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An Inertial Forward-Backward Splitting Method for Zero Point Problem in Banach Spaces

Liang-cai Zhao

College of Mathematics, Yibin University
Yibin, Sichuan 644000, China

Shih-sen Chang[1]

Center for General Educatin, China Medical University
Taichung, 40402, Taiwan

Abstract

In this paper, interest is to establish a modified viscosity iterative forward-backward algorithm involving the inertial technique to find zeros of the sum of two accretive operators in the setting of Banach spaces. We shall prove the strong convergence of the method under mild conditions. We also discuss applications of these methods to variational inequalities, convex minimization problem.

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1. Introduction

Let X be a real Banach space.We study the following zero point problem: find $x^* \in X$ such that

$$0 \in Ax^* + Bx^* \tag{1.1}$$

[1]Corresponding author

where $A : X \to X$ is an operator and $B : X \to 2^X$ is a set-valued operator. This problem includes, as special cases, convex programming, variational inequalities, split feasibility problem and minimization problem. To be more precise, some concrete problems in machine learning, image processing and linear inverse problem can be modeled mathematically as this form (1.1). For example:

Example 1 A stationary solution to the initial value problem of the evolution equation

$$0 \in \frac{\partial u}{\partial \partial t} + Fu, \quad u(0) = u_0, \tag{1.2}$$

can be rewritten as (1) when the governing maximal monotone F is of the form $F = A + B$.

Example 2 If $B = \partial\phi : H \to 2^H$, where $\phi : H \to (-\infty, +\infty)$ is a proper convex and lower semicontinuous, and $\partial\phi$ is the subdifferential of ϕ, then problem (1) is equivalent to find $x^* \in H$ such that

$$\langle Ax^*, v - x^* \rangle + \phi(v) - \phi(x^*) \geq 0, \quad \forall v \in H, \tag{1.3}$$

which is said to be the mixed quasi-variational inequality.

Example 3 In Example 2, if ϕ is the indicator function of C, i.e.,

$$\phi(x) = \begin{cases} 0, & if\ x \in C \\ +\infty & if\ x \notin C. \end{cases}$$

then problem (1.3) is equivalent to the classical variational inequality problem, denoted by $VI(C; A)$, i.e., to find $x^* \in C$ such that

$$\langle Ax^*, v - x^* \rangle \geq 0, \quad \forall v \in C, \tag{1.4}$$

It is easy to see that (1.4) is equivalent to finding a point $x^* \in C$ such that

$$0 \in (A + B)x^*,$$

where B is the subdifferential of the indicator of C.

For solving the problem (1.1), the forward-backward splitting method [4, 9, 14, 15, 20, 27] is usually employed and is defined by the following manner: $x_1 \in H$ and

$$x_{n+1} = (I + rB)^{-1}(x_n - rAx_n), \quad n \leq 1, \tag{1.5}$$

where $r > 0$. In this case, each step of iterates involves only with A as the forward step and B as the backward step, but not the sum of operators. This method includes, as special cases, the proximal point algorithm [6, 26] and the gradient method. In [13], Lions and Mercier introduced the following splitting iterative methods in a real Hilbert space:

$$x_{n+1} = (2J_r^A - I)(2J_R^B - I)x_n, \quad n \geq 1 \tag{1.6}$$

and

$$x_{n+1} = J_r^A(2J_r^B - I)x_n + (I - J_r^B)x_n, \quad n \geq 1 \tag{1.7}$$

where $J_r^T = (I + rT)^{-1}$ with $r > 0$. The first one is often called Peaceman-Rachford algorithm [21] and the second one is called Douglas-Rachford algorithm [10]. We note that both algorithms are weakly convergent in general [3, 5, 13].

In particular, if $A := \nabla f$ and $B := \partial g$, where ∇f is the gradient of f and ∂g is the subdifferential of g which is defined by $\partial g(x) := \{s \in H : g(y) \geq g(x) + \langle s, y - x\rangle, \ \forall y \in H\}$, then problem (1.1) becomes the following minimization problem:

$$\min_{x \in H} f(x) + g(x) \tag{1.8}$$

and (1.5) also becomes

$$x_{n+1} = prox_{rg}(x_n - r\nabla f(x_n)), \ \ \forall n \geq 1, \tag{1.9}$$

where $r > 0$ is the stepsize and $prox_{rg} = (I + r\partial g)^{-1}$ is the proximity operator of g.

In [1], Alvarez and Attouch employed the heavy ball method which was studied in [22,23] for maximal monotone operators by the proximal point algorithm. This algorithm is called the inertial proximal point algorithm and it is of the following form:

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}) \\ x_{n+1} = (I + r_n B)^{-1} y_n, \ \ n \geq 1. \end{cases} \tag{1.10}$$

It was proved that if $\{r_n\}$ is non-decreasing and $\{\theta_n\} \subset [0, 1)$ with

$$\sum_{n=1}^{\infty} \theta_n ||x_n - x_{n-1}||^2 < \infty, \tag{1.11}$$

then algorithm (1.10) converges weakly to a zero of B. In particular, condition (1.11) is true for $\theta_n < 1/3$. Here, θ_n is an extrapolation factor and the inertia is represented by the term $\theta_n(x_n - x_{n-1})$. It is remarkable that the inertial methodology greatly improves the performance of the algorithm and has a nice convergence properties [11,15,18,19].

Very recently, Cholamjiak W, Cholamjiak P and Suantai [7] proposed the following inertial a modified forward-backward algorithm for monotone operators:

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}) \\ x_{n+1} = \alpha_n u + \beta_n y_n + \gamma_n J_{r_n}^B (y_n - r_n A y_n), \ n \geq 1. \end{cases} \tag{1.12}$$

They proved that the sequence $\{x_n\}$ generated by (1.14) strongly converges to a solution of the inclusion problem (1.1) in Hilbert spaces.

Motivated and inspired by the works in the literature. In this paper, interest is to establish a modified viscosity iterative forward-backward algorithm involving the inertial technique for solving the inclusion problems such that the strong convergence is obtained in the framework of Banach spaces. We also discuss applications of these methods to variational inequalities, convex minimization problem.

2. Preliminaries

In order to prove the main results of the paper, we need the following basic concepts, notations and lemmas.

In what follows, we always assume that X is a uniformly convex and q-uniformly smooth Banach space for some $q \in (1, 2]$ (the definitions and properties, see, for example [8]).

Recall that the generalized duality mapping $J_q : X \to 2^{X^*}$ is defined by

$$J_q(x) = \{j_q(x) \in X^* : \langle j_q(x), x\rangle = ||x||^q,\ ||j_q(x)|| = ||x||^{q-1}\},$$

and the following subdifferential inequality holds: for any $x, y \in X$,

$$||x + y||^q \leq ||x||^q + q\langle y, j_q(x + y)\rangle,\ \ j_q(x + y) \in J_q(x + y). \tag{2.1}$$

Recall that [9] if X is q-uniformly smooth, $q \in (1, 2]$, then there exists a constant $k_q > 0$ such that

$$||x + y||^q \leq ||x||^q + q\langle y, j_q(x)\rangle + k_q||y||^q,\ \ x, y \in X. \tag{2.2}$$

The best constant k_q satisfying (2.2) will be called the q-uniform smoothness coefficient of X.

Proposition 1 ([8]). Let $1 < q \leq 2$. Then the following conclusions hold:

(1) Banach space X is smooth if and only if the duality mapping J_q is single valued.

(2) Banach space X is uniformly smooth if and only if the duality mapping J_q is single valued and norm-to-norm uniformly continuous on bounded sets of X.

Recall that a set-valued operator $A : X \to 2^X$ with the domain D(A) and the range R(A) is said to be accretive if, for each $x, y \in D(A)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle u - v, j(x - y)\rangle \geq 0, \forall u \in Ax \text{ and } v \in Ay. \tag{2.3}$$

An accretive operator A is said to be m-accretive if the range $R(I + \lambda A) = X, \forall \lambda > 0$.

For any $\alpha > 0$ and $q \in (1, 2]$, we say that an accretive operator A is α-inverse strongly accretive (shortly, -isa) of order q, if for each $x, y \in D(A)$, there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\langle u - v, j_q(x - y)\rangle \geq \alpha||u - v||^q, \forall u \in Ax \text{ and } v \in Ay. \tag{2.4}$$

Let C be a nonempty closed and convex subset of a real Banach space X and K be a nonempty subset of C. A mapping $T : C \to K$ is called a retraction of

C onto K if $Tx = x$ for all $x \in K$. We say that T is sunny if, for each $x \in C$ and $t \geq 0$,

$$T(tx + (1-t)Tx) = Tx, \tag{2.5}$$

whenever $tx + (1-t)Tx \in C$. A sunny nonexpansive retraction is a sunny retraction which is also nonexpansive.

Proposition 2 ([13, 28]) Let X be a uniformly smooth Banach space, $T : C \to C$ be a nonexpansive mapping with a fixed point and $f : C \to C$ be a contraction mapping. For each $t \in (0,1)$ the unique fixed point $x_t \in C$ of the contractive mapping, $tf + (1-t)T : C \to C$, converges strongly as $t \to 0$ to the unique fixed point z of T with $z = Qf(z)$, where $Q : C \to Fix(T)$ is the unique sunny nonexpansive retraction from C onto $Fix(T)$.

Lemma 2.1 ([16]) Let $\{a_n\}$, $\{c_n\} \subset R^+$, $\{\alpha_n\} \subset (0,1)$ and $\{b_n\} \subset R$ be the sequences such that

$$a_{n+} \leq (1-\alpha_n)a_n + b_n + c_n, \quad \forall n \geq 1.$$

Assume that $\sum_{n=1}^{\infty} c_n < \infty$. Then the following results hold:

(1) If $b_n \leq \alpha_n M$, where $M \geq 0$, then $\{a_n\}$ is bounded.

(2) If $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup_{n\to\infty} \frac{b_n}{\alpha_n} \leq 0$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2 ([12]) Let $\{s_n\}$ be a sequence of nonnegative real numbers such that

$$s_{n+1} \leq (1-\gamma_n)s_n + \gamma_n \tau_n$$

and

$$s_{n+1} \leq s_n - \eta_n + \rho_n, \ \forall n \geq 1,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$, $\{\eta_n\}$ is a sequence of nonnegative real numbers, $\{\tau_n\}$ and $\{\rho_n\}$ are real sequences such that

(a) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(b) $\lim_{n\to\infty} \rho_n = 0$;

(c) $\lim_{k\to\infty} \eta_{n_k} = 0$ implies $\limsup_{k\to\infty} \tau_{n_k} \leq 0$ for any subsequence $\{n_k\} \subset \{n\}$. Then $\lim_{n\to\infty} s_n = 0$.

It is easy to prove the following conclusion holds.

Lemma 2.3 For any $r > 0$, if

$$T_r := J_r^B(I - rA) = (I + rB)^{-1}(I - rA),$$

then $Fix(T_r) = (A+B)^{-1}(0)$.

Lemma 2.4 ([14]) For any $s \in (0, r]$ and $x \in X$, we have

$$||x - T_s x|| \leq 2||x - T_r x||.$$

Lemma 2.5 ([14]) Let X be a uniformly convex and q-uniformly smooth Banach space with $q \in (1,2]$. Assume that A is a single-valued α-isa of order

q on X. Then, for any $r > 0$, there exists a continuous, strictly increasing and convex function $\phi_q : R^+ \to R^+$ with $\phi_q(0) = 0$ such that for all $x, y \in B_r$,

$$\begin{aligned} ||T_r x - T_r y||^q \leq & ||x-y||^q - r(\alpha q - r^{q-1}k_q)||Ax - Ay||^q \\ & - \phi_q(||(I - J_r^B)(I - rA)x - (I - J_r^B)(I - rA)y||), \end{aligned} \tag{2.6}$$

where k_q is the q-uniform smoothness coefficient of X.

It is easy to prove that the following inequality holds.

Proposition 3 Let $1 < q \leq 2$ and let X be a real smooth Banach space with the generalized duality mapping j_q . Let m be a fixed positive integer. Let $\{x_i\}_{i=1}^m \subset X$ and $t_i \geq 0$ for all $i = 1, 2, \cdots, m$ with $\sum_{i=1}^m t_i \leq 1$. Then we have

$$||\sum_{i=1}^m t_i x_i||^q \leq \sum_{i=1}^m t_i ||x_i||^q. \tag{2.7}$$

Lemma 2.6 ([17]) Let $q > 1$. Then the following inequality holds:

$$ab \leq \frac{1}{q}a^q + \frac{q-1}{q}b^{\frac{q}{q-1}} \tag{2.8}$$

for arbitrarily positive real numbers a and b.

3. Main Results

In this section, we are in position to prove the strong convergence of a Halpern-type forward-backward method involving the inertial technique in Banach space.

Theorem 3.1 Let X be a uniformly convex and $q-$uniformly smooth Banach space, $q \in (1, 2]$. Let $A : X \to X$ be an $\alpha-$isa of order q and $B : X \to 2^X$ be an m-accretive operator such that $\Gamma := (A+B)^{-1}(0) \neq \emptyset$. Let $f : X \to X$ be a contractive mapping with contractive constant $\xi \in (0, 1/q)$. Let $\{x_n\}$ be a sequence generated by x_0, $x_1 \in X$ and

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}) \\ x_{n+1} = \alpha_n f(y_n) + \beta_n y_n + \delta_n J_{r_n}^B (y_n - r_n A y_n), \ n \geq 1. \end{cases} \tag{3.1}$$

where $J_{r_n}^B = (I + r_n B)^{-1}$, k_q is the q-uniform smoothness coefficient of X, $\{\theta_n\} \subset [0, \theta]$ with $\theta \in [0, 1)$, $0 < r_n \leq (\frac{\alpha q}{k_q})^{1/(q-1)}$ and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ are sequences in $[0, 1]$ with $\alpha_n + \beta_n + \delta_n = 1$. Assume that the following conditions hold:

(i) $\sum_{n=1}^\infty \theta_n ||x_n - x_{n-1}|| < \infty$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n \leq (\alpha q/k_q)^{1/(q-1)}$

(iv) $\liminf_{n\to\infty} \delta_n > 0$;

then $\{x_n\}$ converges strongly to $z = Qf(z)$, where Q is a sunny nonexpansive retraction of X onto Γ .

Proof For each $n \geq 1$, we put $T_n = J^B_{r_n}(I - r_nA)$ and let the sequence z_n be defined by

$$z_{n+1} = \alpha_n f(z_n) + \beta_n z_n + \delta_n T_n z_n. \tag{3.2}$$

By the condition $0 < r_n \leq (\frac{\alpha q}{k_q})^{1/(q-1)}$ and Lemma 2.5, we know that T_n is a nonexpansive mapping. Hence we have

$$\begin{aligned} ||x_{n+1} - z_{n+1}|| &\leq \alpha_n \xi ||y_n - z_n|| + \beta_n ||y_n - z_n|| + \delta_n ||y_n - z_n|| \\ &= (1 - \alpha_n(1 - \xi))||x_n + \theta_n(x_n - x_{n-1}) - z_n|| \\ &\leq (1 - \alpha_n(1 - \xi))||x_n - z_n|| + \theta_n ||x_n - x_{n-1}||. \end{aligned} \tag{3.3}$$

By Lemma 2.1 (2), we conclude that $\lim_{n\to\infty} ||x_n - z_n|| = 0$.

Next we show that $\{z_n\}$ is bounded. Indeed, let $z = Qf(z) \in \Gamma$. By Lemma 2.3 this implies that $z \in (A + B)^{-1}(0) = Fix(T_n)$, $\forall n \geq 1$. Hence we have By Lemma 2.1 (2), we conclude that $\lim_{n\to\infty} ||x_n - z_n|| = 0$.

Next we show that $\{z_n\}$ is bounded. Indeed, let $z = Qf(z) \in \Gamma$. By Lemma 2.3 this implies that $z \in (A + B)^{-1}(0) = Fix(T_n)$, $\forall n \geq 1$. Hence we have

$$\begin{aligned} ||z_{n+1} - z|| &= ||\alpha_n(f(z_n) - z) + \beta_n(z_n - z) + \delta_n(T_n z_n - z)|| \\ &\leq \alpha_n ||f(z_n) - f(z)|| + \alpha_n ||f(z) - z|| + \beta_n ||z_n - z|| + \delta_n ||z_n - z|| \\ &\leq (1 - \alpha_n(1 - \xi))||z_n - z|| + \alpha_n ||f(z) - z||. \end{aligned} \tag{3.4}$$

By Lemma 2.1 (1), $\{z_n\}$ is bounded, hence $\{x_n\}$ and $\{y_n\}$ are also bounded. In fact, from (2.1), we have

$$\begin{aligned} ||y_n - z||^q &= ||x_n - z + \theta_n(x_n - x_{n-1})||^q \\ &\leq ||x_n - z||^q + q\theta_n \langle x_n - x_{n-1}, j_q(y_n - z)\rangle. \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} ||x_{n+1} - z||^q &= ||\alpha_n(f(y_n) - z) + \beta_n(y_n - z) + \delta_n(T_n y_n - z)||^q \\ &\leq ||\beta_n(y_n - z) + \delta_n(T_n y_n - z)||^q \\ &\quad + q\alpha_n \langle f(y_n) - z, j_q(x_{n+1} - z)\rangle. \end{aligned} \tag{3.6}$$

Since $z = Qf(z) \in \Gamma = Fix(T_n)$, $\forall n \geq 1$, from Proposition 3 and Lemma 2.5 we have

$$\begin{aligned} &||\beta_n(y_n - z) + \delta_n(T_n y_n - z)||^q \leq \beta_n ||y_n - z||^q + \delta_n ||T_n y_n - T_n z||^q \\ &\quad \leq \beta_n ||y_n - z||^q + \delta_n\{||y_n - z||^q - r_n(\alpha q - r_n^{q-1}k_q)||Ay_n - Az||^q \\ &\qquad - \phi_q(||y_n - r_nAy_n - T_n y_n + r_nAz||)\} \\ &\quad = (1 - \alpha_n)||y_n - z||^q - \delta_n r_n(\alpha q - r_n^{q-1}k_q)||Ay_n - Az||^q \\ &\qquad - \delta_n \phi_q(||y_n - r_nAy_n - T_n y_n + r_nAz||). \end{aligned} \tag{3.7}$$

Also by Lemma 2.6, we have

$$\begin{aligned}
q\alpha_n\langle f(y_n)-z,\, j_q(x_{n+1}-z)\rangle &= q\alpha_n\langle f(y_n)-f(z)+f(z)-z,\, j_q(x_{n+1}-z)\rangle\\
&\le q\alpha_n\xi||y_n-z||\cdot||x_{n+1}-z||^{q-1}+q\alpha_n\langle f(z)-z,\, j_q(x_{n+1}-z)\rangle\\
&\le q\alpha_n\xi\left(\frac{1}{q}||y_n-z||^q+\frac{q-1}{q}||x_{n+1}-z||^q\right)+q\alpha_n\langle f(z)-z,\, j_q(x_{n+1}-z)\rangle\\
&= \alpha_n\xi||y_n-z||^q+\alpha_n\xi(q-1)||x_{n+1}-z||^q+q\alpha_n\langle f(z)-z,\, j_q(x_{n+1}-z)\rangle.
\end{aligned} \tag{3.8}$$

Substituting (3.5), (3.7) and (3.8) into (3.6), simplifying, we have

$$\begin{aligned}
||x_{n+1}-z||^q \le{}& \frac{(1-\alpha_n(1-\xi))}{1-\alpha_n\xi(q-1)}(||x_n-z||^q+q\theta_n\langle x_n-x_{n-1},\, j_q(y_n-z)\rangle)\\
&-\frac{1}{1-\alpha_n\xi(q-1)}\Big(\delta_n r_n(\alpha q-r_n^{q-1}k_q)||Ay_n-Az||^q\\
&+\delta_n\phi_q(||y_n-r_nAy_n-T_ny_n+r_nAz||)\Big)\\
&+\frac{q\alpha_n}{1-\alpha_n\xi(q-1)}\langle f(z)-z,\, j_q(x_{n+1}-z)\rangle.
\end{aligned} \tag{3.9}$$

Since

$$\frac{1-\alpha_n(1-\xi)}{1-\alpha_n\xi(q-1)}=1-\frac{\alpha_n(1-\xi q)}{1-\alpha_n\xi(q-1)}\le 1-\alpha_n(1-\xi q).$$

(3.9) can be written as

$$\begin{aligned}
||x_{n+1}-z||^q \le{}& (1-\alpha_n(1-\xi q))||x_n-z||^q\\
&+\frac{q\alpha_n}{1-\alpha_n\xi(q-1)}\langle f(z)-z,\, j_q(x_{n+1}-z)\rangle\\
&+(1-\alpha_n(1-\xi q))q\theta_n\langle x_n-x_{n-1},\, j_q(y_n-z)\rangle\\
&-\frac{1}{1-\alpha_n\xi(q-1)}\Big(\delta_n r_n(\alpha q-r_n^{q-1}k_q)||Ay_n-Az||^q\\
&+\delta_n\phi_q(||y_n-r_nAy_n-T_ny_n+r_nAz||)\Big).
\end{aligned} \tag{3.10}$$

Since $\alpha q-r_n^{q-1}k_q>0$, we have

$$\begin{aligned}
||x_{n+1}-z||^q \le{}& (1-\alpha_n(1-\xi q))||x_n-z||^q\\
&+\alpha_n(1-\xi q)\Bigg(\frac{q}{[1-\alpha_n\xi(q-1)](1-\xi q)}\langle f(z)-z,\, j_q(x_{n+1}-z)\rangle\\
&+\frac{(1-\alpha_n(1-\xi q))q\theta_n}{\alpha_n(1-\xi q)}\langle x_n-x_{n-1},\, j_q(y_n-z)\rangle\Bigg).
\end{aligned} \tag{3.11}$$

and

$$
\begin{aligned}
||x_{n+1} - z||^q \leq (1 - \alpha_n(1 - \xi q))||x_n - z||^q& \\
- \frac{1}{1 - \alpha_n \xi(q-1)} \Big(\delta_n r_n (\alpha q - r_n^{q-1} k_q)||Ay_n - Az||^q& \\
+ \delta_n \phi_q(||y_n - r_n Ay_n - T_n y_n + r_n Az||) \Big)& \qquad (3.12) \\
+ \frac{q\alpha_n}{1 - \alpha_n \xi(q-1)} \langle f(z) - z, \, j_q(x_{n+1} - z) \rangle& \\
+ (1 - \alpha_n(1 - \xi q)) q\theta_n \langle x_n - x_{n-1}, \, j_q(y_n - z) \rangle.&
\end{aligned}
$$

For each $n \geq 1$, let

$$
\begin{aligned}
s_n &= ||x_{n+1} - z||^q \, ; \qquad \gamma_n = \alpha_n(1 - \xi q); \\
\tau_n &= \frac{q}{[1 - \alpha_n \xi(q-1)](1 - \xi q)} \langle f(z) - z, \, j_q(x_{n+1} - z) \rangle \\
&\quad + \frac{(1 - \alpha_n(1 - \xi q)) q\theta_n}{\alpha_n(1 - \xi q)} \langle x_n - x_{n-1}, \, j_q(y_n - z) \rangle \\
\eta_n &= \frac{1}{1 - \alpha_n \xi(q-1)} \Big(\delta_n r_n (\alpha q - r_n^{q-1} k_q)||Ay_n - Az||^q \\
&\quad + \delta_n \phi_q(||y_n - r_n Ay_n - T_n y_n + r_n Az||) \Big) \\
\rho_n &= \frac{q\alpha_n}{1 - \alpha_n \xi(q-1)} \langle f(z) - z, \, j_q(x_{n+1} - z) \rangle \\
&\quad + (1 - \alpha_n(1 - \xi q)) q\theta_n \langle x_n - x_{n-1}, \, j_q(y_n - z) \rangle.
\end{aligned}
$$

Then, (3.11) and (3.12) are reduced to the following:

$$
s_{n+1} \leq (1 - \gamma_n) s_n + \gamma_n \tau_n \qquad (3.13)
$$

and

$$
s_{n+1} \leq s_n - \eta_n + \rho_n. \qquad (3.14)
$$

Since $\alpha_n \in (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. It follows that $\gamma_n \in (0,1)$, $\sum_{n=1}^{\infty} \gamma_n = \infty$. By the boundedness of $\{y_n\}$ and $\{x_n\}$, we see that $\lim_{n\to\infty} \rho_n = 0$. In order to prove $\lim_{n\to\infty} s_n = 0$, by Lemma 2.2, it is sufficient to prove that for any subsequence $\{n_k\} \subset \{n\}$, if $\lim_{k\to\infty} \eta_{n_k} = 0$, then $\limsup_{k\to\infty} \tau_{n_k} \leq 0$.

Indeed, if $\{n_k\}$ is a subsequence of $\{n\}$ such that $\lim_{k\to\infty} \eta_{n_k} = 0$, then by the assumptions and the property of ϕ_q , we can deduce that

$$
\begin{cases}
\lim\limits_{k\to\infty} ||Ay_{n_k} - Az|| = 0 \\
\lim\limits_{k\to\infty} ||y_{n_k} - r_{n_k} Ay_{n_k} - T_{n_k} y_{n_k} + r_{n_k} Az|| = 0.
\end{cases} \qquad (3.15)
$$

This implies, by the triangle inequality, that

$$\lim_{k\to\infty} ||T_{n_k} y_{n_k} - y_{n_k}|| = 0. \tag{3.16}$$

Since $\liminf_{n\to\infty} r_n > 0$, there is $r > 0$ such that $r_n \geq r$ for all $n \geq 1$. In particular, $r_{n_k} \geq r$ for all $k \geq 1$. It follows from Lemma 2.4 and (3.16) that

$$\limsup_{k\to\infty} ||T_r y_{n_k} - y_{n_k}|| \leq 2 \limsup_{k\to\infty} ||T_{n_k} y_{n_k} - y_{n_k}|| = 0. \tag{3.17}$$

which implies that

$$\lim_{k\to\infty} ||T_r y_{n_k} - y_{n_k}|| = 0. \tag{3.18}$$

On the other hand, we have

$$\begin{aligned} ||T_r y_{n_k} - x_{n_k}|| &\leq ||T_r y_{n_k} - y_{n_k}|| + ||y_{n_k} - x_{n_k}|| \\ &\leq ||T_r y_{n_k} - y_{n_k}|| + \theta_{n_k} ||x_{n_k} - x_{n_k-1}||. \end{aligned} \tag{3.19}$$

By condition (i) and (3.18), we get

$$\lim_{k\to\infty} ||T_r y_{n_k} - x_{n_k}|| = 0. \tag{3.20}$$

Put $z_t = tf(z_t) + (1-t)T_r z_t, \quad t \in (0,1)$. By Proposition 2, z_t converges strongly as $t \to 0$ to the unique fixed point $z = Qf(z) \in Fix(T_r) = (A+B)^{-1}(0)$, where $Q : X \to Fix(T_r)$ is the unique sunny nonexpansive retraction from X onto $Fix(T_r) = (A+B)^{-1}(0)$. So we obtain

$$\begin{aligned} ||z_t - x_{n_k}||^q &= ||t(f(z_t) - x_{n_k}) + (1-t)(T_r z_t - x_{n_k})||^q \\ &\leq (1-t)^q ||T_r z_t - x_{n_k}||^q + qt\langle f(z_t) - z_t, j_q(z_t - x_{n_k})\rangle \\ &\quad + qt\langle z_t - x_{n_k}, j_q(z_t - x_{n_k})\rangle \\ &\leq (1-t)^q (||z_t - x_{n_k}|| + \theta_{n_k} ||x_{n_k} - x_{n_k-1}|| + ||T_r y_{n_k} - x_{n_k}||)^q \\ &\quad + qt\langle f(z_t) - z_t, j_q(z_t - x_{n_k})\rangle + qt||z_t - x_{n_k}||^q. \end{aligned}$$

After simplifying we have

$$\begin{aligned} \langle z_t - f(z_t), j_q(z_t - x_{n_k})\rangle &\leq \frac{1}{qt}\{(1-t)^q (||z_t - x_{n_k}|| \\ &+ \theta_{n_k} ||x_{n_k} - x_{n_k-1}|| + ||T_r y_{n_k} - x_{n_k}||)^q + (qt-1)||z_t - x_{n_k}||^q\}. \end{aligned} \tag{3.21}$$

It follows from condition (i), (3.20) and (3.21) that

$$\limsup_{k\to\infty} \langle z_t - f(z_t), j_q(z_t - x_{n_k})\rangle \leq \frac{1}{qt}[(1-t)^q + (qt-1)]M. \tag{3.22}$$

where $M = \sup_{k\geq 1, t\in(0,1)} (||z_t - x_{n_k}|| + \theta_{n_k} ||x_{n_k} - x_{n_k-1}|| + ||T_r y_{n_k} - x_{n_k}||)^q$. Since $\lim_{t\to 0} \frac{1}{qt}[(1-t)^q + (qt-1)] = 0$, $z_t \to z = Qf(z)$ as $t \to 0$ and by Proposition 1(2) j_q is norm-to-norm uniformly continuous on bounded subsets of X, we have

$$||j_q(z_t - x_{n_k}) - j_q(z - x_{n_k})|| \to 0 \ \ (as \ \ t \to 0). \tag{3.23}$$

Observe that

$$\begin{aligned}&|\langle z_t - f(z_t), j_q(z_t - x_{n_k})\rangle - \langle z - f(z_t), j_q(z - x_{n_k})\rangle|\\ &\quad \leq ||z_t - z|| \cdot ||z_t - x_{n_k}||^{q-1} + ||z - f(z_t)|| \cdot ||j_q(z_t - x_{n_k}) - j_q(z - x_{n_k})||.\end{aligned}$$

This together with (3.22) and (3.23) shows that

$$\begin{aligned}\limsup_{k\to\infty}\langle z - f(z), j_q(z - x_{n_k})\rangle &= \limsup_{k\to\infty}\limsup_{t\to 0}\langle z_t - f(z_t), j_q(z_t - x_{n_k})\rangle\\ &\leq \limsup_{k\to\infty}\limsup_{t\to 0}(\frac{1}{qt}[(1-t)^q + (qt-1)]M)\\ &= 0.\end{aligned} \tag{3.24}$$

On the other hand, we have

$$||x_{n_k+1} - x_{n_k}|| \leq \alpha_{n_k}||f(y_{n_k}) - x_{n_k}|| + \beta_{n_k}\theta_{n_k}||x_{n_k} - x_{n_k-1}|| + \delta_{n_k}||T_{n_k}y_{n_k} - x_{n_k}||. \tag{3.25}$$

By (i), (ii), (3.20) and (3.25), we see that

$$\lim_{k\to\infty}||x_{n_k+1} - x_{n_k}|| = 0. \tag{3.26}$$

Combining (3.24) and (3.26), we get that

$$\limsup_{k\to\infty}\langle z - f(z), j_q(z - x_{n_k+1})\rangle \leq 0. \tag{3.27}$$

It also follows from (i) that $\limsup_{k\to\infty}\tau_{n_k} \leq 0$. By Lemma 2.2, $x_n \to z$ (as $n \to \infty$). This completes the proof of Theorem 3.1.

As well known, if X is a real Hilbert space, then it is a uniformly convex and 2-uniformly smooth Banach space, with the 2-uniform smoothness coefficient $k_2 = 1$. And note that in this case the sunny nonexpansive retraction Q of X onto $\Gamma := (A+B)^{-1}(0)$ is the metric projection P_Γ, and the concept of monotonicity coincides with the concept of accretivity. Hence from Theorem 3.1 we can obtain the following result.

Theorem 3.2 Let X be a real Hilbert space, $A : X \to X$ be an $\alpha-$inverse strongly monotone operator of order 2 and $B : X \to 2^X$ be be a maximal monotone operator such that $\Gamma := (A+B)^{-1}(0) \neq \emptyset$. Let $f : X \to X$ be a contractive mapping with contractive constant $\xi \in (0, 1/2)$ and $\{x_n\}$ be the same as in Theorem 3.1. If the following conditions are satisfied:

(i)$\{\alpha_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ are sequences in $[0,1]$ with $\alpha_n + \beta_n + \delta_n = 1$.

(ii) $\sum_{n=1}^{\infty}\theta_n||x_n - x_{n-1}|| < \infty$;

(iii) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(iv) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n \leq 2\alpha$

(v) $\liminf_{n\to\infty}\delta_n > 0$;

then $\{x_n\}$ converges strongly to $z = P_\Gamma f(z)$, which is a solution of problem of problem (1.1).

4. Applications

In this section we shall utilize the forward-backward methods mentioned above to study monotone variational inequalities, convex minimization problem and convexly constrained linear inverse problem.

Throughout this section, let C be a nonempty closed and convex subset of a real Hilbert space H. Note that in this case the concept of monotonicity coincides with the concept of accretivity.

4.1. Application to Monotone Variational Inequality Problems

A monotone variational inequality problem (VIP) is formulated as the problem of finding a point $x^* \in C$ such that:

$$\langle Ax, y - x\rangle \geq 0 \quad \forall y \in C, \tag{4.1}$$

where $A : C \to H$ is a nonlinear monotone operator. We shall denote by Γ the solution set of (4.1) and assume $\Gamma \neq \emptyset$. In Example 3, we have pointed out that VI(C; A) (4.1) is equivalent to finding a point x^* so that

$$0 \in (A + B)x^*, \tag{4.2}$$

where B is the subdifferential of the indicator of C. and it is a maximal monotone operator. By [25] in this case, the resolvent of B is nothing but the projection operator P_C. Therefore the following result can be obtained from Theorem 3 immediately.

Theorem 4.1 Let $A : C \to H$ be an α-inverse strongly monotone operator of order 2 and let f be the same as in Theorem 3.1. Let $\{x_n\}$ be the sequence generated by $x_0,\ x_1 \in C$ and

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}) \\ x_{n+1} = \alpha_n f(y_n) + \beta_n y_n + \delta_n P_C(y_n - r_n A y_n),\ n \geq 1. \end{cases} \tag{4.3}$$

If the following conditions are satisfied:

(i)$\{\alpha_n\},\ \{\beta_n\}$ and $\{\delta_n\}$ are sequences in $[0, 1]$ with $\alpha_n + \beta_n + \delta_n = 1$.

(ii) $\sum_{n=1}^{\infty} \theta_n ||x_n - x_{n-1}|| < \infty$;

(iii) $\lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=1}^{\infty} \alpha_n = \infty$;

(iv) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n \leq 2\alpha$

(v) $\liminf_{n\to\infty} \delta_n > 0$;

then$\{x_n\}$ converges strongly to a solution z of monotone variational inequality (4.1).

4.2 Application to convex minimization problem

Let $\phi : H \to R$ be a convex function, which is also Fréchet differentiable. Let C be a closed convex subset of H.

Recall that the normal cone to C at $u \in C$ is defined by

$$N_C(u) = \{z \in H : \langle z, y - u\rangle \leq 0,\ \forall y \in C\}.$$

It is well known that N_C is a maximal monotone operator. In this case, we have $J_\lambda^B = P_C$ (the metric projection of H onto C).

By setting $A := \nabla\phi$, the gradient of ϕ, and $B = N_C$, then the problem of finding $x^* \in (A+B)^{-1}0$ is equivalent to find a point $x^* \in C$ such that

$$\langle \nabla\phi(x^*),\ x - x^*\rangle \geq 0,\ \forall x \in C. \tag{4.4}$$

Note that (4.4) is equivalent to the following minimization problem: find $x^* \in C$ such that

$$x^* \in arg\min_{x\in C} \phi(x). \tag{4.5}$$

.

By the above consideration, problem (4.5) is equivalent to the following monotone variational inclusion problem: to find a point $x^* \in C$ such that

$$\begin{aligned}
&0 \in \nabla\phi(x^*) + N_C(x^*) \Leftrightarrow x^* \in (\nabla\phi + N_C)^{-1}(0)\\
&\Leftrightarrow x^* \in Fix(P_C(I - \lambda\nabla\phi))\\
&\Leftrightarrow x^* \in \Omega_2 := \{u \in C : u \in (arg\min_{x\in C} \phi(x))\},
\end{aligned}$$

where Ω_2 is the solution set of problem (4.5).Therefore the following theorem can be obtained from Theorem 3.1 immediately

Theorem 4.2 Let H, C, N_C, and Ω_2 be the same as above. Let $\phi : H \to \mathbb{R}$ be a convex and Fréchet differentiable function and $f : H \to H$ be a contractive mapping with contractive constant $\xi \in (0, \frac{1}{2})$. Let $\{x_n\}$ be the sequence generated by $x_0,\ x_1 \in C$ and

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}) \\ x_{n+1} = \alpha_n f(y_n) + \beta_n y_n + \delta_n P_C(y_n - r_n\nabla\phi(y_n)),\ n \geq 1. \end{cases} \tag{4.3}$$

If Ω_2 is nonempty, $\nabla\phi$ is L-Lipschitz with $L > 0$ and the following conditions are satisfied:

(i) $\{\alpha_n\},\ \{\beta_n\}$ and $\{\delta_n\}$ are sequences in $[0, 1]$ with $\alpha_n + \beta_n + \delta_n = 1$.

(ii) $\sum_{n=1}^{\infty} \theta_n ||x_n - x_{n-1}|| < \infty$;

(iii) $\lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=1}^{\infty} \alpha_n = \infty$;

(iv) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n \leq \frac{2}{L}$

(v) $\liminf_{n\to\infty} \delta_n > 0$;

then$\{x_n\}$ converges strongly to a solution z of monotone variational inequality (4.5).

Proof. Note that if $\phi : H \to \mathbb{R}$ is convex and Fréchet differentiable, and $\nabla\phi : H \to H$ is L-Lipschitz continuous with $L > 0$, then $\nabla\phi$ is $\alpha = \frac{1}{L}$-ism (see [2]). Thus, the required result can be obtained immediately from Theorem 3.1.

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