapunov exponent can narrow the range for the critical threshold values and detect mutations in the chaotic system. Numerical simulations showed that as the parameter was varied, repeated resonance of successively higher periods occurred leading to homoclinic orbits and the unstable chaotic motion. This extends some results in the literature.

**Mathematics Subject Classification:** 37D45, 34C28

**Keywords:** Double-well Duffing Oscillator, Melnikov method, positive Lyapunov exponent, QR-factorization method, threshold values, MATCAD software

### 1 Introduction

Consider the parametrically excited oscillator;

$$\ddot{x} + \varepsilon \dot{x} - \alpha x + \beta x^3 = \varepsilon \mu \cos(\Omega t)$$

(1.1)

where $\alpha, \beta$ and $\varepsilon$ are positive coefficients, $\mu$ and $\Omega$ are amplitude and frequency of excitation and $0 < \varepsilon < 1$. Under the deterministic initial conditions;

$$x(0) = x(2\pi)$$
$$\dot{x}(0) = \dot{x}(2\pi)$$

(1.2)

The study of chaotic motion in nonlinear system has attracted the attention of many researchers in the past. Many investigations have been carried out on different nonlinear chaotic system to understand the complex behavior of these systems [18]. Duffing oscillator is one of the most useful nonlinear dynamical systems, which is considered as a prototype model for various physical and engineering problems such as dynamics of buckled elastic beam, particle in double-well, particles in plasma and a defect in solids [2].

More currently, Wang et al [20-22] showed that many nonlinear dynamical problem can be reduced to Duffing system for structures subject to both free and forced excitation. Hence, the Duffing is still an interesting model to study for discovering characteristics of chaos in nonlinear system despite lots of investigation have been carried out for so many years. Several researches have been performed on chaotic motion of Duffing equation [12,15,19]. Thus, the nonlinear system is necessary in various engineering applications for example, drag forces in flow induced vibration [2] and vibration isolators [1].
Various researchers have used different methods in obtaining solutions, for instance Ueda [23] used the numerical simulation where changes in attractors were obtained under various parameters. Nonlinear adaptive filtering method [11], singular value decomposition, QR-matrix factorization method and its improvement method [6,8,13].

Melnikov method and Lyapunov exponents are very significant analytical techniques for determining chaos. The main idea of this method is to measure the distance between the stable and unstable manifolds and if the stable and unstable manifolds intensively intersect once, they will intersect infinite times [24]. Thus, according to Smale-Birkhoff theorem [9], it implies the existence of the chaotic behavior in the chaotic behavior in the Smale-horseshoe sense.

Generally speaking, the Melnikov method is very useful for detecting the presence of transverse homoclinic orbits. However, in order to apply the Melnikov theory in our work, the perturbed Duffing equation is presented as a perturbed Hamiltonian system depending on a small parameter [25-26]. The Melnikov theory was firstly used to study chaos in Duffing system by Holmes [14] and generalized Melnikov function was developed by Wiggins [16-17]. This criterion is the necessary condition for the existence of chaos but not sufficient, therefore, it is a sufficient condition for the suppression of chaos [10].

The Lyapunov exponent is an important indicator in determining the sensitivity of chaotic behavior which characterizes the average rate of the system in phase space between adjacent tracks of convergence and divergence [3]. Whether the Lyapunov exponent is greater than zero or not is one of the most straight forward criterion to distinguish the chaotic systems. In order to calculate the Lyapunov exponent, some methods of solutions includes [3], nonlinear adaptive filtering method, QR matrix factorization method and its improvement methods. This paper makes use of two methods, the Melnikov method and improved QR matrix factorization method.

The objective of this paper therefore is to investigate the existence of homoclinic orbits and chaotic behavior in a perturbed double-well Duffing oscillator under parametric excitations.

The rest of the paper is organized as follows; In section 2, the preliminaries to the results are explained, section 3 gives the main results using the Melnikov method and the calculation of the Lyapunov exponent and section 4 is where the numerical simulations are presented and finally some conclusions are given in section 5.

2 Preliminaries

2.1 Melnikov Method
One of the main tools for determining the existence or non-existence of chaos in a perturbed Hamiltonian system is Melnikov method. In this theory, the distance bet-
ween stable and unstable manifolds of the perturbed system is calculated up to the first order term. Melnikov method is a procedure which gives a bound on the parameters of a system such that chaos is predicted not to occur. The Melnikov method investigate the homoclinic bifurcation in the forced Duffing oscillator system with linear and non-damping. It measures the distance between stable and unstable manifolds in the Poincare section \([25]\) and to preserve the homoclinic loops under a perturbation requires that at \(t_0\), if \(M(t_0)\) that is the Melnikov function has a simple zero, then a homoclinic bifurcation occurs, implying that the chaotic motion occurs.

### 2.2 Melnikov Method For Predicting Chaos

Melnikov method gives an analytic tool for establishing the existence of transverse homoclinic points of the Poincare map for a periodic orbit of a perturbed dynamical system of the form;

\[
\dot{x} = f(x) + \epsilon g(x)
\]  

(2.1)

with \(x \in \mathbb{R}^n\). It can also be used to establish the existence of sub-harmonic periodic orbits of perturbed system of the form in (1). Furthermore, it can be used to show the existence of limit cycles and separatix cycles of perturbed planar system with \(x \in \mathbb{R}^2\). For periodically perturbed planar systems, we have the form;

\[
\dot{x} = f(x) + \epsilon g(x, t)
\]

(2.2)

Where \(x \in \mathbb{R}^2\) and \(g\) is periodic with period \(T\) in \(t\). We assume that \(f \in C'(\mathbb{R}^2)\) and \(g \in C'(\mathbb{R}^2 \times \mathbb{R})\) and we make the assumption;

i. For \(\epsilon = 0\), the system (3.2.) has a homoclinic orbit;

\[
\Gamma_0 : X = y_0(t), \quad -\infty < t < \infty
\]

at a hyperbolic saddle point \(X_0\) and,

ii. For \(\epsilon = 0\), the system has a non-parameter family of periodic orbit. Then the Melnikov function \(M(t_0)\) is defined as;

\[
M(t_0) = \int_{-\infty}^{\infty} f(y_0(t))^\epsilon g(y_0(t), t + t_0) dt
\]

(2.3)

The Melnikov method can be interpreted as a derivation in energy from the value on the perturbed separatix. Before stating main result established by Melnikov concerning the existence of transverse homoclinic point of the Poincare section, we need the following lemma and theory which establish the existence of a periodic orbit and hence the existence of the Poincare map with sufficient \(\epsilon\).

**Lemma 2.1**

Under assumption (i) and (ii), for \(\epsilon\) sufficiently small, the system (2.) has a unique hyperbolic periodic orbit; \(y_\epsilon(t) = X_0 + 0(\epsilon)\) of period \(T\). Correspondingly, the Poincare map \(P_\epsilon\) has a unique hyperbolic fixed point of saddle type;

\[
X_\epsilon = X_0 + 0(\epsilon).
\]

(2.4)
Theorem 2.1
Under the assumption (i) and (ii), if the Melnikov function \( M(t_0) \) has a simple zero in \([0,1]\), then for all sufficiently small \( \varepsilon \neq 0 \), the stable and unstable manifold of the Poincare map \( P_\varepsilon \) intersect transversally, that is, \( P_\varepsilon \) has a transverse homoclinic point. This theorem was established by Melnikov [26]. The idea of the proof is that \( M(t_0) \) is a measure of the separation of the stable and unstable manifold of the Poincare map. The theory is an important result because it establishes the existence of transverse homoclinic point for \( P_\varepsilon \). It implies the existence of strange invariant set for some iterate of \( P_\varepsilon \) and the same type of chaotic dynamics for system (2) as for the Smale horseshoe map. Generally, the Melnikov method is very useful for detecting the presence of transverse homoclinic orbits and the occurrence of homoclinic bifurcations.

Theorem 2.2 (Smale-Birkhoff Homoclinic Theorem) [5]
Let \( f \) be a diffeomorphism (\( C^r \)) and suppose \( p \) is a hyperbolic fixed point. A homoclinic point is a point \( q \neq p \) which is in the stable and unstable manifolds. If the stable and unstable manifolds intersect transversally at \( q \), then \( q \) is called transverse. This implies that there is a homoclinic orbit \( \gamma(q) = (q_n) \) such that \( \lim_{n \to \infty} q_n = q \). Since the stable and unstable manifolds are invariant, we have; \( q_n \in W^s(p) \cap W^u(p) \) for all \( n \in \mathbb{Z} \). Moreover, if \( q \) is transversal, so are all \( q_n \) since \( f \) is diffeomorphism.

2.3 Method of Lyapunov Exponent
The method of Lyapunov exponent serves as a useful tool to qualify chaos. Specifically, Lyapunov exponent measures the rate of convergence or divergence of nearby trajectories [4, 7]. Negative Lyapunov exponents indicates convergence while positive Lyapunov exponents demonstrate divergence and chaos. The magnitude of the Lyapunov exponents is an indicator of the time scale on which chaotic behavior can be predicted or transients for the positive and negative cases respectively [3]. Physically, the Lyapunov exponent measures average exponential divergence or convergence between trajectories that differ only in having an infinitesimally small difference in their initial condition. The system is said to be chaotic if the trajectories remain within a bounded set of the dynamics. If one considers a ball of points in \( N \)-dimensional phase space in which each point follows its own trajectory based upon the system equations of motion over time, the ball of points will collapse to a simple point, will stay a ball or will become ellipsoid in shape [16]. The measure of the rate at which this infinitesimal ball collapse or expands is the Lyapunov exponent. For a system written in the state-space form \( \dot{x} = u(x) \), small derivation from trajectory can be expressed by the equation \( \delta \dot{x}_i = \frac{\partial u_i}{\partial x_j} \delta x_j \). The maximal lyapunov exponent is then defined by this equation.
Other useful quantities are the short time Lyapunov exponent and the local Lyapunov exponent. A short time Lyapunov exponent is simply a Lyapunov exponent defined over some finite time interval. The local Lyapunov exponent is a short time Lyapunov exponent in the limit where the time interval approaches zero. Both are dependent on starting points and the short time Lyapunov exponent is also independent on the magnitude of the time interval. If all points in the neighborhood of a trajectory converge towards the same orbit, the attractor is a fixed point or a limit cycle. However, if the attractor is strange, any two trajectories \( x(t) = f'(x_0) \) and \( x(t) + \delta x(t) = f'(x_0 + \delta x_0) \) that starts over very close to each other separate exponentially with time. This sensitive to initial conditions can be quantified as:

\[
\|\delta x(t)\| = e^{\lambda t} \|\delta x_0\| \tag{2.5}
\]

where \( \lambda \), the mean rate of separation of trajectories of the system is called the Lyapunov exponent, which can be estimated for long time \( t \) as:

\[
\lambda = \frac{1}{t} \ln \frac{\|\delta x(t)\|}{\|\delta x_0\|} \tag{2.6}
\]

\[
\lambda_T(X(t), \delta x_0) = \frac{1}{T} \ln \frac{\|\delta x(t+T)\|}{\|\delta x(t)\|} \tag{2.7}
\]

\[
\lambda_{local}(X(t)) = \lim_{T \to 0} \frac{1}{T} \ln \frac{\|\delta X(t+T)\|}{\|\delta X(t)\|} \tag{2.8}
\]

Equations (2.7) and (2.8) are for short and local Lyapunov exponent. The exponent can be positive or negative but at least one must be positive for an attractor to be classified as chaotic. In particular, if \( \lambda < 0 \), the system converges to a stable fixed point or periodic orbits. A negative value of the Lyapunov exponent is characteristic of dissipative or non-conservative systems. If \( \lambda = 0 \), the system is conservative and converges to a stable cycle limit. If \( \lambda > 0 \), the system is unstable and chaotic. Hence, if the system is chaotic, it will have at least one positive Lyapunov exponent. Thus, the definition of chaotic system is based on a positive Lyapunov exponent. Finally, if \( \lambda = \infty \), the system is random.

Generally, the most used measure of sensitive to initial condition is a system characterization by the Lyapunov exponent, which quantifies the rate of separation of infinitesimal close trajectories. For example, consider a one-dimensional system with two trajectories \( x_1(t) \) and \( x_2(t) \) which at some point \( t_0 \) are arbitrary close together and their difference in time tracked by the function;

\[
\delta x(t) = |x_1(t) - x_2(t)|. \]

The sign of the lyapunov exponent characterizes whether or not the system is exhibiting chaotic behavior. If the exponent is negative, the system, at least in that set of initial conditions is said to be stable (like trajectories go to like trajectories). A Lyapunov exponent of zero implies an unstable system which is essentially on the edge stable and chaotic. And of course a positive exponent implies the system is chaotic where trajectories exhibit exponential divergence.
3 Results

3.1 The Double-Well Duffing Oscillator
For the Duffing oscillator with a double-well potential, there are two stable equilibrium points at $x = \pm \sqrt{\alpha/\beta}$ and one unstable equilibrium point at $x = 0$.

The Duffing equation with a double well potential (with a negative linear stiffness) describes the dynamics of a buckled beam as well as plasma oscillator where chaotic motion can be observed. The forced Duffing oscillator with cubic non-linearity is described by the following ordinary differential equation:

$$\ddot{x} + \alpha \dot{x} + \beta x^3 = \mu \cos(\Omega t)$$  \hfill (2.9)

Under a harmonic excitation, the equation is given as;

$$\ddot{x} + \epsilon \alpha \dot{x} + \beta x^3 = \epsilon \mu \cos(\Omega t)$$  \hfill (2.10)

where $\epsilon$ is the perturbation term, $\delta, \alpha, \beta$ are positive coefficient, $\gamma$ and $\omega$ are amplitude and frequency of the excitation. Equation (2.10) can be reformulated to one-order non-autonomous equation as;

$$\dot{x}_1 + \epsilon \dot{x}_2 - \alpha x_1 + \beta x_1^3 = \epsilon \mu \cos(\Omega t)$$  \hfill (2.11)

then,

$$\dot{x}_1 + \epsilon \dot{x}_2 - \alpha x_1 + \beta x_1^3 = \epsilon \mu \cos(\Omega t)$$

Let

$$\dot{x}_1 = x_2, \dot{x}_2 = \dot{x}_1$$

$$\ddot{x}_2 + \epsilon \dot{x}_2 - \alpha x_1 + \beta x_1^3 = \epsilon \mu \cos(\Omega t)$$

$$\dot{x}_2 = -\epsilon \dot{x}_2 + \alpha x_1 - \beta x_1^3 + \epsilon \mu \cos(\Omega t)$$

Hence, we have;

$$\begin{cases} 
\dot{x}_1 = x_2 \\
\dot{x}_2 = \alpha x_1 - \beta x_1^3 + \epsilon(\dot{x}_2 + \mu \cos(\Omega t))
\end{cases}$$  \hfill (2.12)

when $\epsilon = 0$, an unperturbed system can be obtained as follows;

$$\begin{cases} 
\dot{x}_1 = x_2 \\
\dot{x}_2 = \alpha x_1 - \beta x_1^3
\end{cases}$$  \hfill (2.13)

The system corresponds to a Hamiltonian system and the Hamiltonian function is:

$$H(x, y) = \frac{1}{2} x_2^2 - \frac{\alpha}{2} x_1^2 + \frac{\beta}{4} x_1^4$$  \hfill (2.14)

with a corresponding potential function;

$$V(x) = -\frac{\alpha}{2} x_2^2 + \frac{\beta}{4} x_1^4$$  \hfill (2.15)

The solution of the unperturbed system is given as;

$$x_0(t) = \sqrt{2} \sech t$$

$$y_0(t) = -\sqrt{2} \tanh t$$

3.2 Melnikov Function for the Perturbed Double-Well Duffing Equation
The Melnikov function for the perturbed system is obtain as follows;

$$M(t_0) = \int_{-\infty}^{\infty} x_0(t) [\mu \cos(\Omega t - t_0) - \delta x_0(t)] dt$$  \hfill (2.16)
\[ M(t_0) = -\sqrt{2}\mu \int_{-\infty}^{\infty} \text{sech} t \text{ tanh} t \cos \Omega(t - t_0) \, dt \]  
(2.16)

\[ M(t_0) = -2\mu \int_{-\infty}^{\infty} \text{sech}^2 t \text{ tanh}^2 t \, dt \]  
(2.17)

Then, by method of integral residue, the last integral yields;

\[ M(t_0) = -\frac{4\hat{\epsilon}}{3} + \sqrt{2}\mu\Omega \text{ sech} \left( \frac{\pi \Omega}{2} \right) \sin \Omega t_0 \]  
(2.18)

Thus, the duffing equation is chaotic for \( \hat{\epsilon}, \mu \) sufficiently small provided;

\[ \left| \frac{\hat{\epsilon}}{\mu} \right| < \frac{3\sqrt{2\pi|\Omega|}}{4} \text{ sech} \left( \frac{\pi \Omega}{2} \right) \]  
(2.19)

3.3 The Lyapunov Exponent Of A Double-Well Duffing Oscillator

Consider the Duffing equation below;

\[ \ddot{x} + \hat{\epsilon}\dot{x} - \alpha x + \beta x^3 = \mu \cos(\Omega t) \]  
(2.20)

Where \( \ddot{x}, \dot{x} \) are second-order and first-order derivative, \( \alpha \in \mathbb{R}^n, \beta \in \mathbb{R}^n, \hat{\epsilon} \) is the damping, \( \mu \) is the amplitude of the circle, \( \Omega \) is the angular frequency of the driven circle. In order to determine whether the system is in chaotic state, we need to calculate the Lyapunov exponent using the QR factorization method.

Let \( y = \dot{x}, g(x, y) = -\hat{\epsilon}y + \alpha x - \beta x^3 \)

Then equation (2.20) is equivalent to;

\[ \begin{cases} 
\dot{x} = y \\
\dot{y} = g(x, y) + f(t) 
\end{cases} \]

Which is written in matrix form as;

\[ \dot{Y} = F(Y) \]  
(2.21)

According to the variational principle, its variational equations are;

\[ \dot{Y}(t) = f(t)Y(t), \quad Y(0) = I \]  
(2.22)

Where \( Y(t) \) is a 2 by 2 matrix, \( I \) is a 2 by 2 unit matrix, \( f(t) \) is the Jacobian matrix of the system and its expression is;

\[
\begin{pmatrix}
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\
\frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}
\end{pmatrix} = \begin{pmatrix} 0 & 1 \\
\alpha - 3\beta x^2 & -\hat{\epsilon}
\end{pmatrix}
\]

Then, QR factorization of \( Y(t) \) can be written as;

\[ Y(t) = QR \]  
(2.23)

Where \( Q \) is orthogonal matrix, \( R \) is upper triangular matrix. Substituting (2.23) in (2.22), we obtain the variational equation;

\[ \begin{cases} 
\dot{Q}R + Q\dot{R} = fQR \\
Q(0)R(0) = I 
\end{cases} \]  
(2.24)

Left multiplied equ (22) by \( Q^T \) and right multiply by \( R^{-1} \)

\[ Q^T \dot{Q} + \dot{R}R^{-1} = Q^TfQ \]

\[ Q(0) = I, R(0) = I \]  
(2.25)

The orthogonal matrix \( Q \) is written as a function of angle variables. To the Duffing equation, its orthogonal matrix \( Q \) can be expressed by one angle \( \theta \).
Existence of homoclinic orbits and conditions for the onset of chaotic behavior

\[ Q = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \]

The upper triangular matrix \( R \) can be expressed as;

\[ R = \begin{bmatrix} e^{\lambda_1(t)} & r_{12} \\ 0 & e^{\lambda_2(t)} \end{bmatrix} \]

Where \( \theta \) is the angle variable, \( \lambda_i(t) \) is the value associated with the lyapunov exponent. Then,

\[ Q^T = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \]

\[ R^{-1} = \begin{bmatrix} e^{-\lambda_1(t)} & -\frac{r_{12}}{e^{(\lambda_1+\lambda_2)}} \\ 0 & e^{-\lambda_2(t)} \end{bmatrix} \]

Then putting \( Q^T, R^{-1}, Q \) and \( R \) into equ (2.25), we have;

\[ \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} -\sin \theta & \cos \theta \\ -\cos \theta & -\sin \theta \end{bmatrix} + \begin{bmatrix} \frac{d e^{\lambda_1(t)}}{d t} & \frac{d r_{12}}{d t} \\ 0 & \frac{d e^{\lambda_2(t)}}{d t} \end{bmatrix} \begin{bmatrix} e^{-\lambda_1(t)} & -\frac{r_{12}}{e^{(\lambda_1+\lambda_2)}} \\ 0 & e^{-\lambda_2(t)} \end{bmatrix} \]

\[ = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -\alpha - 3 \beta x^2 & -k \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \]

(2.26)

The correspondent matrix elements on both sides of (2.26) are equal, so we get

\[ \begin{align*}
\frac{d \lambda_1(t)}{d t} &= \frac{\partial g}{\partial y} \sin^2 \theta - \frac{1}{2} \left[ 1 + \frac{\partial g}{\partial x} \right] \sin(2\theta) \\
\frac{d \lambda_2(t)}{d t} &= \frac{\partial g}{\partial y} \cos^2 \theta + \frac{1}{2} \left[ 1 + \frac{\partial g}{\partial x} \right] \sin(2\theta) \\
\frac{d \theta(t)}{d t} &= -\frac{1}{2} \frac{\partial g}{\partial y} \sin 2\theta + \sin^2 \theta - \frac{\partial g}{\partial x} \cos^2 \theta
\end{align*} \]

(2.27)

We add and subtract the first two differential equation and get a new differential equation. Together with the third differential equation, we obtain three new equations;

\[ \begin{align*}
\frac{d v_1}{d t} &= \frac{\partial g}{\partial y} \\
\frac{d v_2}{d t} &= \frac{\partial g}{\partial y} \cos 2\theta - \frac{1}{2} \left[ 1 + \frac{\partial g}{\partial x} \right] \sin(2\theta) \\
\frac{d \theta(t)}{d t} &= -\frac{1}{2} \frac{\partial g}{\partial y} \sin 2\theta + \sin^2 \theta - \frac{\partial g}{\partial x} \cos^2 \theta
\end{align*} \]

(2.28)

Then from;

\[ \begin{align*}
\frac{d v_1}{d t} &= \frac{d \lambda_1}{d t} + \frac{d \lambda_2}{d t} \\
\frac{d v_2}{d t} &= \frac{d \lambda_1}{d t} - \frac{d \lambda_2}{d t}
\end{align*} \]

(2.29)

We obtain;

\[ \lambda_1(t) = \frac{[v_1(t)+v_2(t)]}{2} \]

\[ \lambda_2(t) = \frac{[v_1(t)-v_2(t)]}{2} \]

(2.30)

The time evolution of the Lyapunov exponent are;
\[ f_1(t) = \frac{\lambda_1(t)}{t} \]
\[ f_2(t) = \frac{\lambda_2(t)}{t} \]

Then, the Lyapunov exponent are:
\[ \lambda_1 = \lim_{t \to \infty} \frac{\lambda_1(t)}{t} \]
\[ \lambda_2 = \lim_{t \to \infty} \frac{\lambda_2(t)}{t} \]

4 Numerical Simulation for Double-Well Duffing Oscillator

Simulation of double-well duffing oscillator
\[ \ddot{x} + \varepsilon \dot{x}^2 - ax + bx^3 = \varepsilon \mu x \cos(\Omega t) \]
\[ \varepsilon := 0.01, \ c := 1, \ b := 0.5, \ a := 0.5, \ \mu := 0.1, \ \Omega := 0.4 \]
\[ \varepsilon := 0.01, \ c := 1, \ b := 0.5, \ a := 0.5 \]

Define a function that determines a vector of derivatives values at any solution point \((t, Y)\):
\[ D(t, X) := \begin{bmatrix} X_1 \\ \varepsilon.\mu.\dot{X}_0.\cos(\Omega. t) - \varepsilon. c. X_1 + a. \dot{X}_0 - b. (X_0)^3 \end{bmatrix} \]

Define an additional arguments for the ODE solver:
\[ t_0 := 0 \quad \text{Initial value of independent variable} \]
\[ t_1 := 100 \quad \text{Final value of independent variable} \]
\[ X_0 := \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{Vector of initial function values} \]
\[ N := 1500 \quad \text{Numbers of solution values on } [t_0, t_1] \]
\[ S := (X_0, t_0, t_1, N, D) \]
\[ t := S^{(0)} \quad \text{Independent variables values} \]
\[ X_1 := S^{(1)} \quad \text{First solution function values} \]
\[ X_2 := S^{(2)} \quad \text{Second solution function values} \]

Solution Matrix

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<td>1.109</td>
</tr>
<tr>
<td>15</td>
<td>1</td>
<td>1.053</td>
<td>( \ldots )</td>
</tr>
</tbody>
</table>

Table 1.
Existence of homoclinic orbits and conditions for the onset of chaotic behavior

Solution matrix for the independent variable values.

Figure 1. Trajectory $x(t)$ as a function of time, showing the relationship between the first solution and the independent variable values.

Figure 2. Velocity $\dot{x}(t)$ as a function of time, showing the mutations of the second solution values with independent variable values.
Figure 3. Phase portrait showing no intersections of the manifold.

Figure 4. Trajectory $x(t)$ as a function of time, indicating the variation between the two variables values.
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Figure 5. Velocity $\dot{x}(t)$ as a function of time, showing the mutations.

Figure 6. Phase portrait of the nested curves that indicated the periodic doubling.

5 Conclusions

In the present study, the existence of the homoclinic orbits and conditions for the onset of chaotic behavior in a perturbed double-well oscillator has been investigated using Melnikov method and Lyapunov exponent. Similar results obtained from the two methods shows that the behavior of the perturbed Duffing oscillator is chaotic and highly unstable with repeated resonance of successively higher periods. As a result, the threshold values and the mutations in the chaotic system were illustrated using numerical simulations.
References


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